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Research Article

A New Approach for Solving the Complex Cubic-Quintic Duffing Oscillator Equation for Given Arbitrary Initial Conditions

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The nonlinear differential equation governing the periodic motion of the one-dimensional, undamped, and unforced cubic-quintic Duffing oscillator is solved exactly, providing exact expressions for the period and the solution. The period as well as the exact analytic solution is given in terms of the famous Weierstrass elliptic function. An integrable case of a damped cubic-quintic equation is presented. Mathematica code for solving both cubic and cubic-quintic Duffing equations is given in Appendix at the end.

1. Introduction

It is known that most phenomena in nature have a nonlinear character, i.e., their laws of evolution are governed by either nonlinear ordinary or nonlinear partial differential equations. In many situations, it is desiderable to make an analytical study of the behavior of the equation solutions by means of the stability analysis of some associated linear systems (for example, for hyperbolic equilibria, Hartman–Grobman Theorem). This "linearization" is not possible in all cases. This is the reason why analytical techniques are required to analyze the behavior of these solutions. There are analytical methods that give necessary and sufficient conditions for the existence and uniqueness of solution to nonlinear equations (say Lie Groups, Sobolev spaces, etc.). However, we are investigating analytical methods that allow us to obtain exact solutions to this type of equations. In that sense, we meet in literature different techniques for integrating nonlinear equations, such as parameter perturbation techniques and homotopic perturbation methods, among others. As a contribution to the literature, in this article, we present the exact solution to the cubic-quintic Duffing oscillator equation by means of the famous Weierstrass elliptic function. The approach we present here is different from other approaches known in the literature [1–3]. A Mathematica code is included in Appendix at the end.

This paper is organized as follows. In Section 2, we give the solution to the cubic Duffing equation in terms of Jacobi elliptic functions. In Section 3, we give the solution to the cubic Duffing equation by means of the Weierstrass elliptic function. Section 4 is related to the solution of the cubic-

quintic Duffing equation for given initial conditions. Section 5 is related to applications of the obtained theoretical results for solving the nonlinear cubic-quintic nonlinear Schrodinger equation and the nonlinear cubic-quintic reaction-diffusion equation. A PHP script for solving the damped cubic-quintic equation may be found at http://fizmako.com/duffing35.php.

2. The Cubic Duffing Oscillator Equation

The equation is as follows:

$$v''(t) + \alpha v(t) + \beta v(t)^3 = 0, \quad \beta \neq 0. \tag{1}$$

In the case when $\beta > 0$, this oscillator may be interpreted as a forced oscillator having a spring whose restoring force F reads

$$F = -\alpha v - \beta v^3. \tag{2}$$

This spring may be hardening or softening depending on the sign of β. If $\beta > 0$, we have a hardening spring, while for $\beta < 0$, we deal with a softening spring. This last interpretation is valid only for small v. In this last case, the Duffing oscillator describes the dynamics of a point mass in a double-well potential. Chaotic motions can be observed in this case [4, 5]. Duffing equation is closely related to the pendulum equation [6, 7], and it has many important applications in soliton theory [8]. Other physical interpretations may be found in [9]. [10, 11] describe stability analysis for the Duffing equation.

The general solution to this ode reads

$$v(t) = c_1 cn\left(\sqrt{\alpha + \beta c_1^2}\, t + c_2,\ \frac{\beta}{2(\alpha + \beta c_1^2)}\right). \tag{3}$$

The values of the constants c_1 and c_2 are determined from the initial conditions. Let us consider the i.v.p

$$\begin{aligned} &v''(t) + \alpha v(t) + \beta v(t)^3 = 0, \\ &\text{subjected to} \quad v(0) = v_0, \quad v'(0) = \dot{v}_0. \end{aligned} \tag{4}$$

The number $\Delta = (\alpha + \beta v_0^2)^2 + 2\beta \dot{v}_0^2$ is called the discriminant of i.v.p (4). If $(\alpha + \beta v_0^2)^2 + 2\beta \dot{v}_0^2 > 0$ (or in the case when α, β are complex and $(\alpha + \beta v_0^2)^2 + 2\beta \dot{v}_0^2 \neq 0$), the solution to initial value problem (4) is given by

$$v(t) = \frac{v_0 cn(\sqrt{\omega}\, t \mid m) + (\dot{v}_0/\sqrt{\omega})\, sn(\sqrt{\omega}\, t \mid m) dn(\sqrt{\omega}\, t \mid m)}{1 + \mu sn(\sqrt{\omega}\, t \mid m)^2}, \tag{5}$$

where

$$\begin{cases} \omega = \sqrt{(\alpha + \beta v_0^2)^2 + 2\beta \dot{v}_0^2}, \\ m = \dfrac{1}{2} - \dfrac{\alpha}{2\sqrt{(\alpha + \beta v_0^2)^2 + 2\beta \dot{v}_0^2}}, \\ \mu = \dfrac{\alpha - \sqrt{(\alpha + \beta v_0^2)^2 + 2\beta \dot{v}_0^2} + \beta v_0^2}{2\sqrt{(\alpha + \beta v_0^2)^2 + 2\beta \dot{v}_0^2}}. \end{cases} \tag{6}$$

Here, cn, sn, and dn are the elliptic Jacobi functions. In the case when $(\alpha + \beta v_0^2)^2 + 2\beta \dot{v}_0^2 = 0$, the solution reads

$$\begin{aligned} v(t) &= -\sqrt{-\frac{\alpha}{\beta}} \tanh\left(\sqrt{\frac{\alpha}{2}} t - \quad \tanh^{-1}\left(\sqrt{-\frac{\beta}{\alpha}} u_0\right)\right), \\ v'(0)^2 &= \dot{v}_0^2 = \frac{(\alpha + \beta v_0^2)^2}{-2\beta}. \end{aligned} \tag{7}$$

Finally, when $(\alpha + \beta v_0^2)^2 + 2\beta \dot{v}_0^2 < 0$,

$$v = v(t) = c - \frac{2c}{1 + ds \quad c\left(t - (1/\sqrt{\omega})\, sc^{-1}\left(((a + v_0)/d(v_0 - \alpha)) \mid m\right)\sqrt{\omega} \mid m\right)}, \tag{8}$$

where

$$\begin{cases} \omega = -\dfrac{\alpha + c^2 \quad \beta}{4d^2}, \\ d = \sqrt{\dfrac{2\sqrt{2}\sqrt{c^2\beta\left(\alpha^2\beta - \alpha\right)} - \alpha + 3c^2\beta}{\alpha + c^2\beta}}, \\ m = 2 + \dfrac{2d^2\left(\alpha - 3c^2\beta\right)}{\alpha + c^2\beta}, \\ c = \operatorname{sgn}\left(\dot{y}_0\right)[4]\sqrt{\dfrac{2\alpha y_0^2 + \beta y_0^4 + 2\dot{y}_0^2}{-\beta}}. \end{cases} \tag{9}$$

3. Solution to the Duffing Equation in terms of the Weierstrass Elliptic Function

Our next aim is to solve initial value problem (1). Let

$$u(t) = \lim_{\tau->t}\left(A + \frac{B}{1 + C\wp\left(\tau;\, g_2, g_3\right)}\right), \tag{10}$$

where A, B, C, g_2, and g_3 are some constants to be determined. Here, $\wp(t, g_2, g_3)$ stands for the elliptic Weierstrass function. This function satisfies the ode

$$\wp'\left(t, g_2, g_3\right)^2 = 4\wp\left(t, g_2, g_3\right)^3 - g_2\wp\left(t, g_2, g_3\right) - g_3. \tag{11}$$

From (10), it is clear that

$$\begin{aligned} u(0) &= A, \\ u'(0) &= 0. \end{aligned} \tag{12}$$

Inserting ansatz (10) into the equation $u''(t) + \alpha u(t) + \beta u^3(t) = 0$ gives

$$\begin{aligned} &2C^2\left(2B + AC\alpha + A^3C\beta\right)\wp^3 + 2C\left(-6B + 3AC\alpha\right. \\ &\quad \left. + BC\alpha + 3A^3C\beta + 3A^2BC\beta\right)\wp^2 \\ &\quad + C\left(6A\alpha + 4B\alpha + 6A^3\beta + 12A^2B\beta + 6AB^2\beta - 3BCg_2\right)\wp \\ &\quad + \left(2A\alpha + 2B\alpha + 2A^3\beta + 6A^2B\beta + 6AB^2\beta\right. \\ &\quad \left. + 2B^3\beta + BCg_2 - 4BC^2g_3\right) = 0, \end{aligned} \tag{13}$$

where $\wp = \wp(t) = \lim_{\tau->t}\wp(\tau;\, g_2, g_3)$.

Equating the coefficients of $\wp^j$ to zero $(j = 0, 1, 2, 3)$ gives a nonlinear system of algebraic equations. Solving it gives

$$\begin{aligned} B &= -\frac{6A\left(A^2\beta + \alpha\right)}{3A^2\beta + \alpha}, \\ C &= \frac{12}{3A^2\beta + \alpha}. \end{aligned} \tag{14}$$

In expression (14), the quantity A is arbitrary withy $3A^2\beta + \alpha \neq 0$.

$$\begin{aligned} g_2 &= \frac{1}{12}\left(-3A^4\beta^2 - 6A^2\alpha\beta + \alpha^2\right), \\ g_3 &= \frac{1}{216}\alpha\left(9A^4\beta^2 + 18A^2\alpha\beta + \alpha^2\right). \end{aligned} \tag{15}$$

Thus, the solution to the initial value problem

$$\begin{aligned} &u''(t) + \alpha u(t) + \beta u(t)^3 = 0, \\ &\text{subjected to} \quad u(0) = A, \quad u'(0) = 0, \end{aligned} \tag{16}$$

is given for $\alpha + 3A^2\beta \neq 0$ by

$$u(t) = \lim_{\tau\longrightarrow t}\left(A - \frac{6A\left(\alpha + A^2\beta\right)/\left(\alpha + 3A^2\beta\right)}{1 + \left(12/\left(\alpha + 3A^2\beta\right)\right)\wp\left(\tau;\, (1/12)\left(\alpha^2 - 6A^2\alpha\beta - 3A^4\beta^2\right),\, (1/216)\alpha\left(\alpha^2 + 18A^2\alpha\beta + 9A^4\beta^2\right)\right)}\right). \tag{17}$$

Observe that the function

$$v(t) = u\left(t + t_0\right) \tag{18}$$

is also a solution to the equation $u''(t) + \alpha u(t) + \beta u^3(t) = 0$ for any constant t_0 (real or complex). Our aim is to solve initial value problem (4).To this end, we will make use of the addition formula

$$\begin{aligned} \wp\left(\tau + t_0;\, g_2, g_3\right) &= \frac{1}{4}\left(\frac{\wp'\left(\tau; g_2, g_3\right) - \wp'\left(t_0; g_2, g_3\right)}{\wp\left(\tau; g_2, g_3\right) - \wp\left(t_0; g_2, g_3\right)}\right)^2 \\ &\quad - \wp\left(\tau; g_2, g_3\right) - \wp\left(t_0; g_2, g_3\right), \end{aligned} \tag{19}$$

and then

$$v(t) = \lim_{\tau->t}\left(A + \frac{B}{1 + C\left((1/4)\left(\left(\wp'(\tau; g_2, g_3) - \lambda_2\right)/\left(\wp(\tau; g_2, g_3) - \lambda_1\right)\right)^2 - \wp(\tau; g_2, g_3) - \lambda_1\right)}\right). \tag{20}$$

We already know the values of the constants B, C, g_2, and g_3 (from (14) and (15)). We must find the values of the constants A, λ_1, and λ_2. We will determine them from the conditions

$$\begin{aligned} v(0) &= v_0, \\ v'(0) &= v_0, \\ v''(0) + \alpha v(0) + \beta v(0)^3 &= 0. \end{aligned} \tag{21}$$

Solving the last system gives

$$\begin{aligned} \lambda_1 &= \frac{3A^3\beta + 3A^2\beta v_0 + 5A\alpha + \alpha v_0}{12A - 12v_0}, \\ \lambda_2 &= \frac{Av_0\left(A^2\beta + \alpha\right)}{2\left(A - v_0\right)^2}, \\ A &= \sqrt{\frac{\sqrt{\left(\alpha + \beta v_0^2\right)^2 + 2\beta v_0^2} - \alpha}{\beta}}. \end{aligned} \tag{22}$$

In the case of periodic solution, the period of oscillations is that of the Weierstrass function $\wp(t; g_2, g_3)$, and it may be calculated by means of the formula

$$T = 2\int_{e_1}^{\infty} \frac{1}{\sqrt{4t^3 - g_2 t - g_3}}, \tag{23}$$

where e_1 is the first root to the cubic

$$z^3 - g_2 z - g_3 = 0. \tag{24}$$

We have proved the following.

Theorem 1. *The solution to initial value problem (4) is given by*

$$v(t) = v_{v_0, v_0}(t) = \lim_{\tau->t}\left(A + \frac{B}{1 + C\left((1/4)\left(\left(\wp'(\tau; g_2, g_3) - \lambda_2\right)/\left(\wp(\tau; g_2, g_3) - \lambda_1\right)\right)^2 - \wp(\tau; g_2, g_3) - \lambda_1\right)}\right). \tag{25}$$

The respective constants are evaluated by formulas (??), (15), and (22). The flow $\phi_t(x, v)$ of the nonlinear dynamical system,

$$\begin{aligned} \dot{x} &= -\alpha v - \beta v^3, \\ v &= x, \end{aligned} \tag{26}$$

reads

$$\phi_t(x, v) = \begin{bmatrix} \dfrac{\mathrm{d}}{\mathrm{d}t} v_{x,v}(t) \\ \\ v_{x,v}(t) \end{bmatrix}. \tag{27}$$

There is a more general equation called the generalized Duffing equation or Helmholtz–Duffing equation:

$$v''(t) + n + \alpha v(t) + \beta v(t)^2 + \gamma v(t)^3 = 0, \tag{28}$$

with real or complex constant coefficients. The solution to this equation may be found in [8].

4. The Analytic Solution to the Complex Cubic-Quintic Equation for Given Initial Conditions

In this section, we make use of results in Section 2 in order to solve the cubic-quintic oscillator equation. We show that the cubic-quintic Duffing equation is reduced to the cubic Duffing equation. That is, knowing the flow of the dynamical system associated with the cubic oscillator is enough to find that of the cubic-quintic. Indeed, let p, q, r, y_0, and $\dot{y}_0$ be arbitrary complex numbers with $r \neq 0$. We will solve the initial value problem

$$y''(t) + py(t) + qy(t)^3 + ry(t)^5 = 0,$$
$$\text{subjected to} \quad y(0) = y_0, \quad y'(0) = \dot{y}_0. \tag{29}$$

Let

$$y(t) = \frac{v(t)}{\sqrt{1 + \lambda v(t)^2}}, \tag{30}$$

where the function $v = v(t)$ is the solution to some Duffing equations given by (4). For small r, we may consider that equation (29) represents a small perturbation of equation (1). In that sense, equation (29) has a physical meaning similar to that of (1).

Multiplying equation (29) by $y'(t)$ and integrating it with respect to t give

$$\frac{1}{2}y'(t)^2 + \frac{p}{2}y(t)^2 + \frac{q}{4}y(t)^4 + \frac{r}{6}y(t)^6 = \frac{1}{2}\dot{y}_0^2 + \frac{p}{2}y_0^2 + \frac{q}{4}y_0^4 + \frac{r}{6}y_0^6. \tag{31}$$

In a similar way, from equation $v''(t) + \alpha v(t) + \beta v(t)^3 = 0$, we obtain

$$v'(t)^2 = \alpha v_0^2 + \frac{\beta v_0^4}{2} + \dot{v}_0^2 - \alpha v(t)^2 - \frac{1}{2}\beta v(t)^4. \tag{32}$$

Let

$$R(t) = \frac{1}{2}y'(t)^2 + \frac{p}{2}y(t)^2 + \frac{q}{4}y(t)^4 + \frac{r}{6}y(t)^6 - \left(\frac{1}{2}\dot{y}_0^2 + \frac{p}{2}y_0^2 + \frac{q}{4}y_0^4 + \frac{r}{6}y_0^6\right): \text{residual}. \tag{33}$$

Inserting ansatz (30) into (33) and taking into account (32) give

$$\begin{aligned} R(t) = {} & \left(6\lambda^2 p - 6\lambda^3 p y_0^2 + 3\lambda q - 3\lambda^3 q y_0^4 - 2\lambda^3 r y_0^6 + 2r - 6\lambda^3 \dot{y}_0^2\right) y(t)^6 \\ & - 3\left(\beta - 4\lambda p + 6\lambda^2 p y_0^2 + 3\lambda^2 q y_0^4 - q + 2\lambda^2 r y_0^6 + 6\lambda^2 \dot{y}_0^2\right) y(t)^4 \\ & - 3\left(2\alpha + 6\lambda p y_0^2 - 2p + 3\lambda q y_0^4 + 2\lambda r y_0^6 + 6\lambda \dot{y}_0^2\right) y(t)^2 \\ & - 6p y_0^2 - 3q y_0^4 - 2r y_0^6 + 6\alpha v_0^2 + 3\beta v_0^4 + 6\dot{v}_0^2 - 6\dot{y}_0^2. \end{aligned} \tag{34}$$

Equating to zero, the coefficients of $y(t)^j$ $(j = 0, 2, 4, 6)$ give the following nonlinear algebraic system:

$$\begin{cases} 6\lambda^2 p - 6\lambda^3 p y_0^2 + 3\lambda q - 3\lambda^3 q y_0^4 - 2\lambda^3 r y_0^6 + 2r - 6\lambda^3 \dot{y}_0^2 = 0, \\ \beta - 4\lambda p + 6\lambda^2 p y_0^2 + 3\lambda^2 q y_0^4 - q + 2\lambda^2 r y_0^6 + 6\lambda^2 \dot{y}_0^2 = 0, \\ 2\alpha + 6\lambda p y_0^2 - 2p + 3\lambda q y_0^4 + 2\lambda r y_0^6 + 6\lambda \dot{y}_0^2 = 0, \\ -6p y_0^2 - 3q y_0^4 - 2r y_0^6 + 6\alpha v_0^2 + 3\beta v_0^4 + 6\dot{v}_0^2 - 6\dot{y}_0^2 = 0. \end{cases} \tag{35}$$

We now eliminate the variables v_0 and $\dot{v}_0$ taking into account that $v(0) = v_0$ and $v'(0) = \dot{v}_0$, and we obtain

$$\begin{cases} 2\alpha + \lambda\left(6p y_0^2 + 3q y_0^4 + 2r y_0^6 + 6\dot{y}_0^2\right) = 2p, \\ \beta + \lambda^2\left(6p y_0^2 + 3q y_0^4 + 2r y_0^6 + 6\dot{y}_0^2\right) = 4\lambda p + q, \\ \left(6p y_0^2 + 3q y_0^4 + 2r y_0^6 + 6\dot{y}_0^2\right)\lambda^3 = 3\lambda(2\lambda p + q) + 2r. \end{cases} \tag{36}$$

From the first two equations of system (36), it follows that

$$\begin{cases} \alpha = p - \frac{1}{2}\lambda\left(6p y_0^2 + 3q y_0^4 + 2r y_0^6 + 6\dot{y}_0^2\right), \\ \beta = 4\lambda p + q - \left(6p y_0^2 + 3q y_0^4 + 2r y_0^6 + 6\dot{y}_0^2\right)\lambda^2. \end{cases} \tag{37}$$

The number λ is obtained by solving the cubic equation

$$\left(6p y_0^2 + 3q y_0^4 + 2r y_0^6 + 6\dot{y}_0^2\right)\lambda^3 - 6p\lambda^2 - 3q\lambda - 2r = 0. \tag{38}$$

The values of v_0 and $\dot{v}_0$ are found from the equations $y(0) = y_0$ and $y'(0) = \dot{y}_0$, and they read

$$\begin{cases} v_0 = \pm \dfrac{y_0}{\sqrt{1 - \lambda y_0^2}}, \\ \dot{v}_0 = \pm \dfrac{\dot{y}_0}{\sqrt{-\lambda^3 y_0^6 + 3\lambda^2 y_0^4 - 3\lambda y_0^2 + 1}}. \end{cases} \tag{39}$$

We have proved the following.

Theorem 2. *The solution to the initial value problem*

$$y''(t) + py(t) + qy(t)^3 + ry(t)^5 = 0,$$
$$\text{subjected to} \quad y(0) = y_0, \quad y'(0) = \dot{y}_0, \tag{40}$$

is given by

$$y(t) = \frac{v(t)}{\sqrt{1 + \lambda v(t)^2}}, \tag{41}$$

where

$$v(t) = \lim_{\tau \to t} \left(A + \frac{B}{1 + C\left((1/4)\left(\left(\wp'(\tau; g_2, g_3) - \lambda_2\right)/\left(\wp(\tau; g_2, g_3) - \lambda_1\right)\right)^2 - \wp(\tau; g_2, g_3) - \lambda_1\right)} \right). \tag{42}$$

The respective constants are evaluated by formulas (16), (23), and (37)–(39) - (??).

5. Applications

5.1. Nonlinear Cubic-Quintic Nonlinear Schrodinger (CQNLSE) Equation. This equation reads

$$\sqrt{-1}\ \frac{\partial \varphi}{\partial Z} + A\frac{\partial^2 \varphi}{\partial T^2} + B|\varphi|^2\varphi + C|\varphi|^4\varphi = 0. \tag{43}$$

In the case when $A = -(k/2)$, $B = -(k/2)\chi$, and $C = 0$, the number χ represents a dimensionless positive parameter characterizing the medium that describes wave propagation in fluids, plasmas, and nonlinear optics, while k is the wave number of propagating waves. Let

$$\varphi(Z, T) = \exp\left(\sqrt{-1}\ (cT + bZ)\right) y(aT - 2aAcZ). \tag{44}$$

This transformation gives

$$y''(\xi) - \frac{(b + Ac^2)}{a^2 A} y(\xi) + \frac{B}{a^2 A} y(\xi)^3 + \frac{C}{a^2 A} y(\xi)^5 = 0, \tag{45}$$

which is a cubic-quintic Duffing equation.

5.2. Nonlinear Reaction-Diffusion (NLRD) Equation. The dimensionless form of the variable coefficients in the nonlinear reaction-diffusion (NLRD) equation is

$$u_t + v(t)u_x = Du_{xx} + \alpha u - \beta u^3 + \gamma u^5, \tag{46}$$

where $u = u(x, t)$ is the concentration or density variable depending on the phenomena under study; D is the diffusion coefficient; $v(t)$ is the convection term coefficient; and α, β, and γ are the reaction term coefficients. Making the traveling wave transformation $\xi = kx + \eta(t)$ and letting $u(\xi) = a(t)y(\xi)$ give

$$a_t y + \left(a\eta'(t) + vak\right)y'(\xi) = Da \quad k^2 y''(\xi) + \alpha a y(\xi) - \beta a^3 y^3(\xi) + \gamma a^5 y^5(\xi). \tag{47}$$

By selecting $a =$ constant and $\eta(t) = (2\delta/a) - k\int v(t)\mathrm{d}t$, equation (47) turns out to be

$$y''(\xi) + 2\delta y'(\xi) + py(\xi) + qy^3(\xi) + ry^5(\xi) = 0, \tag{48}$$

where

$$p = \frac{\alpha}{Dk^2},$$
$$q = \frac{\beta a^2}{Dk^2}, \tag{49}$$
$$r = \frac{\gamma a^4}{Dk^2}.$$

In the case $\delta = 0$, we have a cubic-quintic Duffing equation. If $\delta \neq 0$ and

$$\begin{cases} p = -\dfrac{1}{4}k(k - 2n), \\ r = -\dfrac{3k^2}{4A^4}, \end{cases} \tag{50}$$

then

$$y(\xi) = \frac{A}{\sqrt{1 + B \quad \exp(k\xi)}} \tag{51}$$

is a solution to equation (48) for any constants A, B, and k. Finally, when $q = 0$ and $p = (3/4\delta)^2$, we have the damped Duffing equation

$$y''(\xi) + 2\delta y'(\xi) + (3/4\delta)^2 y(\xi) + ry^5(\xi) = 0. \tag{52}$$

Using Lie group theory, it is possible to prove that equation (52) admits a solution of the form

$$y(\xi) = [4]\sqrt{\frac{\gamma}{r}}\sqrt{c_1}e^{-(n\xi/4)}v\left(c_2 - \frac{2c_1}{n}e^{-(n\xi/2)}\right), \tag{53}$$

where the function $v = v(t)$ is a solution to the quintic Duffing equation $v''(\xi) + \gamma v(\xi)^5 = 0$.

Equation (52) may be solved numerically for given initial conditions $y(0) = y_0$ and $y'(0) = y_0'$ at the authors' website http://fizmako.com/duffing35.php.

We think that some formulas given here are new in the literature. The Mathematica code for solving either symbolically or numerically both cubic and cubic-quintic oscillator complex oscillator equations is given in Appendix.

Appendix

Mathematica code:

$$
\begin{aligned}
&v[\alpha\_, \beta\_, v_{0\_}, v_{1\_}][t\_] := \text{Module}\Bigg[\{\Delta, \rho, \mu, \kappa, g_2, g_3, wp, wpp, v, \text{solution}\}\\
&\Bigg\{\Bigg\{\Delta = 2v_1^2\beta + \left(\alpha + v_0^2\beta\right)^2,\ \rho = v_0\left(3\sqrt{\Delta} - 2\alpha\right) + \sqrt{\frac{-\alpha + \sqrt{\Delta}}{\beta}}\left(3\sqrt{\Delta} + 2\alpha\right), \mu = \frac{-\alpha + \sqrt{\Delta}}{\beta},\\
&\kappa = -2\alpha + 3\sqrt{\Delta}, g_2 = \frac{\alpha^2}{3} - \frac{\Delta}{4}, g_3 = -\frac{\alpha^3}{27} + \frac{\alpha\Delta}{24},\\
&wp = \text{WeierstrassP}[t, \{g_2, g_3\}],\ wpp = \text{WeierstrassPPrime}[t, \{g_2, g_3\}],\\
&v = \frac{\rho}{12v_0 - 12\sqrt{\mu}} - wp + \frac{9\left(v_1\sqrt{\Delta}\sqrt{\mu} - 2\left(v_0 - \sqrt{\mu}\right)^2 wpp\right)^2}{\left(v_0 - \sqrt{\mu}\right)^2\left(\rho + 12\left(v_0 - \sqrt{\mu}\right)wp\right)^2}\\
&\text{solution} = \sqrt{\mu}\left(1 - (6(\alpha + \beta\mu))/(\kappa(1 + 12(v/\kappa)))\right)\Bigg\};\Bigg\}; \text{solution}\Bigg];\\
&y[p\_, q\_, r\_, y_{0\_}, y_{1\_}][t\_] := \text{Module}\Bigg[\{\psi, \zeta, H, v_0, v_1, \text{sol}v_0v_1\},\\
&\Bigg\{\Bigg\{\zeta = \text{Root}\left[-2r - 3q\#1 - 6p\#1^2 + 6py_0^2\#1^3 + 3qy_0^4\#1^3 + 2ry_0^6\#1^3 + 6y_1^2\#1^3\&, 1\right],\\
&H[x\_] := \psi''[x] + p\psi[x] + q\psi[x]^3 + r\psi[x]^5;\\
&\text{sol}v_0v_1 = \text{Solve}\left[\left\{v_0 == y_0\sqrt{1 + v_0^2\zeta},\ v_1 == y_1\left(1 + v_0^2\zeta\right)^{3/2}\right\}, \{v_0, v_1\}\right]//\text{Last};\\
&\psi[x\_] := \frac{v\left[-\left(\left(2r + 3q\zeta + 4p\zeta^2\right)/2\zeta^2\right), -\left(2\left(r + q\zeta + p\zeta^2\right)/\zeta\right), v_0, v_1\right][x]}{\sqrt{1 + \zeta v\left[-\left(\left(2r + 3q\zeta + 4p\zeta^2\right)/2\zeta^2\right), -\left(2\left(r + q\zeta + p\zeta^2\right)/\zeta\right), v_0, v_1\right][x]^2}}//.\text{sol}v_0v_1,\\
&H[x\_] := \psi''[x] + p\psi[x] + q\psi[x]^3 + r\psi[x]^5\Bigg\};\Bigg\}; \psi[t]//.\text{sol}v_0v_1\Bigg].
\end{aligned}
\tag{A.1}
$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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