A New Approach to Analytical Solution of Cantilever Beam Vibration With Nonlinear Boundary Condition

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This paper presents the application of novel and reliable exact equivalent function (EF) for deadzone nonlinearity in an analytical investigation of nonlinear differential equations. A highly nonlinear equation of cantilever beam vibration with a deadzone nonlinear boundary condition is used to indicate the effectiveness of this EF. To obtain the analytical solution of dynamic behavior of the mentioned system, a powerful method, called He's parameter expanding method (HPEM) is used. Comparison of the obtained solutions using a numerical method reveals the accuracy of this analytical EF. [DOI: 10.1115/1.4005924]

Keywords: exact equivalent function, cantilever beam, deadzone nonlinearity, Bubnov-Galerkin, He's parameter expanding method

1 Introduction

Over many decades, the dynamics of vibrations of nonlinear beam structures has been a subject of great interest in the broad field of structural mechanics. The nonlinear vibration of beams has received considerable attention by many researchers [1–8]. The sources of nonlinearity of vibration systems are generally considered to be due to the following aspects: (1) the physical nonlinearity, (2) the geometric nonlinearity, and (3) the nonlinearity of boundary conditions. In the case of discontinuous nonlinear boundary conditions, the analytical solution of such problems becomes very complex. Deadzone nonlinearity, as a discontinuous nonlinear boundary condition, due to its inherent difficulty, has not been modeled exactly by researchers until the present.

As it is reported in many research papers, the deadzone nonlinearity is a nondifferentiable function. This input characteristic is quite common in a wide range of mechanical and electrical components such as valves, gear vibration, DC servomotors, and other devices. However, approximation of this nonlinear condition to obtain an analytical solution of the behavior of mentioned systems is always a major difficulty in an engineer's computations. Marcio and Leandro [9] used the error function as an approximation of deadzone-type nonlinearity in deriving analytical models for the least mean square (LMS) adaptive algorithm. Chengwu and Rajendra [10] used the arctangent function to approximate the nonanalytical deadzone relationship in preloaded springs in a mechanical oscillator. Hakimi and Moradi [11] modeled the contact between the drillstring and formation wall by a series of springs with deadband gap using DQM. Recently, considerable attention has been directed towards analytical solutions for nonlinear equations without small parameters. There have been several classical approaches employed to solve the governing nonlinear differential equations to study the nonlinear vibrations including the perturbation methods [12], form function approximations [13], artificial small parameter [14], min-max method [15], multiple scales method [16], variational iteration method [17], HAM [18,19], and homotopy perturbation method (HPM) [20] are used to solve nonlinear problems. He's parameter expanding method is the most effective and convenient method to analytically solve nonlinear differential equations. HPEM has been shown to effectively, easily, and accurately solve large nonlinear problems with components that converge rapidly to accurate solutions. He [21] proposed an elementary introduction to the basic solution procedure of the He's parameter expanding method. The asymptotic analysis of nonlinear beam vibration with preload boundary condition using HPEM has been studied by Sedighi et al. [22,23].

This paper intends to obtain an analytical solution for the nonlinear vibration of a cantilever beam using HPEM, with deadzone nonlinear boundary condition, by introducing novel and efficient exact EF. First, the nonlinear partial differential equation of motion has been reduced by implementation of the Bubnov-Galerkin method, and then mentioned EF has been used to define the deadzone nonlinear boundary condition. The results presented in this paper demonstrate that the method is very effective and convenient for nonlinear oscillators for which the highly nonlinear boundary condition exists. To certify the introduced EF application, it is shown that one term in series expansions is sufficient to obtain a highly accurate solution to the problem.

2 Governing Equation

Consider a cantilever beam of length L, cross-sectional area A, mass per unit length m, moment of inertia I, and a modulus of elasticity E as shown in Fig. 1. A linear spring with constant K is in contact at the free end of the cantilever beam with a deadzone

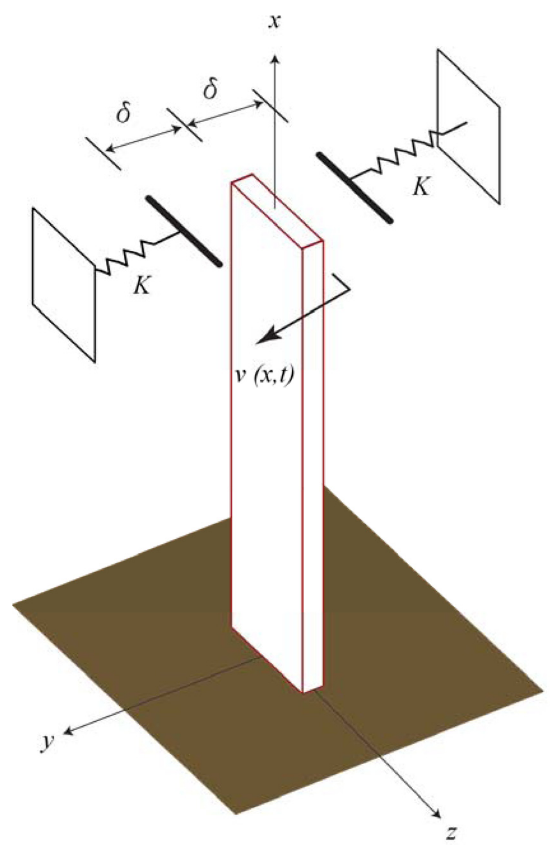

Fig. 1 Cantilever beam with deadzone nonlinear boundary condition

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clearance δ. Assume that the beam considered here is the Euler–Bernoulli beam. The symbol v denotes the displacement of a point in the middle plane of the flexible beam in the y direction.

The nonlinear governing equation of motion for the uniform cantilever beam with deadzone boundary condition and without damping is given by [24]

$$m\ddot{v} + EIv^{iv} + EI\left[v'(v'v'')'\right]' + \frac{1}{2}m\left\{v'\int_L^x\left[\frac{\partial^2}{\partial t^2}\int_0^x v'^2 dx\right]dx\right\}' = 0 \tag{1}$$

The corresponding boundary conditions are

$$v(0,t) = \frac{\partial v}{\partial x}(0,t) = 0,\quad \frac{\partial^2 v}{\partial x^2}(L,t) = 0,\quad EI\frac{\partial^3 v}{\partial x^3}(L,t) = F_{dz}(L,t) \tag{2}$$

where $F_{dz}(L,t)$ is the boundary condition at the free end and is described by the following nonlinear deadzone formula

$$F_{dz}(L,t) = \begin{cases} K(v(L,t)-\delta) & v(L,t) > \delta \\ 0 & -\delta \le v(L,t) \le \delta \\ K(v(L,t)+\delta) & v(L,t) < -\delta \end{cases} \tag{3}$$

Assuming $v(x,t) = q(t)\,\varphi(x)$, where $\varphi(x)$ is the first eigenmode of the clamped-free beam and can be expressed as

$$\varphi(x) = \cosh(\lambda x) - \cos(\lambda x) - \frac{\cosh(\lambda L) + \cos(\lambda L)}{\sinh(\lambda L) + \sin(\lambda L)}(\sinh(\lambda x) - \sin(\lambda x)) \tag{4}$$

where $\lambda = 1.875$ is the root of the characteristic equation for the first eigenmode. Applying the weighted residual Bubnov-Galerkin method yields

$$\int_0^L\left(m\ddot{v} + EIv^{iv} + EI\left[v'(v'v'')'\right]' + \frac{1}{2}m\left\{v'\int_L^x\left[\frac{\partial^2}{\partial t^2}\int_0^x v'^2 dx\right]dx\right\}'\right) \cdot \times \varphi(x)dx = 0 \tag{5}$$

to implement the end nonlinear boundary condition, applying integration by part on Eq. (5), it is converted to the following

$$\int_0^L\left(m\ddot{v} + EI\left[v'(v'v'')'\right]' + \frac{1}{2}m\left\{v'\int_L^x\left[\frac{\partial^2}{\partial t^2}\int_0^x v'^2 dx\right]dx\right\}\right)\varphi(x)dx + \int_0^L EIv^{iv}\varphi(x)\,dx = 0 \tag{6}$$

$$\int_0^L\left(m\ddot{v} + EI\left[v'(v'v'')'\right]' + \frac{1}{2}m\left\{v'\int_L^x\left[\frac{\partial^2}{\partial t^2}\int_0^x v'^2 dx\right]dx\right\}\right)\varphi(x)dx + EIv'''\varphi(x)|_0^L - \int_0^L EIv''' d(\varphi(x)) = 0 \tag{7}$$

In the above equation, the boundary condition term $EIv'''(L,t)$ is replaced by $F_{dz}(L,t)$. So, we can obtain the nonlinear equation in terms of the time-dependent variables as

$$\ddot{q} + \beta_1 q + \beta_2 q^3 + \beta_4 q\dot{q}^2 + \beta_5 q^2\ddot{q} + F_{dz} = 0 \tag{8}$$

where

$$\beta_1 = 12.3624EI/mL^4,\quad \beta_2 = 40.44EI/mL^6,\ \beta_4 = \beta_5 = 4.6/L^2 \tag{9}$$

To solve the nonlinear ordinary Eq. (8) analytically, the deadzone condition F_{dz} must be formulated properly. We introduce a suitable and novel exact equivalent function for this nonlinearity as

$$F_{dz}(u) = {}^{K}\!/_{2}(2u + |u-\delta| - |u+\delta|) \tag{10}$$

Figure 2 shows the equivalent function for F_{dz} with deadzone clearance δ, graphically. The proposed EF is suitable for analytical studies of nonlinear dynamical systems using perturbation based methods such as averaging and homotopy, as well as numerical studies in direct simulations.

Using the new definition of F_{dz}, Eq. (9) is written as follows:

$$\ddot{q} + \beta_1' q + 1.\left[\beta_2 q^3 + \beta_3\{|2q(t)-\delta| - |2q(t)+\delta|\} + \beta_4 q\dot{q}^2 + \beta_5 q^2\ddot{q}\right] = 0 \tag{11}$$

where

$$\beta_1' = \beta_1 + 4K/mL,\quad \beta_3 = K/mL \tag{12}$$

3 Analytical Solution

Consider Eq. (11) for vibration of a cantilever Euler-Bernoulli beam with the following general initial conditions

$$q(0) = A,\quad \dot{q}(0) = 0 \tag{13}$$

The limit-cycles of oscillating systems are periodic motions with the period $T = 2\pi/\omega$, and thus $q(t)$ can be expressed by such a set of base functions

$$\cos(m\omega t),\quad m = 1, 2, 3, \ldots \tag{14}$$

We denote the angular frequency of oscillation by ω and note that one of our major tasks is to determine $\omega(A)$, i.e., the functional behavior of ω as a function of the initial amplitude A. In the HPEM, an artificial perturbation equation is constructed by embedding an artificial parameter $p \in [0,1]$, which is used as an expanding parameter. According to HPEM [17] the solution of Eq. (11) is expanded into a series of p in the form

$$q(t) = q_0(t) + pq_1(t) + p^2 q_2(t) + \ldots \tag{15}$$

The coefficients 1 and β_1' in Eq. (11) are expanded in a similar way.

$$\begin{aligned} 1 &= 1 + pa_1 + p^2 a_2 + \ldots \\ \beta_1' &= \omega^2 - pb_1 - p^2 b_2 + \ldots \\ 1 &= pc_1 + p^2 c_2 + \ldots \end{aligned} \tag{16}$$

where coefficients $a_i, b_i, c_i\ \ (i = 1, 2, 3, \ldots)$ should be determined. Equation (11) becomes a linear differential equation for $p = 0$ and an exact solution can be calculated for $p = 1$. Substituting Eqs. (15) and (16) into Eq. (11)

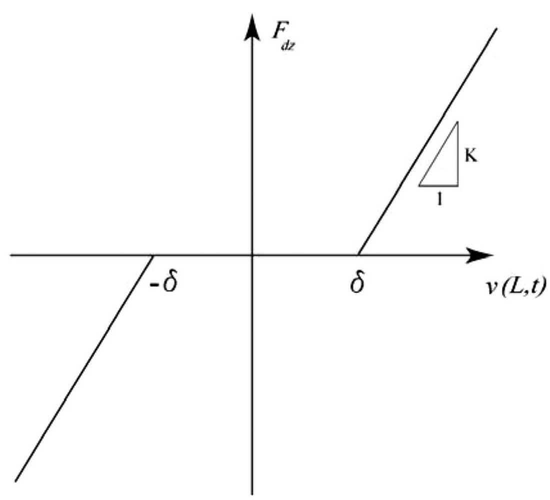

Fig. 2 Plot of EF deadzone nonlinearity

$$(1+pa_1)(\ddot{q}_0+p\ddot{q}_1)+(\omega^2-pb_1)(q_0+pq_1)$$
$$+(pc_1+p^2c_2)\Big[\beta_2(q_0+pq_1)^3+\beta_4(q_0+pq_1)(\dot{q}_0+p\dot{q}_1)^2+\ldots$$
$$\ldots+\beta_5(q_0+pq_1)^2(\ddot{q}_0+p\ddot{q}_1)+\beta_3 f_{dz}(q_0+pq_1)\Big]=0 \quad (17)$$

where

$$f_{dz}(q)=|2q-\delta|-|2q+\delta| \quad (18)$$

In Eq. (18), we have taken into account the following expression:

$$f_{dz}(q)=f_{dz}(q_0+pq_1+p^2q_2+\ldots)$$
$$=\ldots f_{dz}(q_0)+pq_1f'_{dz}(q_0)+p^2\left[q_2f'_{dz}(q_0)+\frac{1}{2}q_1^2f''_{dz}(q_0)\right]$$
$$+O(p^3) \quad (19)$$

where

$$f'_{dz}(q)=\frac{df_{dz}}{dq}=2\frac{|2q-\delta|}{2q-\delta}-2\frac{|2q+\delta|}{2q+\delta}\ \text{and}\ f''_{dz}(q)=f'''_{dz}(q)=\ldots=0 \quad (20)$$

Collecting the terms of the same power of p in Eq. (17), we obtain a series of linear equations. The first equation is

$$\ddot{q}_0(t)+\omega^2q_0(t)=0,\quad q_0(0)=A,\quad \dot{q}_0(0)=0 \quad (21)$$

with the solution

$$q_0(t)=A\cos(\omega t) \quad (22)$$

Substitution of the result into the right-hand side of the second equation gives

$$\ddot{q}_1(t)+\omega^2q_1(t)=\Big(b_1A-\frac{3}{4}c_1\beta_2A^3+4c_1\beta_3A+\frac{3}{4}c_1\beta_5A^3\omega^2$$
$$-\frac{1}{4}c_1\beta_4A^3\omega^2+a_1A\omega^2\Big)\cos(\omega t)$$
$$+\frac{16}{3\pi}c_1\beta_3A\cos(2\omega t)+\frac{1}{4}c_1A^3(\beta_4\omega^2+\beta_5\omega^2-\beta_2)$$
$$\times\cos(3\omega t) \quad (23)$$

In the above equation, the possible following Fourier series expansion have been accomplished:

$$f_{dz}(q_0)=f_{dz}(A\cos(\omega t))=\sum_{n=1}^{\infty}h_n\cos(n\omega t)$$
$$=h_1\cos(\omega t)+h_2\cos(2\omega t)+\ldots$$
$$f'_{dz}(q_0)=f'_{dz}(A\cos(\omega t))=\sum_{n=1}^{\infty}v_n\cos(n\omega t)$$
$$=v_1\cos(\omega t)+v_2\cos(2\omega t)+\ldots \quad (24)$$

where

$$h_n=\frac{2}{\pi}\int_{-\pi/2}^{\pi/2}f_{dz}(A\cos\theta)\cos(n\theta)d\theta$$
$$v_n=\frac{2}{\pi}\int_{-\pi/2}^{\pi/2}f'_{dz}(A\cos\theta)\cos(n\theta)d\theta \quad (25)$$

and the functions f_{dz}, f'_{dz} are substituted from Eqs. (18) and (20). The first terms of the expansion in Eq. (25) are given by

$$h_1=\frac{2}{\pi}\int_{-\pi/2}^{\pi/2}f_{dz}(A\cos\theta)\cos(\theta)d\theta=-4A \quad (26a)$$
$$v_1=\frac{2}{\pi}\int_{-\pi/2}^{\pi/2}f'_{dz}(A\cos\theta)\cos(\theta)d\theta=-\frac{16}{\pi} \quad (26b)$$

The solution of Eq. (23) should not contain the so-called secular term $\cos(\omega t)$. So, the right-hand side of this equation should not contain the terms cos, i.e., the coefficients of cos must be zero.

$$b_1A-\frac{3}{4}c_1\beta_2A^3+4c_1\beta_3A+\frac{3}{4}c_1\beta_5A^3\omega^2-\frac{1}{4}c_1\beta_4A^3\omega^2+a_1A\omega^2=0 \quad (27)$$

Equation (16) for one term approximation of the series with respect to p and for $p=1$ yields

$$a_1=0,\quad b_1=\omega^2-\beta'_1,\quad c_1=1 \quad (28)$$

From Eqs. (27) and (28), it can be easily found that the solution ω is

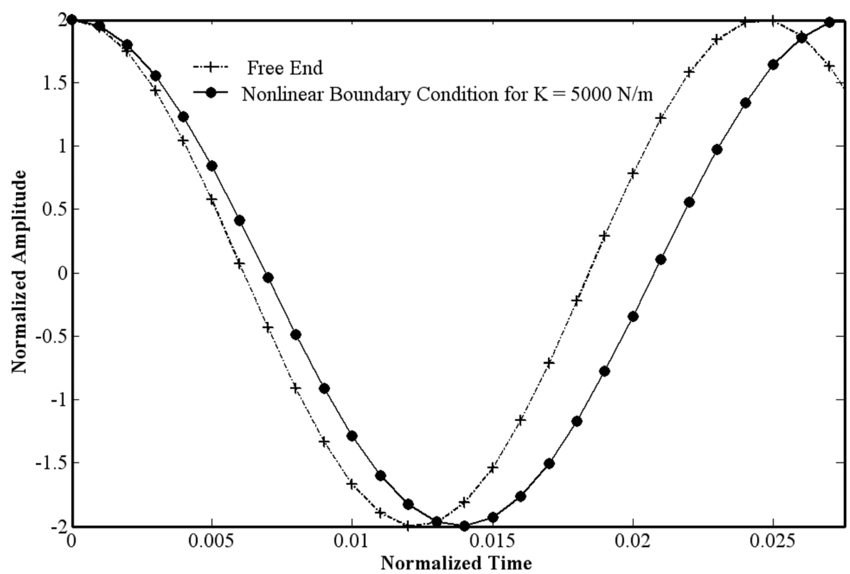

Fig. 3 The impact of nonlinear boundary condition on the response: first order periodic solution (continuous line) with the numerical solution (symbols)

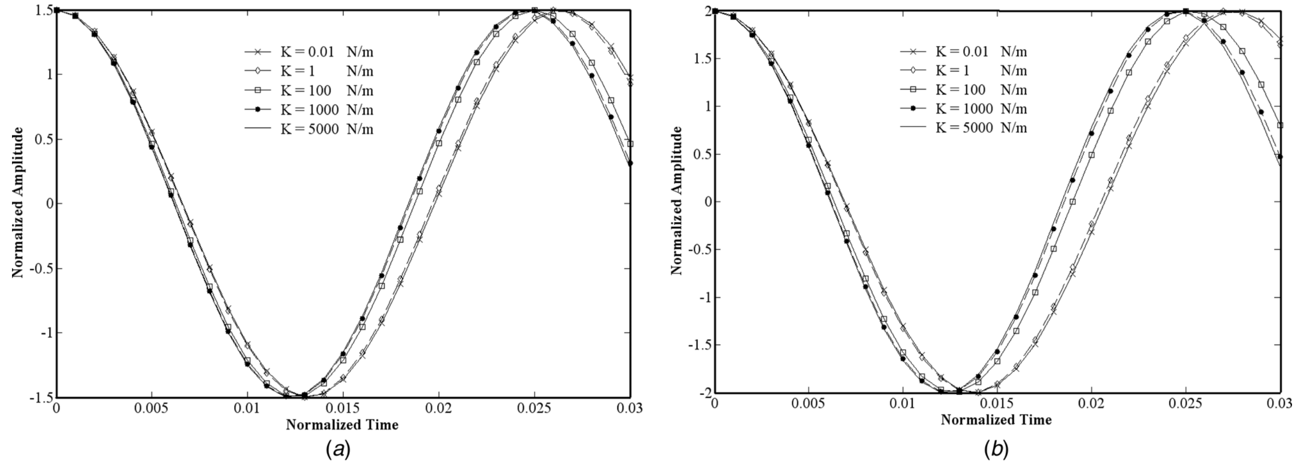

Fig. 4 (*a*) Comparison of the approximate first order periodic solution (continuous line) with the numerical solution (circles) with $A/\delta = 1.5$. (*b*) Comparison of the approximate first order periodic solution (continuous line) with the numerical solution (circles) with $A/\delta = 2$.

$$\omega(A) = \pm\sqrt{\frac{\beta_1' + \frac{3}{4}\beta_2 A^2 - 4\beta_3}{1 + \frac{3}{4}\beta_5 A^2 - \frac{1}{4}\beta_4 A^2}} \tag{29}$$

Replacing ω from Eq. (29) into Eq. (22) yields

$$q(t) \approx q_0(t) = A\cos\left(\sqrt{\frac{\beta_1' + \frac{3}{4}\beta_2 A^2 - 4\beta_3}{1 + \frac{3}{4}\beta_5 A^2 - \frac{1}{4}\beta_4 A^2}}\, t\right) \tag{30}$$

Note that the proposed method can be expanded to predict the beam response in the presence of damping with even nonlinearity, easily. The damping term should be placed in the bracket of Eq. (11). To exhibit the accuracy of the obtained analytical solution, the authors also calculate the variation of nondimensional amplitude A/δ versus $\tau = \omega t$ numerically. Figure 3 illustrates the case of no nonlinear boundary condition for $K = 0$ in comparison with nonlinear deadzone boundary condition for $K = 5000\ N/m$. As can be seen in the Figs. 4(*a*) and 4(*b*), the first order approximation of $q(t)$ using the HPEM with EF for deadzone nonlinearity has an excellent agreement with numerical results using the fourth-order Runge-Kutta method for different values of K.

4 Concluding Remarks

This research introduces the novel and reliable EF for the discontinuous deadzone nonlinearity, as a boundary condition of a cantilever beam, and redefined it exactly using the continuous functions. This new EF is implemented on the nonlinear vibration of a cantilever beam, and an excellent first-order analytical solution using HPEM was obtained. It was demonstrated that the proposed EF can significantly make the analytical investigation of the nonlinear problems easier. The authors believe that the introduced method has special potential to be applied on other strongly nonlinear problems with deadzone nonlinearity.

Nomenclature

System Parameters Used for Numerical Analysis

$A = 25 \times 10^{-4}\ m^{-2}$ cross section area
$E = 200\ Gpa$ modulus of elasticity
$I = 52 \times 10^{-8}\ m^4$ moment of inertia
$K = 5000\ N/m$ spring coefficient
$L = 1\ m$ beam length
$m = 19.65\ kg/m$ mass per unit length

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