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A new approach in interpolation spaces

by

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0. Introduction. In several previous publications (see Peetre [14]-[18]; see also Butzer-Behrens [2], chap. 3, Krée [10], Holmstedt [6] and [7], Oklander [13], Golovkin [4]) we have developed an approach to the theory of interpolation spaces ("K-method") based on the functional

$$K(t, a; \vec{A}) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}), \tag{0.1}$$

where $0 < t < \infty$ and $a \in \sum(\vec{A}) = A_0 + A_1$, $\vec{A} = \{A_0, A_1\}$ being any Banach or more generally quasi-Banach couple (see Section 1). More specifically, we can define interpolation spaces $\vec{A}_{\theta q;K}$ by

$$a \in \vec{A}_{\theta q;K} \Leftrightarrow \left(\int_0^\infty (t^{-\theta} K(t, a; \vec{A}))^q \frac{dt}{t} \right)^{1/q} < \infty. \tag{0.2}$$

Here $0 < \theta < 1$, $0 < q \leqslant \infty$ with the usual interpretation if $q = \infty$. In certain problems, notably the case $A_0 = L_p(\mu_0)$, $A_1 = L_p(\mu_1)$ (see Peetre [19]-[21], Goullaouic [5]) it is more advantageous to use the modified functional

$$K_p(t, a; \vec{A}) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0}^p + t^p \|a_1\|_{A_1}^p)^{1/p}, \tag{0.3}$$

where p thus can be adjusted to the special problem in question. Since $a \to K(t, a; \vec{A})$ and $a \to K_p(t, a)$, when t is fixed, define uniformly in t equivalent quasi-norms in $\sum(\vec{A})$ we obtain the same space, up to an equivalence of quasi-norm, if we in (0.2) replace $K(t, a; \vec{A})$ by $K_p(t, a; \vec{A})$. In the present paper the basic underlying idea is to work with the new functional

$$L(t, a; \vec{A}) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0}^{p_0} + t\|a_1\|_{A_1}^{p_1}). \tag{0.4}$$

In doing so we loose the property of homogeneity in a and in particular we will not get a quasi-norm. And it is not at the start at least clear what is the relation of the new space $\vec{A}_{\theta q;L}$ to the original $\vec{A}_{\theta q;K}$. However, when applied to specific problems, this allows a much greater degree of flexibility. A typical case is $A_0 = L_{p_0}(\mu)$, $A_1 = L_{p_1}(\mu)$ or, more generally, even $A_0 = L_{p_0}(\mu_0)$, $A_1 = L_{p_1}(\mu_1)$, which thus contains the previous special case $p_0 = p_1 = p$.

The plan of the paper is as follows. In Section 1 part of the terminology is explained. In Section 2 we study the relation between the spaces $(A_0, A_1)_{\theta q;K}$ and $(A_0, A_1)_{\theta q;L}$. The main device is lemma 2.1 which gives a connection between $K(t, a; \vec{A})$ and $L(t, a; \vec{A})$ (cf. Holmstedt-Peetre [8], where the corresponding problem for $K(t, a; \vec{A})$ and $K_p(t, a; \vec{A})$ is studied). Sections 4-6 are devoted to applications to various spaces of measurable functions on a measure space. In Section 3 we treat a very general form of the classical theorem of M. Riesz including the case of "field valued" functions. For simplicity we have only treated the so-called diagonal case. But of course the same methods can also be used, at least to some extent, in the general off-diagonal case. The same remark applies to Section 4 where we consider the theorem of Marcinkiewicz (concerning these classical interpolation theorems, cf. Zygmund [28], chap. 12.) In Section 5 we generalize some of the results of Section 3 in such a direction that as a result of our interpolation we get Orlicz spaces. Interpolation in relation to Orlicz spaces has previously been studied by other authors (Simonenko [16] and [17], Rao [24] and [25]) but it is not clear what the connection is with our work. In Section 6 we vary also the endpoints allowing them to be so-called modular function spaces. In doing this we actually leave the domain of quasi-Banach spaces to which case most of the presentation otherwise is adjusted. Indeed the natural domain of the present approach seems to be much broader. Probably not even the abstract modular spaces (see Musielak and Orlicz [12]) put a limit. (Note also that we could likewise have treated the case of a more general field of scalars than R (real numbers), allowing not even the rather trivial case of C (complex numbers) but also the case of local fields (cf. Krée [10], p. 138.) Section 7 contains some remarks on the limiting case when p_0 or $p_1 \to \infty$. In Section 8 we return to the general abstract situation and apply our ideas to give a new simple proof of the so-called parameter theorem (see Peetre [14], [15], and [23]).

Certainly, the ideas of this paper could be developped still further in several directions so this is really just a first announcement and we do intend to pursue the subject in forthcoming publications. This is the reason why we have here cut down many details to a minimum and why certain portions, especially Section 7, are presented in a merely heuristic way, with most of the formal argument left to the reader.

1. Terminology. Let A be a vector space over the real numbers R. By a *quasi-norm* in A we mean a mapping $A \ni a \to \|a\| \in R_+$ (non-negative real numbers) such that

$$\|a\| > 0 \quad \text{if} \quad a \neq 0, \quad \|0\| = 0,$$
$$\|\lambda a\| = |\lambda| \cdot \|a\| \quad \text{if} \quad \lambda \in R,$$
$$\|a+b\| \leqslant \gamma(\|a\|+\|b\|)$$

for some $\gamma \geqslant 1$ independent of a and b (*quasi-triangle inequality.*)

By a *quasi-normed space* we mean a vector space A in which one quasi-norm $\|a\| = \|a\|_A$ has been singled out. If the space is complete in the uniform structure induced by that quasi-norm, we speak of a *quasi-Banach space*. By a *quasi-normed couple* we mean the entity consisting of two quasi-normed spaces A_0 and A_1 together with corresponding continuous embeddings i_0 and i_1 into one and the same Hausdorff topological vector space $\mathscr{A}$. If the spaces are complete we speak of a *quasi-Banach couple*. We shall use the notation $\vec{A} = \{A_0, A_1\}$; the embeddings i_0 and i_1 and the space $\mathscr{A}$ need not be counted for in general. Often one can identify $\mathscr{A}$ with $\sum(\vec{A}) = A_0+A_1$, the sum of A_0 and A_1. The couple $\vec{R} = \{R, R\}$, where R is taken with $\|\lambda\|_R = |\lambda|$ (absolute value), will play a special role.

If $\gamma = 1$ in the quasi-triangle inequality we shall drop the suffix "quasi" and say simply "norm", "normed", "Banach".

Let A and B be quasi-normed spaces. If T is a continuous linear mapping from A into B, which we write symbolically as $T: A \to B$, we denote its quasi-norm by

$$\|T\|_{A,B} = \sup \|Ta\|_B/\|a\|_A.$$

Let $\vec{A} = \{A_0, A_1\}$ and $\vec{B} = \{B_0, B_1\}$ be quasi-normed couples. If both $T: A_0 \to B_0$ and $T: A_1 \to B_1$ hold true (where it is assumed that T is defined in a space containing both A_0 and A_1, and thus $\sum(\vec{A}) = A_0+A_1$), we use the symbol $T: \vec{A} \to \vec{B}$.

Let U be a "measure space" and μ a measure on U. Let $\{E(u)\}_{u\in U}$ be a field of quasi-Banach spaces attached to U. We denote by $L_p(\mu; \{E(u)\})$, where $0 < p \leqslant \infty$, the quasi-Banach spaces corresponding to the quasi-norm

$$\|a\|_{L_p(\mu;\{E(u)\})} = \Big(\int_U \|a(u)\|_{E(u)}^p \, d\mu(u)\Big)^{1/p},$$

with the usual interpretation of $p = \infty$. If we have a constant field, i.e. $E(u) = E$, we write simply $L_p(\mu, E)$ and if $E = R$ just $L_p(\mu)$. We also agree to drop the argument μ whenever possible.

The notation of field often leads to considerable measurability difficulties, which must be treated with great care. Here we adopt the purely formal point of view and disregard all such complications.

Finally we point out that on some occasions (in particular, throughout Section 6) we use $\|a\| = \|a\|_A$ also to denote functionals which are not quasi-norms. We still keep some of the above conventions. Notably we use freely the notation $\|T\|_{A,B}$ in such situations.

The symbol $f \approx g$ ("equivalence") means that $f = O(g)$ and $g = O(f)$, i.e. $f \leqslant Cg$ and $g \leqslant Cf$ for some C.

2. The spaces $\vec{A}_{\theta q;K}$ and $\vec{A}_{\theta q;L}$. Let $\vec{A} = \{A_0, A_1\}$ be any quasi-Banach couple. Let p_0 and p_1 be given numbers, $0 < p_0, p_1 < \infty$, fixed throughout most of the discussion. As in the Introduction ((0.1) and (0.4)) we define

$$K(t, a; \vec{A}) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}), \tag{2.1}$$

$$L(t, a; \vec{A}) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0}^{p_0} + t\|a_1\|_{A_1}^{p_1}), \tag{2.2}$$

where $0 < t < \infty$ and $a \in \sum(\vec{A}) = A_0 + A_1$. Whenever possible we shall drop the last or the two last arguments writing thus $K(t, a)$, $L(t, a)$ or $K(t)$, $L(t)$. Let us also define

$$a \in \vec{A}_{\theta q;K} \Leftrightarrow \|a\|_{\vec{A}_{q;K}} = \left(\int_0^\infty (t^{-\theta} K(t, a))^q \frac{dt}{t}\right)^{1/q} < \infty, \tag{2.3}$$

$$a \in \vec{A}_{q;L} \Leftrightarrow \|a\|_{\vec{A}_{\theta q;L}} = \left(\int_0^\infty (t^{-\theta} L(t, a))^q \frac{dt}{t}\right)^{1/q} < \infty, \tag{2.4}$$

where $0 < \theta < 1$, $0 < q \leqslant \infty$. It is clear that $\vec{A}_{\theta q;K}$ is a quasi-normed space, with $\|a\|_{\vec{A}_{\theta q;K}}$ defined by (2.3), and it is possible to show that it is complete, i.e. a quasi-Banach space. As for $\vec{A}_{\theta q;L}$ the matter is not obvious at all and below (theorem 2.2) we shall see that only a suitable power p of $\|a\|_{\vec{A}_{\theta q;L}}$ is equivalent to a quasi-norm. It is nevertheless convenient, from the notational point of view, to treat $\|a\|_{\vec{A}_{\theta q;L}}$ as if it were a norm. First we consider however the interpolation properties of the two spaces.

THEOREM 2.1. *Let $T: \vec{A} \to \vec{B}$. Then $T: \vec{A}_{\theta q;K} \to \vec{B}_{\theta q;K}$ and $T: \vec{A}_{\eta q;L} \to \vec{B}_{\eta q;L}$ and, moreover, there hold the inequalities*

$$\|T\|_{\vec{A}_{\theta q;K}, \vec{B}_{\theta q;K}} \leqslant \|T\|_{A_0,B_0}^{1-\theta} \|T\|_{A_1,B_1}^{\theta}, \tag{2.5}$$

$$\|T\|_{\vec{A}_{\eta q;L}, \vec{B}_{\eta q;L}} \leqslant \|T\|_{A_0,B_0}^{p_0(1-\eta)} \|T\|_{A_1,B_1}^{p_1\eta}. \tag{2.6}$$

Proof. Write $M_0 = \|T\|_{A_0,B_0}$, $M_1 = \|T\|_{A_1,B_1}$. Using (2.1) and (2.2) we readily see that

$$K(t, Ta; \vec{B}) \leqslant M_0 K\left(\frac{M_1 t}{M_0}, a; \vec{A}\right), \tag{2.7}$$

$$L(t, Ta; \vec{B}) \leqslant M_0^{p_0} L\left(\frac{M_1^{p_1} t}{M_0^{p_0}}, a; \vec{A}\right). \tag{2.8}$$

From (2.3) and (2.7) it follows now that

$$\|Ta\|_{\vec{B}_{\theta q;K}} \leqslant M_0 \left(\int_0^\infty \left(t^{-\theta} K\left(\frac{M_1 t}{M_0}, a; \vec{A}\right)\right)^q \frac{dt}{t}\right)^{1/q}$$
$$= M_0 \left(\frac{M_1}{M_0}\right)^\theta \left(\int_0^\infty (t^{-\theta} K(t, a; \vec{A}))^q \frac{dt}{t}\right)^{1/q} = M_0^{1-\theta} M_1^\theta \|a\|_{\vec{A}_{\theta q;K}}$$

which clearly implies (2.5). In exactly the same way, using (2.4) and (2.8), we prove (2.6).

We now state the main result of this section.

THEOREM 2.2. *For all θ and q holds*

$$\vec{A}_{\theta,pq;K} = \vec{A}_{\eta q;L}, \tag{2.9}$$

where

$$\eta = \frac{\theta p}{p_1} \quad \left(\text{or } 1 - \eta = \frac{(1-\theta)p}{p_0}\right)$$

and

$$p = (1-\eta)p_0 + \eta p_1 \quad \left(\text{or } \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}\right)$$

and the quasi-norm $\|a\|_{\vec{A}_{\theta,pq;K}}$ is equivalent to $\|a\|_{\vec{A}_{\eta,q;L}}^{1/p}$.

Proof. For the proof it is convenient to introduce

$$K^*(t) = K^*(t, a; \vec{A}) = \inf_{a=a_0+a_1} \max(\|a_0\|_{A_0}, t\|a_1\|_{A_1}), \tag{2.10}$$

$$L^*(t) = L^*(t, a; \vec{A}) = \inf_{a=a_0+a_1} \max(\|a_0\|_{A_0}^{p_0}, t\|a_1\|_{A_1}^{p_1}). \tag{2.11}$$

Since

$$K^*(t) \leqslant K(t) \leqslant 2K^*(t), \quad L^*(t) \leqslant L(t) \leqslant 2L^*(t),$$

it is clear that the definition of the spaces is not affected if in (2.3) and (2.4) we introduce $K^*(t)$ and $L^*(t)$ in place of $K(t)$ and $L(t)$. The proof follows now easily from the following

LEMMA 2.1. *If s is defined by*

$$s = t^{p_1}(K^*(t))^{p_0-p_1}, \tag{2.12}$$

then

$$L^*(s) = (K^*(t))^{p_0}. \tag{2.13}$$

Indeed, differentiation of (2.12) yields

$$ds = \left(p_1 t^{p_1-1}(K^*(t))^{p_0-p_1} + t^{p_1}(p_0-p_1)(K^*(t))^{p_0-p_1-1}\frac{dK^*(t)}{dt}\right)dt$$

or

$$\frac{ds}{s} = \left\{p_1 + (p_0-p_1)\frac{t\,dK^*(t)/dt}{K^*(t)}\right\}\frac{dt}{t}. \tag{2.14}$$

It is easily seen that

$$0 \leqslant t\frac{dK^*(t)}{dt} \leqslant K^*(t).$$

Therefore the expression within parenthesis in (2.14) is a point of the interval $[p_0, p_1]$. Applying (2.12), (2.13) and (2.14) we now get (if $q<\infty$, the limiting case $q=\infty$ can be treated directly in a similar way)

$$\int_0^\infty (s^{-\eta}L^*(s))^q\frac{ds}{s} = \int_0^\infty \left(t^{-\eta p_1}(K^*(t))^{-\eta(p_0-p_1)}(K^*(t))^{p_0}\right)^q \times$$

$$\times\left\{p_1+(p_0-p_1)\frac{t\,dK^*(t)/dt}{K^*(t)}\right\}\frac{dt}{t} \approx \int_0^\infty (t^{-\theta}K^*(t))^{qp}\frac{dt}{t}$$

and from this formula the assertion of the theorem clearly follows.

Proof of lemma 2.1. With Gagliardo (cf. e.g. [3]) we introduce the set $\Gamma = \Gamma(a, \vec{A})$ of points $\vec{x} = (x_0, x_1)$ in the positive quadrant of the plane such that there exists a decomposition $a = a_0 + a_1$ with $\|a_0\|_{A_0} \leqslant x_0$, $\|a_1\|_{A_1} \leqslant x_1$. Then we may write (2.10) and (2.11) as

$$K^*(t) = \inf_{\vec{x}\in\Gamma}\max(x_0, tx_1), \tag{2.15}$$

$$L^*(s) = \inf_{\vec{x}\in\Gamma}\max(x_0^{p_0}, sx_1^{p_1}). \tag{2.16}$$

The inf in both (2.15) and (2.16) is assumed at the point $\vec{x}$ if $x_0 = tx_1 = K^*(t)$, $x_0^{p_0} = sx_1^{p_1} = L^*(s)$. Therefore $(K^*(t))^{p_0} = x_0^{p_0} = L^*(s)$ which is (2.13). Similarly

$$(K^*(t))^{p_1} = t^{p_1}x_1^{p_1} = t^{p_1}s^{-1}L^*(s) = t^{p_1}s^{-1}(K^*(t))^{p_0},$$

which yields (2.12).

3. The theorem of M. Riesz. In this section we first take $A_0 = L_{p_0}$, $A_1 = L_{p_1}$ or $\vec{A} = L_{\vec{p}}$ (with $\vec{p} = (p_0, p_1)$) and we shall prove

THEOREM 3.1. $(L_{p_0}, L_{p_1})_{\eta,1;L} = L_p$, *where* $p = p_0(1-\eta)+p_1\eta$. *The quasi-norm in* L_p *is apart from a multiplicative factor equal to* $\|a\|^{1/p}_{(L_{p_0},L_{p_1})_{\eta,1;L}}$.

Proof. We have

$$\begin{aligned} L(t, a; L_{\vec{p}}) &= \inf(\|a_0\|^{p_0}_{L_{p_0}} + \|a_1\|^{p_1}_{L_{p_1}}) \\ &= \inf\int_U (|a_0(u)|^{p_0} + t|a_1(u)|^{p_1})\,d\mu(u) \\ &= \int_U \inf(|a_0(u)|^{p_0} + t|a_1(u)|^{p_1})\,d\mu(u) \\ &= \int_U L(t, a(u); \vec{R})\,d\mu(u). \end{aligned} \tag{3.1}$$

Now it is readily seen that $L(t, a; \vec{R}) = |a|^{p_0}F(t|a|^{p_1-p_0})$ where $F(t) \approx \min(1, t)$ so that

$$c = \int_0^\infty t^{-\eta}F(t)\frac{dt}{t} < \infty.$$

(Note also that

$$L(t, a; L_{\vec{p}}) \approx \int_U \min(|a(u)|^{p_0}, t|a(u)|^{p_1})\,d\mu(u).) \tag{3.2}$$

Therefore

$$L(t, a; L_{\vec{p}}) = \int_U |a(u)|^{p_0}F(t|a(u)|^{p_1-p_0})\,d\mu(u). \tag{3.3}$$

Using Fubini's theorem we thus get

$$\begin{aligned} \|a\|_{(L_{\vec{p}})_{\eta,1;L}} &= \int_0^\infty t^{-\eta}L(t, a; L_{\vec{p}})\frac{dt}{t} \\ &= \int_0^\infty t^{-\eta}\left(\int_U |a(u)|^{p_0}F(t|a(u)|^{p_1-p_0})\,d\mu(u)\right)\frac{dt}{t} \\ &= \int_U |a(u)|^{p_0}\left(\int_0^\infty t^{-\eta}F(t|a(u)|^{p_1-p_0})\frac{dt}{t}\right)d\mu(u) \\ &= \int_U |a(u)|^{p_0}|a(u)|^{(p_1-p_0)\eta}\,d\mu(u)\int_0^\infty t^{-\eta}F(t)\frac{dt}{t} = c\|a\|^p_{L_p}. \end{aligned} \tag{3.4}$$

COROLLARY 3.1 (M. Riesz). *Let* $T\colon \{L_{p_0}, L_{p_1}\} \to \{L_{p_0}, L_{p_1}\}$. *Let* $p \in [p_0, p_1]$. *Then* $T\colon L_p \to L_p$ *and*

$$\|T\|_{L_p,L_p} \leqslant \|T\|^{1-\theta}_{L_{p_0},L_{p_0}}\|T\|^{\theta}_{L_{p_1},L_{p_1}} \quad \left(\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}\right).$$

Proof. This follows easily if we use (the L-part of) theorem 2.1.

Corollary 3.2. $(L_{p_0}, L_{p_1})_{\theta,p;K} = L_p$, *where* $1/p = (1-\theta)/p_0+\theta/p_1$. *This quasi-norm in* L_p *is equivalent to* $\|a\|_{(L_{p_0},L_{p_1})_{\theta,p;K}}$.

Proof. This follows at once from theorem 2.2.

Remark 3.1. Note that corollary 3.1 comprises also the truly quasi-normed case (p_0 or $p_1<1$) not contained in the original formulation of the theorem of M. Riesz. As explained in the Introduction we treat here only the diagonal case. Corollary 3.2 follows also from a result by Lions-Peetre [11], chap. VII, combined with the "parameter theorem" of Peetre [14], [15] (see Section 8).

The method of theorem 3.1 can also immediately be extended to the "field valued" case even with change of measure. Thus we take now

$$A_0 = L_{p_0}(\mu_0, \{E_0(u)\}),\quad A_1 = L_{p_1}(\mu_1, \{E_1(u)\})\quad \text{or}\quad \vec{A} = L_{\vec{p}}(\vec{\mu}, \{\vec{E}(u)\}),$$

where $\vec{p} = (p_0, p_1)$, $\vec{\mu} = (\mu_0, \mu_1)$, $\vec{E}(u) = \{E_0(u), E_1(u)\}$.

Theorem 3.2. *We have*

$$\big(L_{p_0}(\mu_0, \{E_0(u)\}), L_{p_1}(\mu_1, \{E_1(u)\})\big)_{\eta,1;L} = L_1\big(\mu_0^{1-\eta}\mu_1^{\eta}; \{(E_0(u), E_1(u))_{\eta,1;L}\}\big).$$

Proof. The change of measure is indeed illusoric, for we may replace μ_0 and μ_1 by a measure μ with respect to it both are absolutely continuous, and at the same time modify the quasi-norms of $E_0(u)$ and $E_1(u)$ by suitable multiplicative factors, without changing the situation at all. Assuming this that $\mu_0=\mu_1=\mu$ we obtain now as in (3.1)

$$(3.5)\qquad L\big(t, a; L_{\vec{p}}(\{\vec{E}(u)\})\big) = \int_U L\big(t, a(u); \vec{E}(u)\big)\,d\mu(u),$$

which is the substitute of (3.3), from which again follows (cf. (3.4))

$$\begin{aligned}\|a\|_{(L_{\vec{p}}(\{\vec{E}(u)\}))_{\eta,1;L}} &= \int_0^\infty t^{-\eta} L\big(t, a; L_{\vec{p}}(\{\vec{E}(u)\})\big)\frac{dt}{t}\\ &= \int_0^\infty t^{-\eta}\Big(\int_U L\big(t, a(u); \vec{E}(u)\big)\,d\mu(u)\Big)\frac{dt}{t}\\ &= \int_U\Big(\int_0^\infty t^{-\eta} L\big(t, a(u); \vec{E}(u)\big)\frac{dt}{t}\Big)\,d\mu(u)\\ &= \int \|a(u)\|_{(\vec{E}(u))_{\eta,1;L}}\,d\mu(u) = \|a\|_{L_1(\{(\vec{E}(u))_{\eta,1;L}\})}.\end{aligned}$$

Corollary 3.3. *Analogous to corollary 3.1.*

Corollary 3.4. *We have*

$$\big(L_{p_0}(\mu_0, \{E_0(u)\}), L_{p_1}(\mu_1, \{E_1(u)\})\big)_{\theta,p;K} = L_1\big(\mu_0^{1-\eta}\mu_1^{\eta}; \{(E_0(u), E_1(u))_{\eta,1;L}\}\big),$$

where $\eta = \theta p/p_1$ *and* $p = (1-\eta)p_0+\eta p_1$.

Proof. Same as for corollary 3.2.

The case of only change of measure diserves nevertheless special attention.

Corollary 3.5. $(L_p(\mu_0), L_p(\mu_1))_{\theta,p;K} = L_p(\mu_0^{1-\theta}\mu_1^{\theta})$.

Remark 3.2. Exactly as in the case of corollary 3.2 also corollary 3.4 can be obtained from Lions-Peetre [11], chap. VII, in combination with the parameter theorem. The same applies to corollary 3.5. Note that the present treatment should more properly be considered as the development of technique of Peetre [20].

4. The theorem of Marcinkiewicz. Along with the couple $L_p = \{L_{p_0}, L_{p_1}\}$ we now also consider the couple $L_p^* = \{L_{p_0}^*, L_{p_1}^*\}$. Here L_p^* denotes the space corresponding to the quasi-norm

$$\|a\|_{L_p^*} = \sup_{\sigma>0}\sigma\Big(\int_{\{u\,|\,|a(u)|\geq\sigma\}} d\mu(u)\Big)^{1/p} < \infty.$$

(Recall that $\|a\|_{L_p^*}$ is equivalent to a norm if $p>1$.) Hence $T: L_p\to L_p^*$ means that

$$(4.1)\qquad \mathop{\forall}_{\sigma>0}\ \sigma\Big(\int_{\{u\,|\,|Ta(u)|\geq\sigma\}} d\mu(u)\Big)^{1/p} \leq M^*\|a\|_{L_p},$$

where $M^* = \|T\|_{L_p,L_p^*}$.

Theorem 4.1 (Marcinkiewicz). *Let* $T: \{L_{p_0}, L_{p_1}\}\to\{L_{p_0}^*, L_{p_1}^*\}$. *Let* $p\in(p_0, p_1)$. *Then* $T: L_p\to L_p$ *and*

$$(4.2)\qquad \|T\|_{L_p,L_p} \leq C\frac{1}{\big(\theta(1-\theta)\big)^{1/p}}\|T\|_{L_{p_0},L_{p_0}^*}^{1-\theta}\|T\|_{L_{p_1},L_{p_1}^*}^{\theta},$$

where C *depends only on* p_0 *and* p_1, *and* $1/p = (1-\theta)/p_0+\theta/p_1$.

Proof. We shall base the proof on the following lemma where (4.3) should be compared to (2.8):

Lemma 4.1. *Let* T *be as in theorem 4.1. Then there holds*

$$(4.3)\qquad L(t, Ta; L_{\vec{p}}) \leq C\int_0^\infty \min\Big(1, \frac{t}{s}\Big) M_0^{*p_0} L\Big(\frac{M_1^{*p_1}s}{M_0^{*p_0}}, a\Big)\frac{ds}{s},$$

$$M_0^* = \|T\|_{L_{p_0},L_{p_0}^*},\qquad M_1^* = \|T\|_{L_{p_1},L_{p_1}^*}$$

with C *depending only on* p_0 *and* p_1.

Indeed, using (4.3) and theorem 3.1 (cf. notably (3.4)), we obtain

$$\|Ta\|_{L_p}^p = c^{-1}\int_0^\infty t^{-\eta} L(t, Ta)\frac{dt}{t}$$

$$\leqslant c^{-1}C\int_0^\infty t^{-1}\left(\int_0^\infty \min\left(1,\frac{t}{s}\right) M_0^{*p_0} L\left(\frac{M_1^{*p_1}s}{M_0^{*p_0}}, a\right)\frac{ds}{s}\right)\frac{dt}{t}$$

$$= c^{-1}C\int_0^\infty M_0^{*p_0}L\left(\frac{M_1^{*p_1}s}{M_0^{*p_0}},a\right)\left(\int_0^\infty t^{-\eta}\min\left(1,\frac{t}{s}\right)\frac{dt}{t}\right)\frac{ds}{s}$$

$$= c^{-1}C\frac{1}{\eta(1-\eta)}\int_0^\infty M_0^{*p_0}s^{-\eta}L\left(\frac{M_1^{*p_1}s}{M_0^{p_0}},a\right)\frac{ds}{s}$$

$$= c^{-1}C\frac{1}{\eta(1-\eta)}M_0^{*p_0}(M_1^*)^{p_1\eta}(M_0^*)^{-p_0\eta}\int_0^\infty s^{-\eta}L(s,a)\frac{ds}{s}$$

$$=\frac{C}{\eta(1-\eta)}M_0^{*p(1-\theta)}M_1^{*p\theta}\|a\|_{L_p}^p,$$

where we have used the relations

$$\eta=\frac{\theta p}{p_1},\qquad 1-\eta=\frac{(1-\theta)p}{p_0}.$$

These relations also show that $\eta(1-\eta)\approx\theta(1-\theta)$. Thus (4.2) follows.

Proof of lemma 4.1. It suffices to prove for $j = 0, \pm 1, \pm 2, \ldots$ the estimate

(4.4) $$\int_{U_j}\min\left(|Ta(u)|^{p_0}, t|Ta(u)|^{p_1}\right)d\mu(u)$$
$$\leqslant C\min(1, t2^{j(p_1-p_0)})\, M_0^{*p_0}L\left(\frac{M_1^{*p_1}2^{j(p_0-p_1)}}{M_0^{*p_0}}, a\right)$$

with $U_j = \{u\ \ 2^j \leqslant |Ta(u)| < 2^{j+1}\}$.

Indeed, taking the sum in (4.4) we get

(4.5) $$\int_U \min\left(|Ta(u)|^{p_0}, t|Ta(u)|^{p_1}\right)d\mu(u)$$
$$\leqslant C\sum_{j=-\infty}^{\infty}\min(1, t2^{j(p_1-p_0)})M_0^{*p_0}L\left(\frac{M_1^{*p_1}2^{j(p_0-p_1}}{M_0^{*p_0}}, a\right).$$

But in view of (3.2) the left-hand side of (4.5) is equivalent to $L(t, Ta)$ and, moreover, as is readily seen, the right-hand is equivalent to the integral in (4.3). Thus (4.3) follows. It remains to prove (4.4). Consider to this end any decomposition $a = a_0+a_1$. Since $Ta = Ta_0+Ta_1$, it follows that

$$U_j\subset\{u|\ |Ta_0(u)|\geqslant 2^{j-1}\}\cup\{u|\ |Ta_1(u)|\geqslant 2^{j-1}\}.$$

Hence using also (4.2) we get

$$\int_{U_j}\min\left(|Ta(u)|^{p_0}, t|Ta(u)|^{p_1}\right)d\mu(u)$$
$$\leqslant C\min(2^{jp_0}, t2^{jp_1})\Big(\int_{\{u|\ |Ta_0(u)|\geqslant 2^{j-1}\}}d\mu(u)+\int_{\{u|\ |Ta_1(u)|\geqslant 2^{j-1}\}}d\mu(u)\Big)$$
$$\leqslant C\min(2^{jp_0}, t2^{jp_1})(M_0^{*p_0}2^{-jp_0}\|a_0\|_{L_{p_0}}^{p_0}+M_1^{*p_1}2^{-jp_1}\|a_1\|_{L_{p_1}}^{p_1})$$
$$\leqslant C\min(1, t2^{j(p_1-p_0)})M_0^{*p_0}\left(\|a_0\|_{L_{p_0}}^{p_0}+\frac{M_1^{*p_1}2^{j(p_0-p_1)}}{M_0^{*p_0}}\|a_1\|_{L_{p_1}}^{p_1}\right)$$

and this in view of (2.2) clearly implies (4.4).

Remark 4.1. We believe that the above is the "abstract" proof which comes closest to the classical proofs of the theorem of Marcinkiewicz (see Zygmund [28], chap. 12). We do not consider the problem to which extent the method can be carried over to the "field valued" case.

5. The Orlicz case. Let again $A_0 = L_{p_0}$, $A_1 = L_{p_1}$ or $\vec{A} = L_{\vec{p}}$.

We shall try to extend the proof of theorem 3.1 so that it can be applied to more general cases than L_p, namely "Orlicz classes" O_H and "Orlicz spaces" $\hat{O}_H$ (cf. notably Krasnoselskij-Rutitskij [9]). From the abstract point of view this means that we have to replace the spaces $(A_0, A_1)_{\eta q;L}$ by more general interpolation spaces. Let H be any increasing mapping from R_+ into R_+ vanishing at 0 only. We define:

(5.1) $$a\in O_H \Leftrightarrow \|a\|_{O_H} = \int_U H(|a(u)|)\,d\mu(u)<\infty,$$

(5.2) $$a\in \hat{O}_H \Leftrightarrow \|a\|_{\hat{O}_H} = \inf\left\{\alpha\,|\ \int_U H\left(\frac{|a(u)|}{\alpha}\right)d\mu(u)\leqslant 1\right\}.$$

Here $\|a\|_{O_H}$ cannot be expected to be a quasi-norm (and indeed O_H need not even be a vector space). However $\|a\|_{\hat{O}_H}$ is a quasi-norm under suitable assumptions on H, e.g. if H is quasi-convex in the sense that

$$H((1-\lambda)x+\lambda y)\leqslant \lambda H(\gamma x)+(1-\lambda)H(\gamma y)\qquad (0\leqslant\lambda\leqslant 1)$$

for some $\gamma\geqslant 1$ independent of x, y, and λ. In particular, if $\gamma = 1$, H is convex and we get a norm. Note that $O_H = \hat{O}_H = L_p$ if $H(x) = x^p$ $(0<p<\infty)$.

Let us thus return to the proof of theorem 3.1. More specifically, let us in (3.4) replace the special measure $t^{-\eta}dt/t$ by a general measure $d\xi(t)$. We obtain

$$(5.3)\qquad \int_0^\infty L(t, a; L_{\vec{p}})\,d\xi(t) = \int_U |a(u)|^{p_0}\Big(\int_0^\infty F\big(t|a(u)|^{p_1-p_0}\big)\,d\xi(t)\Big)\,d\mu(u)$$

$$\approx \int_U |a(u)|^{p_0}\Big(\int_0^\infty \min\big(1, t|a(u)|^{p_1-p_0}\big)\,d\xi(t)\Big)\,d\mu(u)$$

$$= \int_U |a(u)|^{p_0} h\big(|a(u)|^{p_1-p_0}\big)\,d\mu(u) = \int_U H\big(|a(u)|\big)\,d\mu(u),$$

where we have set

$$(5.4)\qquad h(x) = \int_0^\infty \min(1, tx)\,d\xi(t),$$

$$(5.5)\qquad H(x) = x^{p_0} h(x^{p_1-p_0}).$$

Now the equivalence in (5.3) is of course still valid if we replace h by any equivalent function. There arises the question which are the functions h, which are equivalent to a function admitting the representation (5.4). This problem is solved in Peetre [20], [21] (cf. Peetre [22]). It is shown there that a necessary and sufficient condition is that h satisfies the inequality

$$(5.6)\qquad h(\lambda x) \leqslant C\max(1, \lambda)\,h(x).$$

We call such functions *pseudo-concave*. We can now easily complete the proof of

THEOREM 5.1. *Let* $T\colon \{L_{p_0}, L_{p_1}\} \to \{L_{p_0}, L_{p_1}\}$. *Let* H *be any function of the form* (5.5) *with* h *pseudo-concave. Then* $T\colon O_H \to O_H$, $T\colon \hat{O}_H \to \hat{O}_H$ *and*

$$(5.7)\qquad \left\|\frac{T}{M}\right\|_{O_H, O_H} \leqslant C, \qquad \|T\|_{O_{\hat{H}}, O_{\hat{H}}} \leqslant C\cdot M,$$

where $M = \max(M_0, M_1)$, $M_0 = \|T\|_{L_{p_0}}$, $M_1 = \|T\|_{L_{p_1}}$ *and* C *is a constant independent of* T.

Proof (end). As in the proof of theorem 2.1 (cf. notably (2.8)) we have

$$(5.8)\qquad L(t, Ta) \leqslant M^{p_0} L(M^{p_1-p_0}t, a).$$

From (5.8) and (5.3) it follows that

$$(5.9)\qquad \left\|\frac{Ta}{M}\right\|_{O_H} = \int_U H\left(\frac{|Ta(u)|}{M}\right) d\mu(u)$$

$$\leqslant c\int_0^\infty L\left(t, \frac{Ta}{M}\right) d\xi(t) \leqslant c\int_0^\infty M^{p_0} L\left(M^{p_1-p_0}t, \frac{a}{M}\right) d\xi(t)$$

$$= c\int_0^\infty L(t, a)\,d\xi(t) \leqslant c\int_U H\big(|a(u)|\big)\,d\mu(u) = c\|a\|_{O_H},$$

which proves the first inequality in (5.7). For the second one we note that in view of (5.6)

$$\|\lambda a\|_{O_H} \leqslant c\max(\lambda^{p_0}, \lambda^{p_1})\,\|a\|_{O_H}.$$

Therefore (5.9) can also expressed in the form

$$\left\|\frac{Ta}{CM}\right\|_{O_H} \leqslant \|a\|_{O_H}.$$

Then there holds also for every a

$$\left\|\frac{Ta}{CMa}\right\|_{O_H} \leqslant \left\|\frac{a}{a}\right\|_{O_H},$$

which by (5.2) implies

$$\|Ta\|_{\hat{O}_H} \leqslant CM\|a\|_{\hat{O}_H}$$

establishing thus the second inequality of (5.7).

Next we investigate to which extent the condition on H is necessary.

THEOREM 5.2. *Assume that the conclusions of theorem* 5.1 *hold for all measure spaces* U *and all measures* μ. *Then* H *must be of the form* (5.5) *with* h *pseudo-concave.*

Proof. We clearly may take $U = (0, \infty)$, $d\mu(u) = du$. We define T by $Ta(u) = a(\sigma u)$. Then $M_0 = \sigma^{-1/p_0}$, $M_1 = \sigma^{-1/p_1}$, $M = \max(\sigma^{-1/p_0}, \sigma^{-1/p_1})$, so that by the first inequality of (5.7) we get

$$\int_0^\infty H\left(\frac{|a(\sigma u)|}{\max(\sigma^{-1/p_0}, \sigma^{-1/p_1})}\right) du \leqslant C\int_0^\infty H\big(|a(u)|\big)\,du.$$

Let I be any interval of length 1. Choose $a(u) = x$ on I and $a(u) = 0$ elsewhere. Then we get

$$H\big(\min(\sigma^{1/p_0}, \sigma^{1/p_1})x\big) \leqslant C\sigma H(x).$$

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If we now set $\lambda = \min(\sigma^{1/p_0}, \sigma^{1/p_1})$ we clearly have $\sigma = \max(\lambda^{p_0}, \lambda^{p_1})$. Therefore we get $H(\lambda x) \leqslant C\max(\lambda^{p_0}, \lambda^{p_1})H(x)$, which of course in view of (5.5) is equivalent to (5.6).

Finally, we consider in some greater detail the case where the associated h satisfies (5.6) with $C = 1$. The latter restriction is equivalent to

$$0 \leqslant xh'(x) \leqslant h(x). \tag{5.10}$$

Now solving (5.5) for h we get

$$h(x) = x^{-p_0/(p_1-p_0)}H(x^{1/(p_1-p_0)})$$

and upon differentiating this expression

$$h'(x) = -\frac{p_0}{p_1-p_0}x^{-p_0/(p_1-p_0)-1}H(x^{1/(p_1-p_0)}) + \\ + x^{-p_0/(p_1-p_0)}H'(x^{1/(p_1-p_0)})\frac{1}{p-p_0}\cdot x^{1/(p_1-p_0)-1}$$

or

(5.11)

$$xh'(x) = \frac{1}{p_1-p_0}x^{-p_0/(p_1-p_0)}\{-p_0H(x^{1/(p_1-p_0)}) + x^{1/(p_1-p_0)}H'(x^{1/(p_1-p_0)})\}.$$

Inserting (5.11) in (5.10) we get (after replacing x by $x^{p_1-p_0}$)

$$p_0H(x) \leqslant xH'(x) \leqslant p_1H(x) \tag{5.12}$$

(if $p_0 < p_1$; otherwise the reverse inequalities) which is thus equivalent to (5.6) with $x = 1$. When summing up we arrive at the following useful criterion:

THEOREM 5.3. *Assume that* (5.12) *is fulfilled. Then the conclusions of theorem* 5.1 *hold true.*

Proof. Apply theorem 5.1.

6. The modular case. In Section 5 we obtained in comparison to Section 3 more general spaces than L_p as interpolation spaces but we kept the endpoint space L_{p_0} and L_{p_1} fixed. Now we replace also the latter by more general ones. Namely we let

$$A_0 = O_{\{H_0\}}, \quad A_1 = O_{\{H_1\}} \quad \text{or} \quad A = \vec{O}_{\{\vec{H}\}}.$$

Here $O_{\{H\}}$ generally speaking denotes a "modular function space", i.e. H denotes a mapping from $U\times R_+$ into R_+ such that for each $u\in U$ the mapping $x\to H(u, x)$ is decreasing and we have

$$a\in O_{\{H\}} \Leftrightarrow \|a\|_{O_{\{H\}}} = \int_U H(u, |a(u)|)\,d\mu(u) < \infty. \tag{6.1}$$

If the "sections" $x\to H(u, x)$ are independent of $u\in U$, we clearly get an Orlicz class which we as in Section 5 might denote by O_H simply. Of course, $\|a\| = \|a\|_{\{O_H\}}$ is not a quasi-norm in general but it is a "modular" in the sense of Musielak-Orlicz [12], i.e. in place of the quasi-norm properties (see Section 1) we have $\|\lambda a+\mu b\| \leqslant \|a\|+\|b\|$ if $\lambda+\mu = 1, \lambda\geqslant 0$ and $\mu\geqslant 0$.

We have to modify the definition of $L(t, a)$ ((0.4) or (2.2)). Thus in the rest of this section we set

$$L(t, a) = \inf_{a=a_0+a_1}(\|a_0\|_{O_{\{H_0\}}} + t\|a_1\|_{O_{\{H_1\}}}). \tag{6.2}$$

Corresponding to (3.3) we then obtain

$$L(t, a) = \int_U H(t, u, |a(u)|)\,d\mu(u), \tag{6.3}$$

where we have set

$$H(t, u, x) = \inf_{x=x_0+x_1}(H_0(u, x_0)+tH_1(u, x_1)). \tag{6.4}$$

As in the proof of theorem 3.1 or theorem 5.1 we obtain subsequently corresponding to (3.4) or (5.3)

$$\int_0^\infty L(t, a)\,d\xi(t) = \int_U H(u, |a(u)|)\,d\mu(u) \tag{6.5}$$

with

$$H(u, x) = \int_0^\infty H(t, u, x)\,d\xi(t). \tag{6.6}$$

And this again leads to

THEOREM 6.1. *Let* $T\colon \{O_{\{H_0\}}, O_{\{H_1\}}\}\to\{O_{\{H_0\}}, O_{\{H_1\}}\}$. *Let H be any function of the form* (6.6). *Then* $T\colon O_{\{H\}}\to O_{\{H\}}$ *and*

$$\|T\|_{O_{\{H\}},O_{\{H\}}} \leqslant N, \tag{6.7}$$

where $N = \max(N_0, N_1)$, $N_0 = \|T\|_{O_{\{H_0\}},O_{\{H_0\}}}$, $N_1 = \|T\|_{O_{\{H_1\}},O_{\{H_1\}}}$.

Proof (end). We easily find

$$L(t, Ta) \leqslant NL(t, a). \tag{6.8}$$

Using (6.5) and (6.8) we then get $\|Ta\|_{O_{\{H\}}} \leqslant N\|a\|_{O_{\{H\}}}$, which implies formula (6.7).

7. On the limiting case p_0 or $p_1\to\infty$. Hitherto we have assumed that p_0 and p_1 were finite. Now we say a few words, mostly of formal nature, about the limiting case when one of them, say p_1, tends to infinity.

Let us first consider any couple $\vec{A} = \{A_0, A_1\}$. We have the formula (cf. (2.16))

$$L(t) = L(t, a; \vec{A}) = \inf_{\vec{x}\in\Gamma} (x_0^{p_0} + t x_1^{p_1}). \tag{7.1}$$

Let also temporarily assume $t = 1$. Then if $p_1 \to \infty$, the term within parentheses in (7.1) tends to $x_0^{p_0}$ or ∞ according to if $x_1 < 1$ or $x_1 > 1$. (It is convenient to disregard the possibility $x_1 = 1$, which would give the contribution $x_0^{p_0}+1$.) Thus we get as a limiting form of (7.1) (with $t = 1$)

$$M(1) = \lim_{p_1\to\infty} L(1) = \inf_{\vec{x}\in\Gamma} x_0^{p_0}.$$

Nothing essential changes if we pass from the special case $t = 1$ to the case of general t. However, if we first in (7.1) replace t by t^{p_1} and then apply the above limiting procedure, we get

$$M(t) = M(t, a; \vec{A}) = \lim_{p_1\to\infty} L(t^{p_1}) = \inf_{\substack{\vec{x}\in\Gamma,\\ tx_1<1}} x_0^{p_0}, \tag{7.2}$$

which gives the limiting form of $L(t)$ we are looking for.

Now minimizing problems of the type appearing in (7.2) are quite familiar in many contexts, e.g. in problems of statistical estimations. Indeed, if we specialize to the case $A_0 = L_1(\mu_0)$, $A_1 = L_1(\mu_1)$ taking also $p_0 = 1$, our methods for evaluating $L(t)$ when modified to the limiting case $M(t)$ reduce essentially to a proof of the Neyman-Pearson lemma (see the discussion in Beckenbach-Bellman [1], p. 121-123).

Another special case deserves attention, namely $A_0 = L_{p_0}$, $A_1 = L_\infty$. It is, however, convenient to perform the passage to the limit in a somewhat different fashion. We start instead of (7.1) with the formula

$$L(t, a; \{L_{p_0}, L_{p_1}\}) = \inf_{a=a_0+a_1} \Big(\int_U |a_0(u)|^{p_0} d\mu(u) + t\int_U |a_1(u)|^{p_1} d\mu(u)\Big). \tag{7.3}$$

Starting again with the case $t = 1$, a formal passage to the limit yields

$$\lim_{p_1\to\infty} L(1, a; \{L_{p_0}, L_{p_1}\}) = \inf_{a=a_0+a_1} \Big(\int_U |a_0(u)|^{p_0} d\mu(u) + \int_{\{u\,|\,|a_1(u)|=1\}} d\mu(u)\Big).$$

Generalizing this formula to general t in analogous manner as above, we then get

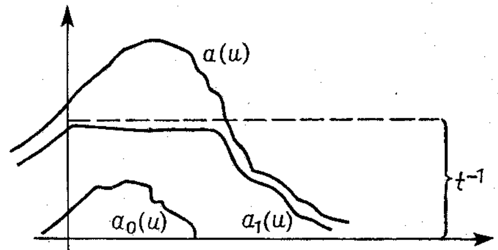
(7.4)

$$\lim_{p_1\to\infty} L(t^{p_1}, a; \{L_{p_0}, L_{p_1}\}) = \inf_{\substack{a=a_0+a_1\\ |a_1(u)|\le t^{-1}}} \Big(\int_U |a_0(u)|^{p_0} d\mu(u) + \int_{\{u\,|\,|a_1(u)|=t^{-1}\}} d\mu(u)\Big).$$

Now it is not hard to see that the inf in (7.4) is assumed if

$$a_0(u) = \begin{cases} a(u) - \dfrac{a(u)}{|a(u)|}\, t^{-1} & \text{if } |a(u)| \geq t^{-1}, \\ 0 & \text{if } |a(u)| < t^{-1}, \end{cases}$$

$$a_1(u) = a(u) - a_0(u).$$



Therefore we get

$$\lim_{p_1\to\infty} L(t^{p_1}, a; \{L_{p_0}, L_{p_1}\}) = \int_{|a(u)|\geq t^{-1}} |a(u)|^{p_0} d\mu(u) \tag{7.5}$$

or, introducing the positive decreasing rearrangement a^* of a,

$$\lim_{p_1\to\infty} L(t^{p_1}, a; \{L_{p_0}, L_{p_1}\}) = \int_0^\tau |a^*(\sigma)|^{p_0} d\sigma \tag{7.6}$$

with $\tau = \int_{|a(u)|\geq t^{-1}} d\mu(u)$.

This should be compared with the formula (due to Peetre [14], [18] if $p_0 = 1$, Krée [10] if p_0 general)

$$K_{p_0}(\tau, a; L_{p_0}, L_\infty) = \Big(\int_0^\tau |a^*(\sigma)|^{p_0} d\sigma\Big)^{1/p_0}, \tag{7.7}$$

on which a great deal of the theory of interpolation in the case $A_0 = L_{p_0}$, $A_1 = L_{p_1}$ can be based (see in particular also Holmstedt [6], [7]). Thus we have found a connection between (7.7) and the formulas for $L(t, a)$ given in the previous sections (see notably (3.2)).

8. On the parameter theorem. Let $\vec{A} = \{A_0, A_1\}$ be any Banach couple. Let us introduce the following space (cf. Lions-Peetre [11]):

$$a \in \vec{A}_{\theta p_0 p_1} \Leftrightarrow \exists\, \vec{v} = \{v_0, v_1\}:\ a = v_0(t) + v_1(t), \tag{8.1}$$

$$|||v||| = |||v_0|||_0 + |||v_1|||_1$$

$$= \Big(\int_0^\infty (t^{-\theta}\|v_0(t)\|_{A_0})^{p_0} \frac{dt}{t}\Big)^{1/p_0} + \Big(\int_0^\infty (t^{1-\theta}\|v_1(t)\|_{A_1})^{p_1} \frac{dt}{t}\Big)^{1/p_1} < \infty \quad (0 < \theta < 1).$$

This is a quasi-Banach space if we set

(8.2) $$\|a\| = \|a\|_{\vec{A}_{\theta p_0 p_1}} = \inf_{\vec{v}} |||\vec{v}|||.$$

The purpose of this section is to provide a new proof of the following theorem of which several proofs are known (Peetre [14], [15], [23], Holmstedt [6], [7]). Our argument is particularly close to the one of Holmstedt.

THEOREM 8.1 (parameter theorem). *We have* $\vec{A}_{\theta p_0 p_1} = \vec{A}_{\theta p;K}$, *where* $1/p = (1-\theta)/p_0 + \theta/p_1$ *or, in view of theorem* 2.2, $\vec{A}_{\theta p_0 p_1} = \vec{A}_{\eta 1;L}$, *where* $\eta = \theta p/p_1$.

The following result is known (Lions-Peetre [11], chap. I, lemma 3.1):

LEMMA 8.1. $\|a\| \approx \inf_{\vec{v}} |||v_0|||_0^{1-\theta} |||v_1|||_1^{\theta}$.

However with the same proof we also obtain

LEMMA 8.2. $\|a\| \approx \inf_{\vec{v}} (|||v_0|||_0^{p_0} + |||v_1|||_1^{p_1})^{1/p}$.

For completeness we indicate the argument.

Proof of lemma 8.1. By the inequality between the arithmetic and the geometric means we get

$$|||v_0|||_0^{1-\theta} |||v_1|||_1^{\theta} \leqslant (1-\theta)|||v_0|||_0 + \theta |||v_1|||_1 \leqslant |||v_0|||_1 + |||v_1|||_1.$$

Therefore, by (8.2), we obtain

$$\inf |||v_0|||^{1-\theta} |||v_1|||^{\theta} \leqslant \|a\|,$$

which is one of the inequalities needed. For the proof of the other one let us note the inequality

(8.3) $$\|a\| \leqslant |||v_0|||_0 + |||v_1|||_1 \leqslant 2\max(|||v_0|||_0, |||v_1|||_1),$$

where $\vec{v} = \vec{v}(t)$ is any function satisfying the relation $a = v_0(t) + v_1(t)$. Let us now with such a given v associate the function $\vec{w}(t) = \vec{w}(\lambda t)$, where λ is any real number > 0. The crucial relation $a = w_0(t) + w_1(t)$ is still satisfied. Therefore (8.3) yields

$$\|a\| \leqslant 2\max(|||w_0|||_0, |||w_1|||_1) \leqslant 2\max(\lambda^{\theta} |||v_0|||_0, \lambda^{\theta-1} |||v_1|||_1).$$

But if we choose λ so that $\lambda^{\theta} |||v_0|||_0 = \lambda^{\theta-1} |||v_1|||_1$, we have $\|a\| \leqslant 2 |||v_0|||_0^{1-\theta} |||v_1|||_1^{\theta}$. Therefore follows upon making $\vec{v}$

$$\|a\| \leqslant 2\inf |||v_0|||_0^{1-\theta} |||v_1|||_1^{\theta},$$

which is precisely what is desired.

Proof of lemma 8.2. The inequality between the arithmetic and the geometric means now yields

$$|||v_0|||_0^{1-\theta} |||v_1|||_1^{\theta} \leqslant \left(\frac{p(1-\theta)}{p_0} |||v_0|||_0^{p_0} + \frac{p\theta}{p_1} |||v_1|||_1^{p_1}\right)^{1/p} \leqslant c(|||v_0|||_0^{p_0} + |||v_1|||_1^{p_1})^{1/p},$$

where we have used the definition of p. Therefore by lemma 8.1 we obtain

$$\|a\| \leqslant C\inf (|||v_0|||_0^{p_0} + |||v_1|||_1^{p_1})^{1/p},$$

which is one of the inequalities needed. The other one is proved exactly as in the proof of lemma 8.1 by means of the function $\vec{w}(t) = \vec{v}(\lambda t)$, but the parameter λ of course has to be chosen differently, namely so that

$$\lambda^{\theta p_0} |||v_0|||_0^{p_0} = \lambda^{(\theta-1)p_1} |||v_1|||_1^{p_1}.$$

It is now easy to prove theorem 8.1.

Proof of theorem 8.1. By lemma 8.2 we get using also (2.2) and (2.4) and the definition of η

$$\begin{aligned}\|a\|^p &\approx \inf |||v_0|||_0^{p_0} + |||v_1|||_1^{p} \\ &= \inf \int_0^\infty \left(\left(t^{-\theta}\|v_0(t)\|_{A_0}\right)^{p_0} + \left(t^{1-\theta}\|v_1(t)\|_{A_1}\right)^{p_1}\right)\frac{dt}{t} \\ &= \int_0^\infty t^{-\theta p_0} \inf\left(\|v_0(t)\|_{A_0}^{p_0} + t^{(1-\theta)p_1+\theta p_0}\|v_1(t)\|_{A_1}^{p_1}\right)\frac{dt}{t} \\ &= \int_0^\infty t^{-\theta p_0} L(t^{(1-\theta)p_1+\theta p_0}, a)\frac{dt}{t} \\ &= c\int_0^\infty t^{-\eta} L(t, a)\frac{dt}{t} = c\|a\|_{\vec{A}_{\eta 1}}.\end{aligned}$$

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The main triangle projection in matrix spaces and its applications

by

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Introduction. The origin of this paper are the following three, at first appearance unrelated, problems:

1. Is the operator $S\colon l_1 \to l_\infty$ given by $S(a(n)) = \sum_{i\leqslant n} a(i)$ (p, q)-absolutely summing for $p > q \geqslant 1$? ([8], Problem 5).

2. Does there exist an unconditional basis in the space of all compact linear operators in an infinite-dimensional Hilbert space?

3. Is every unconditionally convergent series in l_1 of the form $\sum_n P^n x$, where $P^n(a(i)) = (a(i+n))$, absolutely convergent? (S. Mazur, Scottish Book, Problem 89).

It became clear that all these problems reduce to estimation of norms of "the main triangle projections" in corresponding matrix spaces. Let us consider, for example, the linear space of all matrices $a = (a(i,j))$ with the norm

$$\lambda_{2,2}(a) = \sup \sum_{i,j} s(i)t(j)a(i,j),$$

where the supremum is taken over all sequences $(s(i))$, $(t(j))$ of scalars such that $\sum_i |s^2(i)| \leqslant 1$, $\sum_j |t^2(j)| \leqslant 1$ ($\lambda_{2,2}(a)$ is equal to the norm of the operator in l_2 given by the matrix a). The main triangle projection is defined by

$$T_n(a)(i,j) = \begin{cases} a(i,j) & \text{if } i+j \leqslant n+1, \\ 0 & \text{otherwise.} \end{cases}$$

We prove that the norms of these projections grow the same as $\ln n$ when n becomes large. This order of growth is attained for the Hilbert matrices h_n, $h_n(i,j) = (n+1-i-j)^{-1}$ if $i+j \neq n+1$ and $i, j \leqslant n$, $h_n(i,j) = 0$ otherwise.

In the first section the concept of a matrix norm is introduced, and the norms of the main triangle projections with respect to some special matrix norms are estimated. The results of this section applied to the matrix