the premature termination. By the assumption in Section I, $f(z)$ should be the common factor of $F_1(z)$ and $F_2(z)$, or we have $D(z) = f(z)c(z)$ where $c(z)$ is a polynomial function. Then we can continue to test all zeros of $f(z)$ by applying Property 3. Therefore, with Property 2, the sufficient and necessary conditions for all zeros of $f(z)c(z)$ being inside or on the unit circle are that it is always possible to obtain all the real and positive K_i's, for $0 \leqslant i \leqslant n-1$.

Q.E.D.

Property 5 [9]: If no premature termination occurs, then $D(z)$ has r zeros outside the unit circle and $n - r$ zeros inside the unit circle, where r is the number of K_i's in (5), which are negative.

III. A Simple Transformation on a Discrete System

Now let us apply a transformation

$$z \rightarrow \frac{w + w^{-1}}{2}$$

or

$$D(z) \rightarrow \bar{D}(w) = w^n D\left(\frac{w + w^{-1}}{2}\right).$$

Then a real root z_1, with $-1 \leqslant z_1 \leqslant 1$, maps into a pair of complex roots on the unit circle in the W-plane and a pair of complex roots (z_2, z_2^*) inside the unit circle will map into a set of four roots $(w_2', w_2'^*, w_2'', w_2''^*)$ of symmetry about the unit circle in the W-plane, two of them are inside the unit circle, and the others are outside the unit circle. For a stable system we perform the continued fraction expansion in $\phi_{2n}'(w) = \bar{D}(w)/\bar{D}_d(w)$, since $F_2(w) = 0$ for $\bar{D}(w)$ polynomial. Two positive entries in $\{K_i\}$ will be created by a real root z_1 in $D(z)$; two positive and two negative entries in $\{K_i\}$ will be created by a pair of complex roots (z_2, z_2^*).

IV. An Algorithm to Test and an Example

Step 1: Apply the continued fraction expansion on $\phi_n(z)$. If it ends prematurely, apply Property 3, and continue the expansion. The resulting K_i's will be all positive. Then all zeros of the polynomial $f(z)$, which causes the first premature termination of $\phi_n(z)$, appear on the unit circle. It is easy to determine the number of roots on $z = 1$ (or -1) and the pairs of complex roots on the unit circle by testing $f(z)$. Of course, no premature termination occurs for a stable system and we set $f(z) = 1$.

Step 2: Obtain $E(z) = D(z)/f(z)$, which has all its zeros inside the unit circle, and apply another continued fraction expansion on $\phi_{2l}'(w) = \bar{E}(w)/\bar{E}_d(w)$, until all new $2l$ entries can eventually be obtained, where l denotes the degree of $E(z)$. If $2n_1$ negative and $2(n_1 + n_2)$ positive entries appear after the kth premature termination, which should also appear k times before the kth premature termination, there exist n_1 pairs of complex and n_2 real roots of multiplicity $(k+1)$ in $E(z)$. If no termination occurs, it implies that all roots of $E(z)$ are distinct inside the unit circle. Then the number $2l_1$ in $\{K_1\}$, which is negative, corresponds to the number l_1 of pairs of complex roots of $E(z)$ and the number $l - 2l_1$ corresponds to the number of real roots in $E(z)$.

Example [8]: Let $D(z) = 2z^3 + 2z^2 + z$.

Step 1: Then

$$\phi_3(z) = \frac{2z^3 + 3z^2 + 3z + 2}{2z^3 + z^2 - z - 2}.$$

All positive entries are obtained in [8], i.e., $(K_0, K_1, K_2) = (\frac{5}{7}, \frac{49}{16}, \frac{16}{7})$. No premature termination occurs; therefore, all zeros of $D(z)$ are inside the unit circle and set $f(z) = 1$.

Step 2:

$$\phi_6'(w) = \frac{w^6 + 2w^5 + 5w^4 + 4w^3 + 5w^2 + 2w + 1}{8w^6 + 12w^5 + 12w^4 - 12w^2 - 12w - 8}.$$

No premature termination occurs and there are two negative numbers in $(K_0, K_1, K_2, K_3, K_4, K_5) = (\frac{1}{12}, \frac{180}{7}, \frac{49}{96}, \frac{32}{63}, -\frac{567}{896}, -\frac{6272}{441})$; therefore, it has one pair of complex and one pair of real roots in $D(z)$.

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A New Algorithm to Compute the Discrete Cosine Transform

BYEONG GI LEE

Abstract—**A new algorithm is introduced for the 2^m-point discrete cosine transform. This algorithm reduces the number of multiplications to about half of those required by the existing efficient algorithms, and it makes the system simpler.**

Introduction

During the past decade, the discrete cosine transform (DCT) [1] has found applications in speech and image processing. Various fast algorithms have been introduced for reducing the number of multiplications involved in the transform [2]–[6]. In this correspondence we propose an additional algorithm which not only reduces the number of multiplications but also has a simpler structure. We refer to this algorithm as the FCT (fast cosine transform), since it is similar to the FFT (fast Fourier transform). The number of real multiplications it requires is about half that required by the existing efficient algorithms.

Algorithm Derivation

We denote the DCT of the data sequence $x(k)$, $k = 0, 1, \cdots, N-1$, by $X(n)$, $n = 0, 1, \cdots, N-1$. Then we have [1]

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$$x(k) = \sum_{n=0}^{N-1} e(n)\, X(n) \cos [\pi(2k+1)\, n/2N], \qquad k = 0, 1, \cdots, N-1 \tag{1}$$

and

$$X(n) = \frac{2}{N} e(n) \sum_{k=0}^{N-1} x(k) \cos [\pi(2k+1)\, n/2N], \qquad n = 0, 1, \cdots, N-1 \tag{2}$$

where

$$e(n) = \begin{cases} 1/\sqrt{2}, & \text{if } n = 0, \\ 1, & \text{otherwise.} \end{cases} \tag{3}$$

We consider (1), which is the inverse DCT (IDCT), and define C such that

$$C_{2N}^{(2k+1)n} = \cos [\pi(2k+1)\, n/2N]. \tag{4}$$

Then the N-point IDCT becomes

$$x(k) = \sum_{k=0}^{N-1} \hat{X}(n)\, C_{2N}^{(2k+1)n}, \qquad k = 0, 1, \cdots, N-1 \tag{5}$$

where

$$\hat{X}(n) = e(n)\, X(n). \tag{6}$$

Decomposing $x(k)$ into even and odd indexes of n (assuming that N is even), we can rewrite (5) as

$$x(k) = g(k) + h'(k), \tag{7a}$$

$$x(N-1-k) = g(k) - h'(k), \qquad k = 0, 1, \cdots, N/2-1 \tag{7b}$$

where

$$g(k) = \sum_{n=0}^{N/2-1} \hat{X}(2n)\, C_{2N}^{(2k+1)2n} \tag{8a}$$

$$h'(k) = \sum_{n=0}^{N/2-1} \hat{X}(2n+1)\, C_{2N}^{(2k+1)(2n+1)}. \tag{8b}$$

Clearly, $g(k)$, $k = 0, 1, \cdots, N/2-1$, forms an $N/2$-point IDCT, since

$$C_{2N}^{(2k+1)2n} = C_{N}^{(2k+1)n} = C_{2(N/2)}^{(2k+1)n} \tag{9}$$

We rewrite $h'(k)$ in the form

$$h'(k) = \sum_{n=0}^{N/2-1} \hat{X}'(2n+1)\, C_{2(N/2)}^{(2k+1)n} \tag{10}$$

which is another $N/2$-point IDCT. Since

$$2C_{2N}^{(2k+1)}\, C_{2N}^{(2k+1)(2n+1)} = C_{2N}^{(2k+1)2n} + C_{2N}^{(2k+1)2(n+1)} \tag{11}$$

we have

$$2C_{2N}^{(2k+1)}\, h'(k) = \sum_{n=0}^{N/2-1} \hat{X}(2n+1)\, C_{2N}^{(2k+1)2n} + \sum_{n=0}^{N/2-1} \hat{X}(2n+1)\, C_{2N}^{(2k+1)2(n+1)}. \tag{12}$$

So, if we define

$$\hat{X}(2n-1)\big|_{n=0} = 0 \tag{13}$$

then

$$\sum_{n=0}^{N/2-1} \hat{X}(2n+1)\, C_{2N}^{(2k+1)2(n+1)} = \sum_{n=0}^{N/2-1} \hat{X}(2n-1)\, C_{2N}^{(2k+1)2n} \tag{14}$$

because

$$C_{2N}^{(2k+1)2(N/2)} = C_{2}^{(2k+1)} = 0. \tag{15}$$

Thus (12) can be rewritten as

$$2C_{2N}^{(2k+1)}\, h'(k) = \sum_{n=0}^{N/2-1} (\hat{X}(2n+1) + \hat{X}(2n-1))\, C_{2N}^{(2k+1)2n} \tag{16}$$

which has the form of (10). Now we define

$$G(n) = \hat{X}(2n), \tag{17a}$$

$$H(n) = \hat{X}(2n+1) + \hat{X}(2n-1), \qquad n = 0, 1, \cdots, N/2-1, \tag{17b}$$

and

$$g(k) = \sum_{n=0}^{N/2-1} G(n)\, C_{2(N/2)}^{(2k+1)n}, \tag{18a}$$

$$h(k) = \sum_{n=0}^{N/2-1} H(n)\, C_{2(N/2)}^{(2k+1)n}, \qquad k = 0, 1, \cdots, N/2-1. \tag{18b}$$

Then (7), (8), and (16)–(18) finally yield

$$x(k) = g(k) + (1/(2C_{2N}^{(2k+1)}))\, h(k), \tag{19a}$$

$$x(N-1-k) = g(k) - (1/(2C_{2N}^{(2k+1)}))\, h(k), \qquad k = 0, 1, \cdots, N/2-1. \tag{19b}$$

Therefore, we have decomposed the N-point IDCT in (5) into the sum of two $N/2$-point IDCT's in (18). By repeating this process, we can decompose the IDCT further.

We can also decompose the DCT in a similar manner. Alternatively, the DCT can be obtained by "transposing" the IDCT–i.e., reversing the direction of the arrows in the flow graph of IDCT, since the DCT is an *orthogonal* transform.

Example

With $N = 8$, (17)–(19) yield

$$G(n) = \hat{X}(2n), \tag{20a}$$

$$H(n) = \hat{X}(2n+1) + \hat{X}(2n-1), \qquad n = 0, 1, 2, 3 \tag{20b}$$

and

$$g(k) = \sum_{n=0}^{3} G(n)\, C_{8}^{(2k+1)n}, \tag{21a}$$

$$h(k) = \sum_{n=0}^{3} H(n)\, C_{8}^{(2k+1)n}, \tag{21b}$$

$$x(k) = g(k) + (1/(2C_{16}^{2k+1}))\, h(k), \tag{22a}$$

$$x(7-k) = g(k) - (1/(2C_{16}^{2k+1}))\, h(k), \qquad k = 0, 1, 2, 3. \tag{22b}$$

Equations (20) and (22) respectively form the first and the last stages of the flow graph in Fig. 1. By repeating the above steps

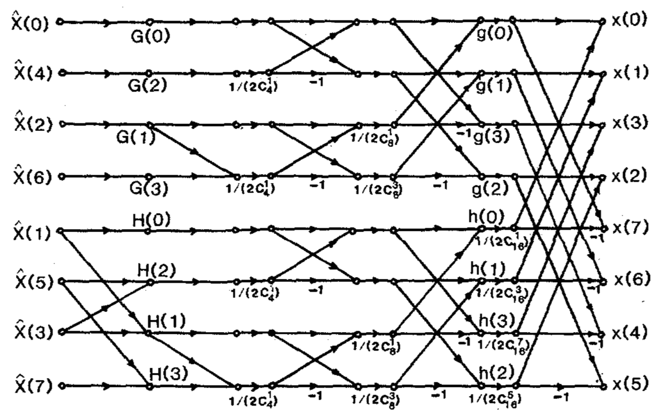

Fig. 1.

TABLE I

m	N	Number of Multiplications		Number of Additions	
		REF[4]	FCT	REF[4]	FCT
3	8	16	12	26	29
4	16	44	32	74	81
5	32	116	80	194	209
6	64	292	192	482	513
7	128	708	448	1154	1217
8	256	1668	1024	2690	2817
9	512	3844	2304	6146	6401
10	1024	8708	5120	13826	14337
11	2048	19460	11264	30722	31745
12	4096	43012	24576	67586	69633

on (21), we obtain the FCT flow graph for an eight-point IDCT as shown in Fig. 1.

Concluding Remarks

It follows from Fig. 1 that the flow graphs of the FCT and FFT are similar. The number of real multiplications thus appears to be $(N/2)\log_2 N$ for an N-point FCT with $N = 2^m$, which is about half the number required by existing efficient algorithms. The number of additions, however, is slightly higher and given by $(3N/2)\log_2 N - N + 1$. See Table I for a comparison with the algorithm in [4].

If Fig. 1 we also note that the input sequence $\hat{X}(n)$ is in bit-reversed order. The order of the output sequence $x(k)$ is generated in the following manner: starting with the set (0, 1), form a set by adding the prefix "0" to each element, and then obtain the rest of the elements by complementing the existing ones. This process results in the set (00, 01, 11, 10), and by repeating it we obtain (000, 001, 011, 010, 111, 110, 100, 101). Thus, we have the output sequence $x(0)$, $x(1)$, $x(3)$, $x(2)$, $x(7)$, $x(6)$, $x(4)$, $x(5)$ for the case $N = 8$; see Fig. 1.

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On the Interrelationships Among a Class of Convolutions

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Abstract—**In this paper some interrelationships among a class of circular operations are investigated based on matrix formulation. It is shown that a class of convolutions representing forward/backward and convolution/correlation of two periodic sequences may be related to each other in terms of discrete transforms having the circular convolution property. The results obtained are useful in efficient realization of adaptive digital filters using fast transforms.**

I. Introduction

The need for computing convolution of two functions arises in many diverse applications. These include digital filtering, spectrum analysis, time delay estimation, computation of discrete Fourier transform (DFT) using circular correlation, multiplication of large integers, polynomial transforms, and so forth [1], [2]. In computation of various convolutions, the fast convolution approach using efficient computational algorithms of discrete transforms has proven to be useful [3].

Recently, discrete transforms based on number theoretic concepts have received considerable attention as a method for efficient and error-free computation of digital convolutions [2]. Unlike the fast Fourier transform (FFT), the number theoretic transform (NTT) does not cause roundoff errors in arithmetic operations. Particularly, the Fermat number transform that is one of the NTT's requires only word shifts and additions, but not multiplications, nor the storage of basis functions. Accordingly, the NTT has several desirable properties in carrying out various convolution operations in comparison to the FFT.

In this correspondence, we consider a class of convolutions that include forward and backward convolutions of two periodic sequences and also forward and backward correlations. Based on matrix formulation, we study their interrelationships. Particularly, we show that they may be related to each other through a discrete transform such as DFT and NTT.

II. Interrelationships Among a Class of Convolutions

Here we discuss a class of circular operations based on matrix formulation. In the following discussion it is assumed that various arithmetic operations including matrix operations are

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