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A new approach for one-dimensional sine-Gordon equation

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Abstract
In this work, we use a reproducing kernel method for investigating the sine-Gordon equation with initial and boundary conditions. Numerical experiments are studied to show the efficiency of the technique. The acquired results are compared with the exact solutions and results obtained by different methods. These results indicate that the reproducing kernel method is very effective.

Keywords: reproducing kernel method; sine-Gordon equation; bounded linear operator; homogenizing

1 Introduction

The nonlinear one-dimensional sine-Gordon (SG) equation came into sight in the differential geometry and attracted a lot of attention because of the collisional behaviors of solitons that arise from this equation. Numerical solutions of the SG equation have been widely investigated in recent years [1–5]. Compact finite difference and diagonally implicit Runge-Kutta-Nyström (DIRKN) methods were used [3]. The authors of [6] introduced a numerical method for solving the SG equation by using collocation and radial basis functions. The boundary integral equation technique is presented in [7]. Bratsos [8, 9] has researhed a numerical technique for solving the one-dimensional SG equation and a third-order numerical technique for the two-dimensional SG equation. A numerical technique using radial basis functions for the solution of the two-dimensional SG equation has been shown in [10]. Some authors advised spectral techniques and Fourier pseudospectral technique for solving nonlinear wave equation taking a discrete Fourier series and Chebyshev orthogonal polynomials [11–13]. Ma and Wu [14] used a meshless technique by using a multiquadric (MQ) quasi-interpolation. In this paper, we investigate the one-dimensional nonlinear sine-Gordon equation

$$\frac{\partial^2 u}{\partial \tau^2}(\eta,\tau) = \frac{\partial^2 u}{\partial \eta^2}(\eta,\tau) - \sin\bigl(u(\eta,\tau)\bigr), \quad 0 \le \eta \le 1, \tau \ge 0, \tag{1}$$

with initial conditions

$$\begin{aligned} &u(\eta,0) = f(\eta), \quad 0 \le \eta \le 1, \\ &\frac{\partial u}{\partial \tau}(\eta,0) = g(\eta), \quad 0 \le \eta \le 1, \end{aligned} \tag{2}$$

and boundary conditions

$$u(0,\tau) = h_1(\tau), \qquad u(1,\tau) = h_2(\tau), \quad \tau \geq 0, \tag{3}$$

by using the reproducing kernel method (RKM). We can get numerical results in very short time. By this method nonlinear problems can be solved easily like linear problems. Reproducing kernel functions are very important for numerical results. We can change the inner product in the spaces and obtain different reproducing kernel functions for better results. These are advantages of this method. Homogenizing the initial and boundary conditions is very significant for this method. We give a general transformation to homogenize the initial and boundary conditions in Section 3.

The theory of reproducing kernels [15] was used for the first time at the beginning of the 20th century by Zaremba. Reproducing kernel theory has important implementations in numerical analysis, differential equations, and probability and statistics [16–25]. The efficiency of the method was used by many authors to research several scientific implementations. The reproducing kernel functions can be represented by piecewise polynomials, and the higher the order of derivatives, the simpler the reproducing kernel function statements. Such statements of reproducing kernel functions are the simplest from the computational viewpoint, and the speed and accuracy can be significantly improved in scientific and engineering implementations. The productivity of such reproducing kernel functions is indicated to be very exhorting by experimental results [26].

This work is arranged as follows. Section 2 introduces several useful reproducing kernel functions. A representation of solution in $W_2^{(3,3)}(\Omega)$ is given in Section 3. Section 4 presents the essential results: exact and approximate solutions of (1)-(3); enhancement of the method to some problems in the reproducing kernel space; and convergence of the approximate solution. Some numerical examples are discussed in Section 5. There are some conclusions in the final section.

2 Reproducing kernel functions

We obtain some useful reproducing kernel functions in this section.

Definition 1 [16] Let E be a nonempty set. A function $K : E \times E \longrightarrow C$ is called a reproducing kernel function of the Hilbert space H if

(a) $\forall \tau \in E, K(\cdot,\tau) \in H$,

(b) $\forall \tau \in E, \forall \varphi \in H, \langle \varphi(\cdot), K(\cdot,\tau)\rangle = \varphi(\tau)$.

Definition 2 [16] A Hilbert space H defined on a nonempty set E is called a reproducing kernel Hilbert space if there exists a reproducing kernel function $K(\eta,\tau)$.

Definition 3 [16] We define the $W_2^3[0,1]$ by

$$W_2^3[0,1] = \left\{ \begin{array}{c} u \mid u, u', u'' \text{ are absolutely continuous real-valued functions in } [0,1], \\ u^{(3)} \in L^2[0,1], \eta \in [0,1], u(0) = 0, u(1) = 0 \end{array} \right\}.$$

The inner product and the norm in $W_2^3[0,1]$ are defined respectively by

$$\langle u,v\rangle_{W_2^3} = u(0)v(0) + u'(0)v'(0) + u'(1)v'(1) + \int_0^1 u^{(3)}(\eta)v^{(3)}(\eta)\,d\eta, \quad u,v \in W_2^3[0,1],$$

and

$$\|u\|_{W_2^3} = \sqrt{\langle u,u\rangle_{W_2^3}}, \quad u \in W_2^3[0,1].$$

Definition 4 [16] We define the space $F_2^3[0,T]$ by

$$F_2^3[0,T] = \left\{\begin{array}{c} u \mid u, u', u'' \text{ are absolutely continuous in } [0,T], \\ u^{(3)} \in L^2[0,T], \tau \in [0,T], u(0) = 0, u'(0) = 0 \end{array}\right\}$$

with the inner product and norm

$$\langle u,v\rangle_{F_2^3} = \sum_{i=0}^{2} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u^{(3)}(T)v^{(3)}(\tau)\,d\tau, \quad u,v \in F_2^3[0,T],$$

and

$$\|u\|_{F_2^3} = \sqrt{\langle u,u\rangle_{F_2^3}}, \quad u \in F_2^3[0,T].$$

The space $F_2^3[0,T]$ is a reproducing kernel space, and its reproducing kernel function r_s is given by

$$r_s(\tau) = \begin{cases} \frac{1}{4}s^2\tau^2 + \frac{1}{12}s^2\tau^3 - \frac{1}{24}s\tau^4 + \frac{1}{120}\tau^5, & \tau \le s, \\ \frac{1}{4}s^2\tau^2 + \frac{1}{12}s^3\tau^2 - \frac{1}{24}\tau s^4 + \frac{1}{120}s^5, & \tau > s. \end{cases}$$

Definition 5 [16] We define the space $H_2^1[0,T]$ by

$$H_2^1[0,T] = \left\{\begin{array}{c} u \mid u \text{ is absolutely continuous in } [0,1], \\ u' \in L^2[0,T], \tau \in [0,T] \end{array}\right\}$$

the inner product and norm

$$\langle u,v\rangle_{H_2^1} = u(0)v(0) + \int_0^\tau u'(T)v'(\tau)\,d\tau, \quad u,v \in H_2^1[0,T],$$

and

$$\|u\|_{H_2^1} = \sqrt{\langle u,u\rangle_{H_2^1}}, \quad u \in H_2^1[0,T].$$

Its reproducing kernel function q_s is

$$q_s(\tau) = \begin{cases} 1+\tau, & \tau \le s, \\ 1+s, & \tau > s. \end{cases}$$

Definition 6 [16] We define the space $G_2^1[0,1]$ by

$$G_2^1[0,1] = \left\{\begin{array}{c} u \mid u \text{ is absolutely continuous in } [0,1], \\ u' \in L^2[0,1], \eta \in [0,1] \end{array}\right\}$$

with the inner product and norm

$$\langle u,v\rangle_{G_2^1} = u(0)v(0) + \int_0^1 u'(\eta)v'(\eta)\,d\eta, \quad u,v \in G_2^1[0,1],$$

and

$$\|u\|_{G_2^1} = \sqrt{\langle u,u\rangle_{G_2^1}}, \quad u \in G_2^1[0,1].$$

The space $G_2^1[0,1]$ is a reproducing kernel space, and its reproducing kernel function Q_y is given by

$$Q_y(\eta) = \begin{cases} 1+\eta, & \eta \le y, \\ 1+y, & \eta > y. \end{cases}$$

Theorem 2.1 *The reproducing kernel function R_y of $W_2^3[0,1]$ is*

$$R_y(\eta) = \begin{cases} \sum_{i=1}^6 c_i(y)\eta^{i-1}, & \eta \le y, \\ \sum_{i=1}^6 d_i(y)\eta^{i-1}, & \eta > y, \end{cases} \tag{4}$$

where

$$\begin{aligned}
&c_1(y) = 0, \qquad c_4(y) = 0, \qquad d_1(y) = \frac{1}{120}y^5, \qquad d_4(y) = -\frac{1}{12}y^2,\\
&c_2(y) = -\frac{1}{122}y^5 + \frac{5}{244}y^4 - \frac{127}{244}y^2 + \frac{31}{61}y,\\
&c_3(y) = -\frac{1}{2{,}928}y^5 + \frac{127}{5{,}856}y^4 - \frac{1}{12}y^3 + \frac{1{,}137}{1{,}952}y^2 - \frac{127}{244}y,\\
&c_5(y) = \frac{1}{2{,}938}y^5 - \frac{5}{5{,}856}y^4 + \frac{127}{5{,}856}y^2 - \frac{31}{1{,}464}y,\\
&c_6(y) = -\frac{1}{7{,}320}y^5 + \frac{1}{2{,}928}y^4 - \frac{1}{2{,}928}y^2 - \frac{1}{122}y + \frac{1}{120},\\
&d_2(y) = -\frac{1}{122}y^5 - \frac{31}{1{,}464}y^4 - \frac{127}{244}y^2 + \frac{31}{61}y,\\
&d_3(y) = -\frac{1}{2{,}928}y^5 + \frac{127}{5{,}856}y^4 + \frac{1{,}137}{1{,}952}y^2 - \frac{127}{244}y,\\
&d_5(y) = \frac{1}{2{,}928}y^5 - \frac{5}{5{,}856}y^4 + \frac{127}{5{,}856}y^2 + \frac{5}{244}y,\\
&d_6(y) = -\frac{1}{7{,}320}y^5 + \frac{1}{2{,}928}y^4 - \frac{1}{2{,}928}y^2 - \frac{1}{122}y.
\end{aligned}$$

Proof Let $u \in W_2^3[0,1]$ and $0 \le y \le 1$. Define R_y by (4). Note that

$$R'_y(\eta) = \begin{cases} \sum_{i=1}^5 i c_{i+1}(y)\eta^{i-1}, & \eta < y, \\ \sum_{i=1}^5 i d_{i+1}(y)\eta^{i-1}, & \eta > y, \end{cases}$$

$$R''_y(\eta) = \begin{cases} \sum_{i=1}^4 i(i+1) c_{i+2}(y)\eta^{i-1}, & \eta < y, \\ \sum_{i=1}^4 i(i+1) d_{i+2}(y)\eta^{i-1}, & \eta > y, \end{cases}$$

$$R_y^{(3)}(\eta) = \begin{cases} \sum_{i=1}^{3} i(i+1)(i+2)c_{i+3}(y)\eta^{i-1}, & \eta < y, \\ \sum_{i=1}^{3} i(i+1)(i+2)d_{i+3}(y)\eta^{i-1}, & \eta > y, \end{cases}$$

$$R_y^{(4)}(\eta) = \begin{cases} \sum_{i=1}^{2} i(i+1)(i+2)(i+3)c_{i+4}(y)\eta^{i-1}, & \eta < y, \\ \sum_{i=1}^{2} i(i+1)(i+2)(i+3)d_{i+4}(y)\eta^{i-1}, & \eta > y, \end{cases}$$

and

$$R_y^{(5)}(\eta) = \begin{cases} 120c_6(y), & \eta < y, \\ 120d_6(y), & \eta > y. \end{cases}$$

By Definition 5 and integration by parts we have

$$\begin{aligned} \langle u(\eta), R_y(\eta)\rangle_{W_2^3} &= u(0)R_y(0) + u'(0)R_y'(0) + u'(1)R_y'(1) + u''(1)R_y^{(3)}(1) \\ &\quad - u''(0)R_y^{(3)}(0) - u'(1)R_y^{(4)}(1) + u'(0)R_y^{(4)}(0) + \int_0^1 u'(\eta)R_y^{(5)}(\eta)\,d\eta \\ &= u'(0)\bigl(R_y'(0) + R_y^{(4)}(0)\bigr) + u'(1)\bigl(R_y'(1) - R_y^{(4)}(1)\bigr) \\ &\quad + u''(1)R_y^{(3)}(1) - u''(0)R_y^{(3)}(0) \\ &\quad + \int_0^y R_y^{(5)}(\eta)u'(\eta)\,d\eta + \int_y^1 R_y^{(5)}(\eta)u'(\eta)\,d\eta \\ &= u'(0)\bigl(c_2(y) + 24c_5(y)\bigr) - u''(0)\bigl(6c_4(y)\bigr) \\ &\quad + u'(1)\bigl(d_2(y) + 2d_3(y) + 3d_4(y) - 20d_5(y) - 115d_6(y)\bigr) \\ &\quad + u''(1)\bigl(6d_4(y) + 24d_5(y) + 60d_6(y)\bigr) \\ &\quad + \int_0^y 120c_6(y)u(\eta)\,d\eta + \int_y^1 120d_6(y)u(\eta)\,d\eta \\ &= 120u(y)\left(\frac{1}{120}\right) = u(y). \end{aligned}$$

This completes the proof. □

Definition 7 [16] We define the space $W_2^{(3,3)}(\Omega)$ by

$$W_2^{(3,3)}(\Omega) = \left\{ \begin{array}{c} u \mid \frac{\partial^4 u}{\partial\eta^2\partial\tau^2} \text{ is completely continuous in } \Omega = [0,1]\times[0,\tau], \\ \frac{\partial^6 u}{\partial\eta^3\partial\tau^3} \in L^2(\Omega), u(\eta,0)=0, \frac{\partial u(\eta,0)}{\partial\tau} = 0, u(0,\tau)=0, u(1,\tau)=0 \end{array} \right\}$$

with the inner product and norm

$$\begin{aligned} \langle u,v\rangle_{W_2^{(3,3)}} &= \sum_{i=0}^{2}\int_0^\tau \left[\frac{\partial^3}{\partial\tau^3}\frac{\partial^i}{\partial\eta^i}u(0,\tau)\frac{\partial^3}{\partial\tau^3}\frac{\partial^i}{\partial\eta^i}v(0,\tau)\right]d\tau \\ &\quad + \sum_{j=0}^{2}\left\langle \frac{\partial^j}{\partial\tau^j}u(\cdot,0), \frac{\partial^j}{\partial\tau^j}v(\cdot,0)\right\rangle_{W_2^3} \\ &\quad + \int_0^1\int_0^\tau \left[\frac{\partial^3}{\partial\eta^3}\frac{\partial^3}{\partial\tau^3}u(\eta,\tau)\frac{\partial^3}{\partial\eta^3}\frac{\partial^3}{\partial\tau^3}v(\eta,\tau)\right]d\tau\,d\eta, \quad u,v\in W_2^{(3,3)}(\Omega) \end{aligned}$$

and

$$\|u\|_W = \sqrt{\langle u, u\rangle_W}, \quad u \in W_2^{(3,3)}(\Omega).$$

Theorem 2.2 *Let $K_{(y,s)}(\eta,\tau)$ be a reproducing kernel function $W_2^{(3,3)}(\Omega)$. We have*

$$K_{(y,s)}(\eta,\tau) = R_y(\eta) r_s(\tau),$$

and for any $u \in W_2^{(3,3)}(\Omega)$,

$$u(y,s) = \langle u(\eta,\tau), K_{(y,s)}(\eta,\tau)\rangle_{W_2^{(3,3)}}$$

and

$$K_{(y,s)}(\eta,\tau) = K_{(\eta,\tau)}(y,s).$$

Definition 8 [16] We define the space $\widehat{W}_2^{(1,1)}(\Omega)$ by

$$\widehat{W}_2^{(1,1)}(\Omega) = \left\{ \begin{array}{c} u \mid u \text{ is completely continuos in } \Omega = [0,1] \times [0,\tau], \\ \frac{\partial^2 u}{\partial\eta\partial\tau} \in L^2(\Omega) \end{array} \right\}$$

with the inner product and norm

$$\begin{aligned} \langle u, v\rangle_{\widehat{W}_2^{(1,1)}} &= \int_0^\tau \left[\frac{\partial}{\partial\tau} u(0,\tau) \frac{\partial}{\partial\tau} v(0,\tau)\right] d\tau + \langle u(\cdot,0), v(\cdot,0)\rangle_{G_2^1} \\ &\quad + \int_0^1 \int_0^\tau \left[\frac{\partial}{\partial\eta}\frac{\partial}{\partial\tau} u(\eta,\tau) \frac{\partial}{\partial\eta}\frac{\partial}{\partial\tau} v(\eta,\tau)\right] d\tau\, d\eta, \quad u,v \in \widehat{W}_2^{(1,1)} \end{aligned}$$

and

$$\|u\|_{\widehat{W}_2^{(1,1)}} = \sqrt{\langle u, u\rangle_{\widehat{W}_2^{(1,1)}}}, \quad u \in \widehat{W}_2^{(1,1)}.$$

$\widehat{W}_2^{(1,1)}(\Omega)$ is a reproducing kernel space, and its reproducing kernel function $G_{(y,s)}(\eta,\tau)$ is given as

$$G_{(y,s)}(\eta,\tau) = Q_y(\eta) q_s(\tau).$$

3 Solutions in $W_2^{(3,3)}(\Omega)$

In this section, we give the solution of (1)-(3) in the reproducing kernel space $W_2^{(3,3)}(\Omega)$. We define the linear operator $L : W_2^{(3,3)}(\Omega) \to \widehat{W}_2^{(1,1)}(\Omega)$ as

$$Lv = \frac{\partial^2 v}{\partial\tau^2}(\eta,\tau) - \frac{\partial^2 v}{\partial\eta^2}(\eta,\tau), \quad v \in W_2^{(3,3)}(\Omega).$$

If we homogenize the conditions of the model problem (1)-(3), then it changes to the following problem:

$$\begin{cases} Lv = M(\eta,\tau), & (\eta,\tau) \in \Omega = [0,1] \times [0,\tau], \\ v(\eta,0) = \frac{\partial v}{\partial\tau}(\eta,0) = v(0,\tau) = v(1,\tau) = 0, \end{cases} \tag{5}$$

where

$$\begin{aligned}
M(\eta,\tau) &= \frac{(\eta-1)f(\eta)h_1''(\tau)}{h_1(0)} - \eta h_2''(\tau) - \frac{2f'(\eta)h_1(\tau)}{h_1(0)} - \frac{(\eta-1)f''(\eta)h_1(\tau)}{h_1(0)} \\
&\quad + 2f'(\eta) + \eta f''(\eta) + \tau\left(g''(\eta) + \frac{2g(0)f'(\eta)}{h_1(0)} + \frac{(\eta-1)f''(\eta)g(0)}{h_1(0)}\right) \\
&\quad - \sin\left[\begin{array}{c} v(\eta,\tau) - \frac{(\eta-1)f(\eta)h_1(\tau)}{h_1(0)} + \eta h_2(\tau) + \eta f(\eta) - \eta h_2(0) \\ +\tau\left(g(\eta) + \frac{(\eta-1)f(\eta)g(0)}{h_1(0)} - \eta g(1)\right) \end{array}\right]
\end{aligned}$$

and

$$\begin{aligned}
v(\eta,\tau) &= u(\eta,\tau) + \frac{(\eta-1)f(\eta)h_1(\tau)}{h_1(0)} - \eta h_2(\tau) - \eta f(\eta) + \eta h_2(0) \\
&\quad - \tau\left(g(\eta) + \frac{(\eta-1)f(\eta)g(0)}{h_1(0)} - \eta g(1)\right)
\end{aligned}$$

with $h_1(0) \neq 0$.

Lemma 3.1 *The operator L is bounded linear.*

Proof By Definition 8 we have

$$\begin{aligned}
\|Lu\|^2_{\widehat{W}_2^{(1,1)}} &= \int_0^\tau \left[\frac{\partial}{\partial\tau}Lu(0,\tau)\right]^2 d\tau + \langle Lu(\cdot,0), Lu(\cdot,0)\rangle_{G_2^1} \\
&\quad + \int_0^1 \int_0^\tau \left[\frac{\partial}{\partial\eta}\frac{\partial}{\partial\tau}Lu(\eta,\tau)\right]^2 d\tau\, d\eta \\
&= \int_0^\tau \left[\frac{\partial}{\partial\tau}Lu(0,\tau)\right]^2 d\tau + \left[Lu(0,0)\right]^2 \\
&\quad + \int_0^1 \left[\frac{\partial}{\partial\eta}Lu(\eta,0)\right]^2 d\eta + \int_0^1 \int_0^\tau \left[\frac{\partial}{\partial\eta}\frac{\partial}{\partial\tau}Lu(\eta,\tau)\right]^2 d\tau\, d\eta.
\end{aligned}$$

Since

$$\begin{aligned}
u(\eta,\tau) &= \langle u(\xi,\gamma), K_{(\eta,\tau)}(\xi,\gamma)\rangle_{W_2^{(3,3)}}, \\
Lu(\eta,\tau) &= \langle u(\xi,\gamma), LK_{(\eta,\tau)}(\xi,\gamma)\rangle_{W_2^{(3,3)}},
\end{aligned}$$

from the continuity of $K_{(\eta,\tau)}(\xi,\gamma)$ we have

$$\left|Lu(\eta,\tau)\right| \leq \|u\|_{W_2^{(3,3)}} \left\|LK_{(\eta,\tau)}(\xi,\gamma)\right\|_{W_2^{(3,3)}} = a_0\|u\|_{W_2^{(3,3)}}.$$

Accordingly, for $i = 0, 1$,

$$\begin{aligned}
\frac{\partial^i}{\partial\eta^i}Lu(\eta,\tau) &= \left\langle u(\xi,\gamma), \frac{\partial^i}{\partial\eta^i}LK_{(\eta,\tau)}(\xi,\gamma)\right\rangle_{W_2^{(3,3)}}, \\
\frac{\partial}{\partial\tau}\frac{\partial^i}{\partial\eta^i}Lu(\eta,\tau) &= \left\langle u(\xi,\gamma), \frac{\partial}{\partial\tau}\frac{\partial^i}{\partial\eta^i}LK_{(\eta,\tau)}(\xi,\gamma)\right\rangle_{W_2^{(3,3)}},
\end{aligned}$$

and then

$$\left|\frac{\partial^i}{\partial\eta^i}Lu(\eta,\tau)\right| \le e_i\|u\|_{W_2^{(3,3)}}, \qquad \left|\frac{\partial}{\partial\tau}\frac{\partial^i}{\partial\eta^i}Lu(\eta,\tau)\right| \le f_i\|u\|_{W_2^{(3,3)}}.$$

Therefore,

$$\|Lu(\eta,\tau)\|^2_{\widehat{W}_2^{(1,1)}} \le \sum_{i=0}^{1}\left(e_i^2+\tau f_i^2\right)\|u\|^2_{W_2^{(3,3)}} = A\|u\|^2_{W_2^{(3,3)}},$$

where $A=\sum_{i=0}^{1}(e_i^2+\tau f_i^2)$. □

Now, choose a countable dense subset $\{(\eta_1,\tau_1),(\eta_2,\tau_2),\ldots\}$ in Ω and define

$$\varphi_i(\eta,\tau)=G_{(\eta_i,\tau_i)}(\eta,\tau), \qquad \Psi_i(\eta,\tau)=L^*\varphi_i(\eta,\tau),$$

where L^* is the adjoint operator of L. The orthonormal system $\{\widehat{\Psi}_i(\eta,\tau)\}_{i=1}^{\infty}$ of $W_2^{(3,3)}(\Omega)$ can be obtained by the Gram-Schmidt orthogonalization of $\{\Psi_i(\eta,\tau)\}_{i=1}^{\infty}$ as

$$\widehat{\Psi}_i(\eta,\tau)=\sum_{k=1}^{i}\beta_{ik}\Psi_k(\eta,\tau).$$

Theorem 3.2 *Assume that $\{(\eta_i,\tau_i)\}_{i=1}^{\infty}$ is dense in Ω. Then $\{\Psi_i(\eta,\tau)\}_{i=1}^{\infty}$ is a complete system in $W_2^{(3,3)}(\Omega)$, and*

$$\Psi_i(\eta,\tau)=L_{(y,s)}K_{(y,s)}(\eta,\tau)|_{(y,s)=(\eta_i,\tau_i)}.$$

Proof We have

$$\begin{aligned}\Psi_i(\eta,\tau) &= \left(L^*\varphi_i\right)(\eta,\tau)=\left\langle\left(L^*\varphi_i\right)(y,s),K_{(\eta,\tau)}(y,s)\right\rangle_{W_2^{(3,3)}}\\ &= \left\langle\varphi_i(y,s),L_{(y,s)}K_{(\eta,\tau)}(y,s)\right\rangle_{\widehat{W}_2^{(1,1)}}\\ &= L_{(y,s)}K_{(\eta,\tau)}(y,s)|_{(y,s)=(\eta_i,\tau_i)}\\ &= L_{(y,s)}K_{(y,s)}(\eta,\tau)|_{(y,s)=(\eta_i,\tau_i)}.\end{aligned}$$

Clearly, $\Psi_i(\eta,\tau)\in W(\Omega)$. For each fixed $u(\eta,\tau)\in W_2^{(3,3)}(\Omega)$, if

$$\left\langle u(\eta,\tau),\Psi_i(\eta,\tau)\right\rangle_{W_2^{(3,3)}}=0, \quad i=1,2,\ldots,$$

then

$$\left\langle u(\eta,\tau),\left(L^*\varphi_i\right)(\eta,\tau)\right\rangle_{W_2^{(3,3)}}=\left\langle Lu(\eta,\tau),\varphi_i(\eta,\tau)\right\rangle_{\widehat{W}_2^{(1,1)}}=(Lu)(\eta_i,\tau_i)=0, \quad i=1,2,\ldots.$$

Since $\{(\eta_i,\tau_i)\}_{i=1}^{\infty}$ is dense in Ω, $(Lu)(\eta,\tau)=0$. Therefore, $u=0$ by the existence of L^{-1}. □

Theorem 3.3 *If $\{(\eta_i,\tau_i)\}_{i=1}^{\infty}$ is dense in Ω, then the solution of* (5) *is*

$$u=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}M(\eta_k,\tau_k)\widehat{\Psi}_i(\eta,\tau). \tag{6}$$

Proof The system $\{\Psi_i(\eta,\tau)\}_{i=1}^{\infty}$ is complete in $W_2^{(3,3)}(\Omega)$. Therefore, we get

$$\begin{aligned} u &= \sum_{i=1}^{\infty} \langle u, \widehat{\Psi}_i \rangle_{W_2^{(3,3)}} \widehat{\Psi}_i = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \langle u, \Psi_k \rangle_{W_2^{(3,3)}} \widehat{\Psi}_i = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \langle u, L^*\varphi_k \rangle_{W_2^{(3,3)}} \widehat{\Psi}_i \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \langle Lu, \varphi_k \rangle_{\widehat{W}_2^{(1,1)}} \widehat{\Psi}_i = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \langle Lu, G_{(\eta_k,\tau_k)} \rangle_{\widehat{W}_2^{(1,1)}} \widehat{\Psi}_i \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} Lu(\eta_k,\tau_k)\widehat{\Psi}_i = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} M(\eta_k,\tau_k)\widehat{\Psi}_i. \end{aligned}$$

This completes the proof. □

Now an approximate solution u_n can be obtained from the n-term intercept of the exact solution u:

$$u_n = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} M(\eta_k,\tau_k)\widehat{\Psi}_i(\eta,\tau).$$

Obviously,

$$\|u_n(\eta,\tau) - u(\eta,\tau)\| \to 0 \quad (n \to \infty).$$

4 The method implementation

If M is linear, then the analytical solution of (5) can be obtained directly by (6). If M is nonlinear, then the solution of (5) can be obtained either by (6) or by an iterative method as follows. We construct an iterative sequence u_n by putting

$$\begin{cases} \text{any fixed } u_0 \in W_2^{(3,3)}, \\ u_n = \sum_{i=1}^{n} A_i \widehat{\Psi}_i, \end{cases} \tag{7}$$

where

$$\begin{cases} A_1 = \beta_{11} M(\eta_k,\tau_k), \\ A_2 = \sum_{k=1}^{2} \beta_{2k} M(\eta_k,\tau_k), \\ \cdots \\ A_n = \sum_{k=1}^{n} \beta_{nk} M(\eta_k,\tau_k). \end{cases} \tag{8}$$

Lemma 4.1 *If*

$$u_n \xrightarrow{\|\cdot\|} \widehat{u}, \quad \|u_n\| \textit{ is bounded}, (\eta_n,\tau_n) \to (y,s), \textit{ and } M(\eta,\tau) \textit{ is continuous},$$

then

$$M(\eta_n,\tau_n) \to M(y,s).$$

Proof By the reproducing property and Cauchy-Schwarz inequality we have

$$\begin{aligned} |u(\eta,\tau)| &= \left|\langle u(y,s), K_{(\eta,\tau)}(y,s)\rangle_{W_2^{(3,3)}}\right| \\ &\le \|u(y,s)\|_{W_2^{(3,3)}} \|K_{(\eta,\tau)}(y,s)\|_{W_2^{(3,3)}} = N_1 \|u(y,s)\|_{W_2^{(3,3)}}. \end{aligned}$$

Thus, we obtain

$$\left|\frac{\partial u(\eta,\tau)}{\partial \eta}\right| = \left|\left\langle u(y,s), \frac{\partial K_{(\eta,\tau)}(y,s)}{\partial \eta}\right\rangle_{W_2^{(3,3)}}\right| \le \|u(y,s)\|_{W_2^{(3,3)}} \left\|\frac{\partial K_{(\eta,\tau)}(y,s)}{\partial \eta}\right\|_{W_2^{(3,3)}}$$
$$= N_2 \|u(y,s)\|_{W_2^{(3,3)}}$$

and

$$\left|\frac{\partial u(\eta,\tau)}{\partial \tau}\right| = \left|\left\langle u(y,s), \frac{\partial K_{(\eta,\tau)}(y,s)}{\partial \tau}\right\rangle_{W_2^{(3,3)}}\right| \le \|u(y,s)\|_{W} \left\|\frac{\partial K_{(\eta,\tau)}(y,s)}{\partial \tau}\right\|_{W_2^{(3,3)}}$$
$$= N_3 \|u(y,s)\|_{W_2^{(3,3)}}.$$

One the one hand, we have

$$\begin{aligned} \left|u_{n-1}(y,s) - \widehat{u}(y,s)\right| &= \left|\langle u_{n-1}(\eta,\tau) - \widehat{u}(\eta,\tau), K_{(y,s)}(\eta,\tau)\rangle_{W_2^{(3,3)}}\right| \\ &\le \left\|u_{n-1}(\eta,\tau) - \widehat{u}(\eta,\tau)\right\|_{W_2^{(3,3)}} \left\|K_{(\eta,\tau)}(y,s)\right\|_{W_2^{(3,3)}} \\ &= N_4 \left\|u_{n-1}(\eta,\tau) - \widehat{u}(\eta,\tau)\right\|_{W_2^{(3,3)}}; \end{aligned}$$

on the other hand, we get

$$\begin{aligned} \left|u_{n-1}(\eta_n,\tau_n) - \widehat{u}(y,s)\right| &= \left|u_{n-1}(\eta_n,\tau_n) - u_{n-1}(y,s) + u_{n-1}(y,s) - \widehat{u}(y,s)\right| \\ &\le \left|\nabla u_{n-1}(\xi,\eta)\right|\left|(\eta_n,\tau_n) - (y,s)\right| + \left|u_{n-1}(y,s) - \widehat{u}(y,s)\right|. \end{aligned}$$

Using these inequalities with $u_n \stackrel{\|\cdot\|}{\to} \widehat{u}$, we find

$$\left|u_{n-1}(y,s) - \widehat{u}(y,s)\right| \to 0, \qquad \left|\nabla u_{n-1}(\xi,\eta)\right| \le \sqrt{c_1^2 + c_2^2}\|u\|_{W_2^{(3,3)}}.$$

Therefore, as $n \to \infty$, using the boundedness of $\|u_n\|$ gives

$$\left|u_{n-1}(\eta_n,\tau_n) - \widehat{u}(y,s)\right| \to 0.$$

As $n \to \infty$, with the continuity of $M(\eta,\tau)$ we get

$$M(\eta_n,\tau_n) \to M(y,s).$$

This completes the proof. □

Theorem 4.2 *Assume that* $\|u_n\|$ *is a bounded in* (7) *and that* (5) *has a unique solution. If* $\{(\eta_i,\tau_i)\}_{i=1}^{\infty}$ *is dense in* $W_2^{(3,3)}(\Omega)$, *then the n-term approximate solutions* $u_n(\eta,\tau)$ *converge to the analytical solution* $u(\eta,\tau)$ *of* (5), *and*

$$u(\eta,\tau) = \sum_{i=1}^{\infty} A_i \widehat{\Psi}_i(\eta,\tau),$$

where A_i *is given by* (8).

Proof First, we prove the convergence of $u_n(\eta,\tau)$. From (7) we infer that

$$u_{n+1}(\eta,\tau) = u_n(\eta,\tau) + A_{n+1}\widehat{\Psi}_{n+1}(\eta,\tau),$$

The orthonormality of $\{\widehat{\Psi}_i\}_{i=1}^{\infty}$ yields that

$$\|u_{n+1}\|^2 = \|u_n\|^2 + A_{n+1}^2 = \sum_{i=1}^{n+1} A_i^2. \tag{9}$$

In terms of (9), we have that $\|u_{n+1}\| > \|u_n\|$. Since $\|u_n\|$ is bounded, $\|u_n\|$ is convergent, and there exists a constant c such that

$$\sum_{i=1}^{\infty} A_i^2 = c.$$

This implies that

$$\{A_i\}_{i=1}^{\infty} \in l^2.$$

If $m > n$, then

$$\begin{aligned}\|u_m - u_n\|^2 &= \|u_m - u_{m-1} + u_{m-1} - u_{m-2} + \cdots + u_{n+1} - u_n\|^2 \\ &\leq \|u_m - u_{m-1}\|^2 + \|u_{m-1} - u_{m-2}\|^2 + \cdots + \|u_{n+1} - u_n\|^2.\end{aligned}$$

Since

$$\|u_m - u_{m-1}\|^2 = A_m^2,$$

we have

$$\|u_m - u_n\|^2 = \sum_{l=n+1}^{m} A_l^2 \to 0 \quad \text{as } n \to \infty.$$

The completeness of $W_2^{(3,3)}(\Omega)$ shows that $u_n \to \widehat{u}$ as $n \to \infty$. We have

$$\widehat{u}(\eta,\tau) = \sum_{i=1}^{\infty} A_i \widehat{\Psi}_i(\eta,\tau).$$

Note that

$$(L\widehat{u})(\eta,\tau) = \sum_{i=1}^{\infty} A_i L\widehat{\Psi}_i(\eta,\tau)$$

and

$$\begin{aligned}(L\widehat{u})(\eta_l,\tau_l) &= \sum_{i=1}^{\infty} A_i L\widehat{\Psi}_i(\eta_l,\tau_l) = \sum_{i=1}^{\infty} A_i \langle L\widehat{\Psi}_i(\eta,\tau), \varphi_l(\eta,\tau)\rangle_{\widehat{W}_2^{(1,1)}} \\ &= \sum_{i=1}^{\infty} A_i \langle \widehat{\Psi}_i(\eta,\tau), L^*\varphi_l(\eta,\tau)\rangle_{W_2^{(3,3)}} = \sum_{i=1}^{\infty} A_i \langle \widehat{\Psi}_i(\eta,\tau), \Psi_l(\eta,\tau)\rangle_{W_2^{(3,3)}}.\end{aligned}$$

Therefore,

$$\sum_{l=1}^{i} \beta_{il}(L\widehat{u})(\eta_l,\tau_l) = \sum_{i=1}^{\infty} B_i \left\langle \widehat{\Psi}_i(\eta,\tau), \sum_{l=1}^{i} \beta_{il}\Psi_l(\eta,\tau) \right\rangle_{W_2^{(3,3)}}$$
$$= \sum_{i=1}^{\infty} B_i \langle \widehat{\Psi}_i(\eta,\tau), \widehat{\Psi}_l(\eta,\tau) \rangle_{W_2^{(3,3)}} = A_l.$$

In view of (8), we have

$$L\widehat{u}(\eta_l,\tau_l) = M(\eta_l,\tau_l).$$

Since $\{(\eta_i,\tau_i)\}_{i=1}^{\infty}$ is dense in Ω, for each $(y,s) \in \Omega$, there exists a subsequence $\{(\eta_{n_j},\tau_{n_j})\}_{j=1}^{\infty}$ such that

$$(\eta_{n_j},\tau_{n_j}) \to (y,s) \quad (j \to \infty).$$

We know that

$$L\widehat{u}(\eta_{n_j},\tau_{n_j}) = M(\eta_{n_j},\tau_{n_j}).$$

Let $j \to \infty$; by the continuity of M we have

$$(L\widehat{u})(y,s) = M(y,s),$$

which proves that $\widehat{u}(\eta,\tau)$ satisfies (5). □

We obtain an approximate solution $\zeta_n(t)$ as

$$\zeta_n(t) = \sum_{i=1}^{n} \sum_{k=1}^{i} \sigma_{ik} z\big(t_k, \zeta(t_k)\big)\widehat{\eta}_i(t). \tag{10}$$

Remark Let consider a countable dense set

$$\big\{(\eta_1,\tau_1),(\eta_2,\tau_2),\dots\big\} \in \Omega$$

and define

$$\varphi_i = G_{(\eta_i,\tau_i)}, \qquad \Psi_i = L^*\varphi_i, \qquad \widehat{\Psi}_i = \sum_{k=1}^{i} \beta_{ik}\Psi_k.$$

Then the coefficients β_{ik} can be found by

$$\beta_{11} = \frac{1}{\|\Psi_1\|}, \quad \beta_{ii} = \frac{1}{\sqrt{\|\Psi_i\|^2 - \sum_{k=1}^{i-1} c_{ik}^2}}, \quad \beta_{ij} = \frac{-\sum_{k=j}^{i-1} c_{ik}\beta_{kj}}{\sqrt{\|\Psi_i\|^2 - \sum_{k=1}^{i-1} c_{ik}^2}}, \quad c_{ik} = \langle \Psi_i, \widehat{\Psi}_k \rangle.$$

5 Numerical experiments

In this section, we solve two examples were solved with RKM. We show our results by tables and figures. The numerical results are compared with exact solutions and existing numerical approximations to illustrate the efficiency and high accuracy of the method. The method presents the solutions in terms of convergent series with easily computable components and improves the convergence of the series solution. The method was used

Table 1 Numerical results for Example 5.1

η	τ	ES	AS	AE	Time CPU (s)
0.1	0.1	0.2938926262	0.2938930965	4.703×10^{-7}	3.860
0.2	0.2	0.4755282582	0.4755313577	3.0995×10^{-6}	3.016
0.3	0.3	0.4755282582	0.4755183355	9.9227×10^{-6}	2.984
0.4	0.4	0.2938926261	0.2939007109	8.0848×10^{-6}	3.000
0.5	0.5	0.0	0.0000282140	2.82140×10^{-5}	3.094
0.6	0.6	−0.2938926264	−0.2939063137	1.36873×10^{-5}	3.031
0.7	0.7	−0.4755282583	−0.4755305759	2.3176×10^{-6}	3.031
0.8	0.8	−0.4755282581	−0.4755277748	4.833×10^{-7}	2.953
0.9	0.9	−0.2938926260	−0.2938966580	4.0320×10^{-6}	3.204
1.0	1.0	0.0	$-3.690702068 \times 10^{-7}$	$3.690702068 \times 10^{-7}$	3.578

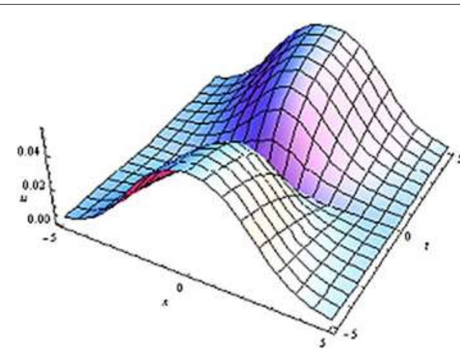
Table 2 Numerical results for Example 5.1 with $\tau = 1$

η	ES AS	AE RE
−0.80	0.5877852522	1.756×10^{-7}
	0.5877854278	$2.987485639 \times 10^{-7}$
−0.40	0.9510565165	7.70×10^{-8}
	0.9510565935	$8.096259125 \times 10^{-8}$
0.40	−0.9510565165	7.70×10^{-8}
	−0.9510565935	$8.096259125 \times 10^{-8}$
0.80	−0.5877852522	1.756×10^{-7}
	−0.5877854278	$2.987485639 \times 10^{-7}$

Table 3 Comparison of AE and RE for Example 5.1

η	AE [27]	AE [RKM]	RE [27]	RE [RKM]
−0.80	1.94*E*−05	1.756*E*−07	3.29*E*−05	2.987485639*E*−7
−0.40	2.84*E*−07	7.700*E*−08	2.98*E*−07	8.096259125*E*−8
0.40	2.84*E*−07	7.700*E*−08	2.98*E*−07	8.096259125*E*−8
0.80	1.94*E*−05	1.756*E*−07	3.29*E*−05	2.987485639*E*−7

Figure 1 Plots of RKM solution for Example 5.1.



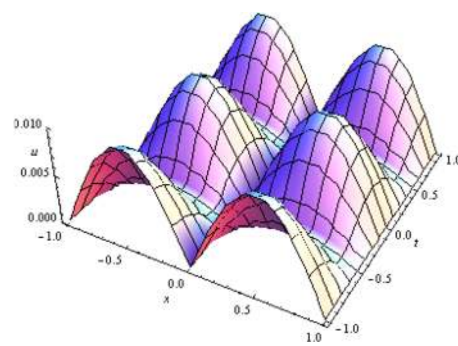


Figure 2 Plots of absolute error for Example 5.1.

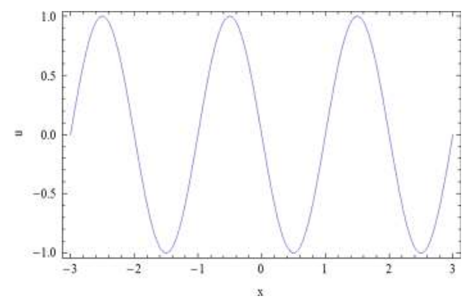


Figure 3 Plots of absolute error for Example 5.1.

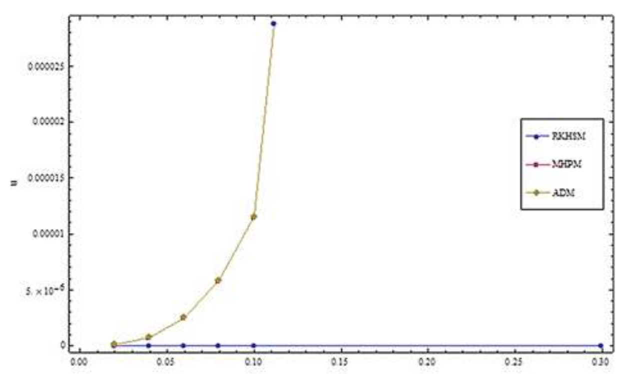


Figure 4 Plots of absolute error for Example 5.1.

in a direct way without using restrictive assumptions. Throughout this work, all computations are implemented by using Maple 16 software package.

Example 5.1 Let us consider the problem with the following initial conditions:

$$u(\eta,0)=\sin(\pi\eta),\qquad \frac{\partial u}{\partial\tau}(\eta,0)=0.$$

The exact solution is [28]

$$u(\eta,\tau)=\frac{1}{2}\bigl(\sin\pi(\eta+\tau)+\sin\pi(\eta-\tau)\bigr).$$

After homogenizing the initial conditions and using our method, we obtain the results presented in Tables 1-3 and Figures 1-4.

Table 4 Numerical results for Example 5.2

η	τ	ES	AS	AE	RE	Time CPU (s)
0.1	0.1	0.396702532289366215	0.39670253228936612	9.43×10^{-17}	2.37×10^{-16}	0.874
0.2	0.2	0.77443854423966038	0.774438544239660056	3.24×10^{-16}	4.19×10^{-16}	0.843
0.3	0.3	1.11790876976715877	1.117908769767159201	4.29×10^{-16}	3.83×10^{-16}	0.874
0.4	0.4	1.41753016381685945	1.417530163816859722	2.65×10^{-16}	1.87×10^{-16}	0.952
0.5	0.5	1.66943886908587781	1.669438869085877479	3.33×10^{-16}	1.99×10^{-16}	0.874
0.6	0.6	1.87415961287513759	1.874159612875137133	4.57×10^{-16}	2.44×10^{-16}	0.904
0.7	0.7	2.03492391748435838	2.034923917484357432	9.47×10^{-16}	4.65×10^{-16}	0.921
0.8	0.8	2.15626165034295019	2.156261650342951155	9.63×10^{-16}	4.46×10^{-16}	0.967
0.9	0.9	2.24305837947587106	2.243058379475871257	1.88×10^{-16}	8.39×10^{-17}	0.999
1.0	1.0	2.30002473031364741	2.300024730313647325	8.66×10^{-17}	3.76×10^{-17}	0.967

Table 5 Numerical results for Example 5.2 with $\tau = 1$

η	ES	AS	AE	RE	Time CPU (s)
–0.80	2.5681097221289163512	2.5681097220865804301	4.2×10^{-11}	1.6×10^{-11}	0.842
–0.40	2.9858433445292456583	2.9858433445285564413	6.8×10^{-13}	2.3×10^{-13}	0.873
0.00	3.1415926535897932385	3.1415926535905335235	7.4×10^{-13}	2.3×10^{-13}	0.936
0.40	2.9858433445292456583	2.9858433445285564413	6.8×10^{-13}	2.3×10^{-13}	0.686
0.80	2.5681097221289163512	2.5681097220865804301	4.2×10^{-11}	1.6×10^{-11}	0.733

Table 6 Comparison of absolute and relative errors for Example 5.2

η	AE [27]	AE [RKM]	RE [27]	RE [RKM]
–0.80	1.53*E*–08	4.23359211*E*–11	5.96*E*–09	1.64852462241777932*E*–11
–0.40	3.54*E*–10	6.89217*E*–13	1.18*E*–10	2.30828252012217808*E*–13
0.00	1.62*E*–10	7.40285*E*–13	5.15*E*–11	2.35640034093567477*E*–13
0.40	3.54*E*–10	6.89217*E*–13	1.18*E*–10	2.30828252012217808*E*–13
0.80	1.53*E*–08	4.23359211*E*–11	5.96*E*–09	1.64852462241777932*E*–11

Table 7 Numerical results for Example 5.2 with $\eta = 2.5$

τ	ES	AS	AE	RE	Time CPU (s)
0.02	0.013045652299470337726	0.013045652299470429228	9.15×10^{-17}	7.01×10^{-15}	1.014
0.04	0.026091027076458045383	0.026091027076458055762	1.03×10^{-17}	3.97×10^{-16}	0.905
0.06	0.039135846843901571207	0.039135846843901591760	$2. \times 10^{-17}$	5.25×10^{-16}	0.873
0.08	0.05217983418557021854	0.052179834185570326022	1.07×10^{-16}	2.05×10^{-15}	0.842
0.1	0.065222711791451326376	0.06522271179145110938	2.16×10^{-16}	3.32×10^{-15}	0.858
0.3	0.19552959072837645953	0.19552959072837616718	2.9×10^{-16}	1.49×10^{-15}	0.749

Table 8 Numerical solutions for Example 5.2 with $\eta = 5.0$

τ	ES	AS	AE	RE	Time CPU (s)
0.02	0.0010780225516042560299	0.0010780225516046950169	4.3×10^{-16}	4.07×10^{-13}	1.233
0.04	0.0021560449466078880312	0.0021560449466103154491	2.4×10^{-15}	1.12×10^{-12}	0.655
0.06	0.003234067028410408468	0.0032340670284064571408	3.9×10^{-15}	1.22×10^{-12}	0.811
0.08	0.0043120886404116027904	0.0043120886404125186434	9.1×10^{-16}	2.12×10^{-13}	0.936
0.1	0.0053901096260116659256	0.0053901096260170153463	5.3×10^{-15}	9.92×10^{-13}	1.092
0.3	0.016170250578558993341	0.016170250578577492240	1.8×10^{-14}	1.14×10^{-12}	1.139

Table 9 Numerical results for Example 5.2 with $\eta = 7.5$

τ	ES	AS	AE	RE	Time CPU (s)
0.02	0.0000884934721388573766	0.0000884934721392853003	4.2×10^{-16}	4.8×10^{-12}	0.827
0.04	0.0001769869441910896592	0.0001769869441950110056	3.9×10^{-15}	2.2×10^{-11}	0.749
0.06	0.0002654804160700717544	0.0002654804160796786727	9.6×10^{-15}	3.6×10^{-11}	0.889
0.08	0.0003539738876891785695	0.0003539738876799533295	9.2×10^{-15}	2.6×10^{-11}	0.967
0.1	0.0004424673589617850136	0.0004424673589703796322	8.5×10^{-15}	1.9×10^{-11}	0.812
0.3	0.0013274020335728111501	0.0013274020333220852196	2.5×10^{-13}	1.8×10^{-10}	1.326

Table 10 Numerical results for Example 5.2 with $\eta = 10.0$

τ	ES	AS	AE	RE	Time CPU (s)
0.02	0.00000726398874701739435	0.0000072639887470122372	5.1×10^{-18}	7.0×10^{-13}	0.874
0.04	0.00001452797749398687768	0.0000145279774939233104	6.3×10^{-17}	4.3×10^{-12}	0.936
0.06	0.00002179196624086053894	0.0000217919662408353413	2.5×10^{-17}	1.1×10^{-12}	0.998
0.08	0.00002905595498759046712	0.0000290559549867947262	7.9×10^{-16}	2.7×10^{-11}	1.170
0.1	0.00003631994373412875118	0.000036319943734399877	2.7×10^{-16}	7.4×10^{-12}	0.858
0.3	0.00010895983117843073892	0.0001089598311794268356	9.9×10^{-16}	9.1×10^{-12}	0.889

Table 11 Comparison of absolute errors for Example 5.2 with $\eta = 2.5$ and $\eta = 5.0$

τ	AE RKM $\eta = 2.5$	AE MHPM $\eta = 2.5$ [29]	AE ADM $\eta = 2.5$ [29]	AE RKM $\eta = 5.0$	AE MHPM $\eta = 5.0$ [29]	AE ADM $\eta = 5.0$ [29]
0.02	9.15*E*−17	9.25104*E*−8	9.25104*E*−8	4.3*E*−16	5.22002*E*−11	5.22002*E*−11
0.04	1.03*E*−17	7.40084*E*−7	7.40084*E*−7	2.4*E*−15	4.17602*E*−10	4.17602*E*−10
0.06	2.*E*−17	2.49778*E*−6	2.49778*E*−6	3.9*E*−15	1.40941*E*−9	1.40941*E*−9
0.08	1.07*E*−16	5.92068*E*−6	5.92068*E*−6	9.1*E*−16	3.34082*E*−9	3.34082*E*−9
0.1	2.16*E*−16	1.15638*E*−5	1.15638*E*−5	5.3*E*−15	6.52506*E*−9	6.52506*E*−9
0.3	2.9*E*−16	3.12304*E*−4	3.12304*E*−4	1.8*E*−14	1.76230*E*−7	1.76230*E*−7

Table 12 Comparison of absolute errors for Example 5.2 with $\eta = 7.5$ and $\eta = 10.0$

τ	AE RKM $\eta = 7.5$	AE MHPM $\eta = 7.5$ [29]	AE ADM $\eta = 7.5$ [29]	AE RKM $\eta = 10.0$	AE MHPM $\eta = 10.0$ [29]	AE ADM $\eta = 10.0$ [29]
0.02	4.2*E*−16	2.88750*E*−14	2.88750*E*−14	5.1*E*−18	1.59700*E*−17	1.59700*E*−17
0.04	3.9*E*−15	2.31000*E*−13	2.31000*E*−13	6.3*E*−17	1.27763*E*−16	1.27763*E*−16
0.06	9.6*E*−15	7.79626*E*−13	7.79626*E*−13	2.5*E*−17	4.31201*E*−16	4.31201*E*−16
0.08	9.2*E*−15	1.84800*E*−12	1.84800*E*−12	7.9*E*−16	1.02210*E*−15	1.02210*E*−15
0.1	8.5*E*−15	3.60939*E*−12	3.60939*E*−12	2.7*E*−16	1.99629*E*−15	1.99629*E*−15
0.3	2.5*E*−13	9.74833*E*−11	9.74833*E*−11	9.9*E*−16	5.39165*E*−14	5.39165*E*−14

Table 13 Numerical results for Example 5.2 with $\eta = 0.06$

τ	ES	AS	AE	RE	Time CPU (s)
0.02	0.07984560896434381352	0.079845608964343853426	3.99×10^{-17}	4.99×10^{-16}	0.889
0.04	0.15962763841261813303	0.15962763841261815794	2.49×10^{-17}	1.56×10^{-16}	1.232
0.06	0.23928281206851416623	0.23928281206851423765	7.14×10^{-17}	2.98×10^{-16}	0.920
0.08	0.31874845652859878735	0.31874845652859888367	9.63×10^{-17}	3.02×10^{-16}	0.904
0.1	0.39796279376194770105	0.39796279376194771082	9.77×10^{-18}	2.45×10^{-17}	0.858

Table 14 Numerical results for Example 5.2 with $\eta = 0.06$

τ	ES	AS	AE	RE	Time CPU (s)
0.02	0.079734118548588251664	0.079734118548588199339	5.23×10^{-17}	6.56×10^{-16}	0.811
0.04	0.15940492340173257962	0.15940492340173286314	2.83×10^{-16}	1.77×10^{-15}	0.796
0.06	0.23894940199533129661	0.23894940199533120491	9.17×10^{-17}	3.83×10^{-16}	0.889
0.08	0.31830514045634341908	0.31830514045634405770	6.38×10^{-16}	2.0×10^{-15}	0.982
0.1	0.39741061409807554739	0.39741061409807569743	1.5×10^{-16}	3.77×10^{-16}	0.811

Table 15 Numerical results for Example 5.2 with $\eta = 0.1$

τ	ES	AS	AE	RE	Time CPU (s)
0.02	0.079591154289758679228	0.079591154289759560107	8.8×10^{-16}	1.1×10^{-14}	0.874
0.04	0.15911933466140392832	0.15911933466140305331	8.75×10^{-16}	5.49×10^{-15}	0.874
0.06	0.23852186564169922344	0.23852186564169936963	1.46×10^{-16}	6.12×10^{-16}	0.920
0.08	0.31773666512002042948	0.31773666512002071579	2.86×10^{-16}	9.01×10^{-16}	1.295
0.1	0.396702532289366215	0.39670253228936612	9.43×10^{-17}	2.37×10^{-16}	0.874

Table 16 Comparison of absolute errors for Example 5.2

τ	AE RKM $\eta = 0.06$	AE MDM $\eta = 0.06$ [30]	AE RKM $\eta = 0.08$	AE MDM $\eta = 0.08$ [30]	AE RKM $\eta = 0.1$	AE MDM $\eta = 0.1$ [30]
0.02	3.99*E*−17	2.22045*E*−16	5.23*E*−17	4.49640*E*−15	8.8*E*−16	4.47420*E*−14
0.04	2.49*E*−17	2.22045*E*−16	2.83*E*−16	4.44089*E*−15	8.75*E*−16	4.44644*E*−14
0.06	7.14*E*−17	1.94289*E*−16	9.17*E*−17	4.38538*E*−15	1.46*E*−16	4.41314*E*−14
0.08	9.63*E*−17	1.94289*E*−16	6.38*E*−16	4.38538*E*−15	2.86*E*−16	4.36318*E*−14
0.1	9.77*E*−18	1.94289*E*−16	1.5*E*−16	4.32987*E*−15	9.43*E*−17	4.29656*E*−14

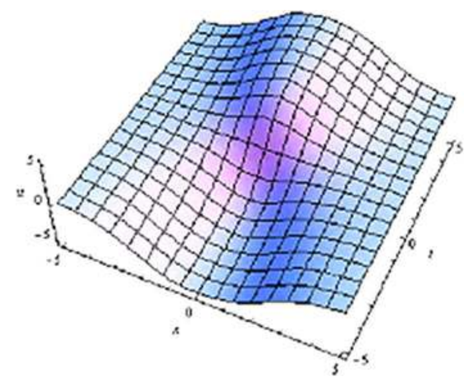


Figure 5 Plots of absolute error for Example 5.1.

Example 5.2 We solve the SG equation (1) in the region Ω with the following initial conditions:

$$u(\eta, 0) = 0, \qquad \frac{\partial u}{\partial \tau}(\eta, 0) = 4 \sec h(\eta).$$

The exact solution is [27]

$$u(\eta, \tau) = 4 \arctan\bigl(\sec h(\eta)\tau\bigr).$$

After homogenizing the initial conditions by RKM, we get the results presented in Tables 4-16 and Figures 5-8.

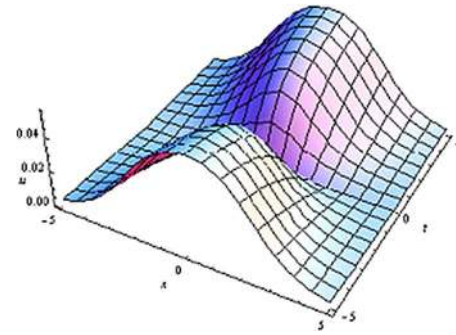


Figure 6 Plots of absolute error for Example 5.1.

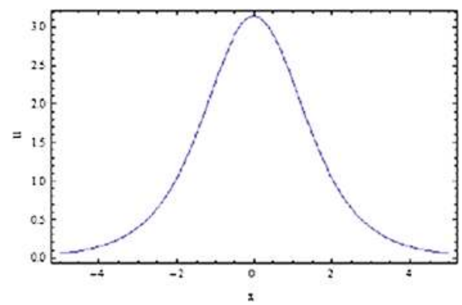


Figure 7 Plots of absolute error for Example 5.1.

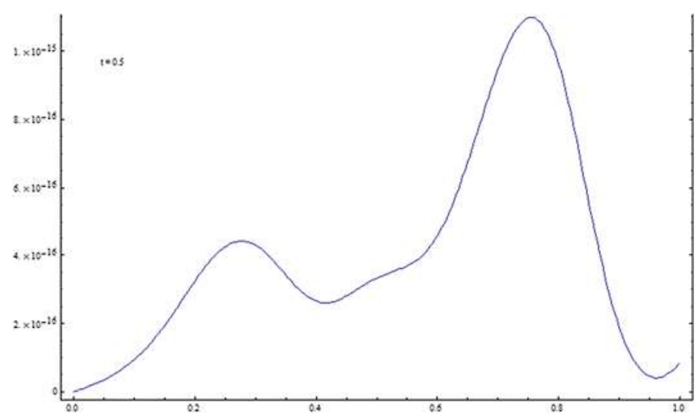


Figure 8 Plots of absolute error for Example 5.1.

Remark In Tables 1-16, we abbreviate the exact solution and the approximate solution by AS and ES, respectively. AE stands for the absolute error, that is, the absolute value of the difference of the exact and approximate solutions, whereas RE indicates the relative error, that is the absolute error divided by the absolute value of the exact solution.

6 Conclusion

Linear and nonlinear SG equations were investigated by RKM in this work. Homogenizing the initial and boundary conditions is very crucial for this method. We gave a general transformation to homogenize the conditions. This transformation will be very useful for researches who study RKM. We obtained very accurate numerical results and showed them by tables and figures. The computational results confirmed the efficiency, reliability, and accuracy of our method, which is easily applicable. RKM produced a rapidly convergent series with easily computable components using symbolic computation software. The results obtained by RKM are very effective and convenient with less computational work and time.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have read and approved the final manuscript.

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