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A NEW APPROACH TO CONFORMAL INVARIANT
FIELD THEORIES

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A B S T R A C T

A new approach to conformal invariant field theories is presented. The physical idea is to introduce a fundamental scale of hadron phenomena by means of a dilatation non-invariant vacuum state in the framework of a scale invariant Lagrangian field theory. A new unconventional feature is that this programme can only be carried out if the "vacuum" state is not translation invariant. The "vacuum" is still invariant under a 10-parameter subgroup of the full conformal group but this subgroup does not coincide with the Poincaré group. A physical interpretation based on an appropriate averaging on an infinite ensemble of equivalent "vacuum" states allows to preserve energy momentum conservation which is defined within a thermodynamical framework.

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1. INTRODUCTION

Many important empirical features of high energy hadron physics seem to point towards the existence of a fundamental physical parameter with the dimension of a length. This is clearly exhibited, for example, by the amazing linearity and parallelism of the Regge trajectories and by the nearly constant average transverse momenta in high energy multiple production. The very beautiful and exciting approach based on dual theory ¹⁾ contains a single parameter with the dimension of a length. It aims at a general explanation of hadronic phenomena within a very tight "bootstrap" program in which the presence of no other free parameter, besides a single universal coupling constant, is allowed.

Despite its success, the dual programme has obvious shortcomings : e.g., the requirement that, at least at first approximation, all resonances should be infinitely narrow, and also the difficulties which are met when one tries to incorporate quark point structures.

In this respect, it is of interest to investigate the possibility of introducing a fundamental length within the scheme of the conventional field theory. This, while retaining some of the good features of dual theory, might allow the possibility of constructing a composite model for hadrons in a field theoretical framework.

If we consider the usual renormalizable field theories in which all coupling constants are dimensionless, we see that the only dimensional parameters are the bare masses of the constituent particles. Those bare masses, which are strongly affected by strong SU_3 breaking, are very unlikely to provide a universal SU_3 independent scale for hadronic interactions.

On the other hand, the recent developments on the problem of the origin of the physical masses of particles have opened very exciting new avenues for progress. It is indeed well known that the presence of some symmetry group forces the bare mass of some particles to be zero. For example, gauge invariance of second kind does not allow for the presence of any photon mass term in the electromagnetic Lagrangian.

The symmetry group, which forces the bare mass of a particle to vanish, does usually force also the physical masses of the same particle to vanish. This is indeed the case of the photon.

However, one of the most exciting theoretical discoveries is the possible existence of abnormal solutions of the field theoretical problem where the presence of a non-symmetric vacuum allows the physical masses of these particles. For example, chiral symmetry requires the bare mass of the nucleon to vanish. Spontaneous breaking of this symmetry allows a non-vanishing nucleon physical mass. More recently, it was possible to reconcile the presence of an exact invariance under non-Abelian gauge transformation with non-vanishing physical masses of the W weak intermediate bosons ²⁾.

The new theoretical possibilities opened by the presence of an un-symmetric vacuum has led to beautiful and exciting developments. We recall in particular the very successful development of soft pion theory and the fascinating possibility of unified renormalizable theories of weak and electromagnetic interactions. If one looks at the specific form of the Lagrangian for unified field theories, one finds an asymmetry in the treatment of different kinds of particles. The presence of invariance properties of the Lagrangian sets to zero the bare masses of the spin $\frac{1}{2}$ and of the vector particles which get their mass from the vacuum mechanism. On the other hand, scalar particles which are essential to the success of the whole scheme, must have a non-vanishing mass.

In this paper, we shall take the extreme point of view that no dimensional constant should appear in the Lagrangian, which will be therefore invariant with respect to the dilatation group. We shall then look for a vacuum state which will not be dilatation invariant and will introduce a single-dimensional constant which will represent the fundamental length of the theory.

Let us examine this programme in more detail.

Field theoretical Lagrangians invariant under dilatation usually exhibit invariance under the larger group of conformal transformations. These transformations, in addition to the Poincaré operators $L_{\mu\nu}$ and P_μ and the dilatation operator D , contain an extra four-vector operator K_μ which is related to P_μ by

$$K_\mu = I P_\mu I \quad (1.1)$$

where I is the operator of the inversion transformation $x_\mu \rightarrow x_\mu/x^2$. The algebraic properties of the conformal group, which is isomorphic to a $O(4,2)$ pseudo-Euclidean rotation group, are discussed in detail in the next section ^{3),4)}.

We thus start from a conformal invariant field theory and look for a class of solutions in which the "vacuum" state is only invariant under a subgroup of the full conformal group. In particular, the "vacuum" should not be invariant under dilatation.

The standard procedure for investigating spontaneously symmetry breaking is to study the "vacuum" expectation of a scalar field ϕ

$$\langle 0 | \phi(x) | 0 \rangle = B(x) \quad (1.2)$$

In the framework of a semiclassical approximation, the function $B(x)$ must be a solution of the classical Lagrange equation

$$\frac{\partial}{\partial x_\mu} \frac{\delta \mathcal{L}}{\delta \left(\frac{\partial \phi}{\partial x_\mu} \right)} - \frac{\delta \mathcal{L}}{\delta \phi} = 0 \quad (1.3)$$

The specific x dependence of the function $B(x)$ tells us which part of the full conformal group is spontaneously broken. For example, the Poincaré invariance obviously requires $B(x)$ to be a constant independent of x .

In order to have a first orientation in this new programme, we have chosen, in this paper, to use a Lagrangian containing only one (or more) scalar field(s).

To be sure about the generality of the result, we allow the freedom of working in a space time with any number $D=s+1$ of space time dimensions. In that case the conformal invariant Lagrangian is :

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - g \phi^{2D/D-2} \quad (1.4)$$

Before proceeding further, let us notice that, in the framework of a full conventional perturbation approach, one might argue that the Lagrangian (1.4) is not conformal invariant, since the coupling constant g contains a hidden dependence on an ultra-violet cut-off parameter. At this stage we shall not worry about this problem, since - within our semiclassical approximation - loop effects will not be present. This touchy question is thus deferred to a further, more exhaustive, study.

Let us go back to the main object of this investigation. The study of the classical Eq. (1.3), following from the Lagrangian (1.4), in search of the solution invariant under the largest possible conformal subgroup, leads to the following results :

- a) for non-zero values of g no constant $B(x)$ solution exists.
- b) it is in general possible to find a Lorentz invariant $B(x)$ which is also invariant under the transformation generated by

$$R_\mu = \frac{1}{2} \left(a P_\mu + \frac{1}{a} K_\mu \right) \quad (1.5)$$

where the constant a has the dimension of a length.

We see that the requirement of a fully conformal invariant Lagrangian, together with a mechanism of spontaneous symmetry breaking leads us to an unconventional and somewhat uncomfortable situation. We see that (in our usual $D=4$ space) the "vacuum state" is invariant under a 10-parameter subgroup, but that this is not the Poincaré group, but the de Sitter group formed by $L_{\mu\nu}$ and R_μ .

We can illustrate this situation in the following way. In the conventional symmetry mechanism, we choose what subgroup of the full original symmetry should survive. In our case, anybody reasonable would keep the Poincaré group alive.

When we resort to the mechanism of spontaneously broken symmetry, it is the Lagrangian which decides through Eq. (1.3) in which way it wants to be broken. In the case of the conformal group, it definitely decides preserving the $O(3,2)$ de Sitter $L_{\mu\nu}$, R_μ group and not the Poincaré group. The very important point that spontaneous symmetry breaking can take place along a few privileged directions has first been made by Michel and Radicati ⁵⁾. The case of spontaneously broken conformal symmetry breaking is a further example of the validity of their point of view.

Our result seems thus to be mathematically sound, and it is fully confirmed by an exact treatment of the exactly soluble case with $D=1$ ⁶⁾. On the other hand, from a physical point of view, very serious questions arise. The presence of a "vacuum state" which is not translationally invariant does raise the frightening possibility of obtaining physical amplitudes which violate the established principle of energy-momentum conservation.

An accurate physical analysis shows that this is indeed not the case. Because of invariance of the Lagrangian under the full conformal group, the classical equation (1.3) exhibits an infinite set of fully equivalent solutions, related to each other by finite translations.

This implies the existence of an infinite number of equivalent "vacuum states" which cannot be distinguished in any physical manner. The true observable amplitude will thus be obtained by means of an appropriate averaging on the full "vacuum ensemble". This will allow to recover full four-momentum conservation and at the same time to obtain the analogue of a finite temperature Green function ⁷⁾.

It is hoped that the arguments given in this paper should be sufficiently convincing to show that this new programme, although somewhat unconventional, is not necessarily unreasonable.

2. THE CONFORMAL INVARIANT LAGRANGIAN

As discussed in the Introduction, we consider a scalar field $\phi(x)$ in a space of D dimensions. We shall of course deal with a single time dimension and $s=D-1$ space dimensions. The metric is

$$\kappa^2 = x_\mu x^\mu = x_0^2 - x_1^2 - \dots - x_s^2$$

where x_D represents the time co-ordinate. The free Lagrangian density has the form

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi)$$

The "canonical dimensionality" of the field is fixed in such a way that the free action

$$A^0 = \frac{1}{2} \int (\partial_\mu \phi)(\partial^\mu \phi) d^D x$$

is dimensionless.

We thus see that the dimensionality of the field ϕ is :

$$(\text{length})^{-\frac{D-2}{2}}$$

In constructing our Lagrangian, we require that the coupling constant g should be dimensionless, independently of D . The interaction term will be proportional to $\phi^{(2D/D-2)}$. We are thus led to the total Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - g \phi^{\frac{2D}{D-2}} \quad (2.1)$$

which, in four dimensions, reduces to the celebrated ϕ^4 Lagrangian.

In that case, positive definiteness of the Hamiltonian suggests to choose positive values of g . We shall in general adopt positive values of g throughout our work.

Two particular cases deserve special attention :

$D=1$ Quantum field theory in one time and zero space dimensions is nothing else but ordinary quantum mechanics. In particular, the Lagrangian

$$\mathcal{L} = \frac{1}{2} \left(\frac{d\phi}{dt} \right)^2 - g \frac{1}{\phi^2} \quad (2.2)$$

corresponds to the celebrated $1/r^2$ potential which, both in its non-relativistic and its Bethe-Salpeter form, has led to the first example of anomalous dimensions.

Since the exact solution of the Lagrangian (2.2) is well known, the $D=1$ is a very good test of our approximate procedure. Its detailed study has been accomplished in collaboration with de Alfaro and Furlan ⁶⁾.

$D=2$ This case is particularly remarkable because the dimensionality of the field ϕ is now zero. This leads to an infinite power of ϕ in the interaction term. It is clear that if one does not want to limit oneself to the trivial case $g=0$ some limiting procedure in D should be adopted. The investigation of this amusing singular case is deferred to a further work.

Before proceeding further, let us write down explicitly the generalization of our Lagrangian to the case in which the field ϕ belongs to an internal multiplet. In the case of the well-known σ model, we have

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \left[(\partial_\mu \phi_\sigma) (\partial^\mu \phi_\sigma) + (\partial_\mu \vec{\phi}_n) (\partial^\mu \vec{\phi}_n) \right] \\ & - g \left[\phi_\sigma^2 + \vec{\phi}_n^2 \right]^{\frac{D}{D-2}} \end{aligned} \quad (2.3)$$

The Lagrangians (2.1) and (2.3) are invariant under the full group of conformal transformations which, in the case of the Lagrangian (2.1), act as follows :

$$\begin{aligned} [\phi(x), M_{\mu\nu}] &= i(x_\mu \partial_\nu - x_\nu \partial_\mu) \phi(x) \\ [\phi(x), D] &= i \left(x^\mu \partial_\mu + \frac{D-2}{2} \right) \phi(x) \\ [\phi(x), P_\mu] &= i \partial_\mu \phi(x) \\ [\phi(x), K_\mu] &= i \left[-x^2 \partial_\mu + 2x_\mu (x^\rho \partial_\rho + \frac{D-2}{2}) \right] \phi(x) \end{aligned} \quad (2.4)$$

The commutation relations among the operators are

$$\begin{aligned}
 [M_{\mu\nu}, M_{\rho\sigma}] &= -i \{ g_{\mu\rho} M_{\nu\sigma} - g_{\nu\rho} M_{\mu\sigma} + g_{\mu\sigma} M_{\rho\nu} - g_{\nu\sigma} M_{\rho\mu} \} \\
 [M_{\mu\nu}, D] &= 0 \quad [P_\mu, P_\nu] = [K_\mu, K_\nu] = 0 \\
 \left[M_{\mu\nu}, \begin{pmatrix} P_\rho \\ K_\rho \end{pmatrix} \right] &= i \left\{ g_{\nu\rho} \begin{pmatrix} P_\mu \\ K_\mu \end{pmatrix} - g_{\mu\rho} \begin{pmatrix} P_\nu \\ K_\nu \end{pmatrix} \right\} \quad (2.5) \\
 \left[D, \begin{pmatrix} P_\mu \\ K_\mu \end{pmatrix} \right] &= -i \begin{pmatrix} P_\mu \\ -K_\mu \end{pmatrix} \\
 [P_\mu, K_\nu] &= 2i (g_{\mu\nu} D - M_{\mu\nu})
 \end{aligned}$$

It is well known that the conformal algebra in D dimensions is isomorphic with an $O(D,2)$ hyperbolic rotation group.

In order to exhibit the group theoretical structure of conformal transformations, let us start by introducing the vector operator

$$R_\mu = \frac{1}{2} \left(a P_\mu + \frac{1}{a} K_\mu \right) \quad (2.6)$$

whose commutation relations with ϕ are :

$$[\phi(x), R_\mu] = \frac{i}{a} \left\{ \frac{a^2 - x^2}{2} \partial_\mu + x_\mu \left(x^2 \partial^2 + \frac{D-2}{2} \right) \right\} \phi(x) \quad (2.7)$$

The commutation relations involving R_μ and the other generators can be simply expressed by introducing a new ${}^\mu(D+1)$ axis, together with the metric :

$$g_{D+1, D+1} = +1 \quad (2.8)$$

If we define

$$R_\mu = M_{D+1, \mu} \quad (2.9)$$

the commutation relations involving R_μ and the generators of the Lorentz group can be written in the compact form

$$\begin{aligned} [M_{ab}, M_{cd}] = -L \{ & g_{ac} M_{bd} - g_{bc} M_{ad} \\ & + g_{ad} M_{cb} - g_{bd} M_{ca} \} \end{aligned} \quad (2.10)$$

(for $a, b, c, d = 1, 2, \dots, D, D+1$).

Equations (2.9) and (2.10) show that the $(M_{\mu\nu}, R)$ generators form an $O(s, 2)$ subgroup of the full conformal group. This subgroup will play a fundamental role in our investigation. The remaining operators can be classified by means of the following definitions

$$\begin{aligned} S_\mu &= \frac{1}{2} (a P_\mu - \frac{1}{a} K_\mu) \\ S_{D+1} &= -D \end{aligned} \quad (2.11)$$

with the commutation relations

$$[L_{ab}, S_c] = -L (g_{ac} S_b - g_{bc} S_a) \quad (2.12a)$$

$$[S_a, S_b] = i L_{ab} \quad (2.12b)$$

Again the commutation relations (2.12a) and (2.12b) can be written in a more compact form.

We introduce a new $(D+2)^{\text{th}}$ axis together with the metric :

$$g_{D+2, D+2} = -1 \quad (2.13)$$

and we define ^{*}) :

$$S_a = M_{D+1, a} \quad (2.14)$$

Equations (2.12a) and (2.12b) can be obtained from a generalized form of Eqs. (2.10) in which we now take $(a, b, c, d = 1, 2, \dots, D, D+1, D+2)$.

Let us consider the commutation relations of the generators with $\phi(x)$. We define the rationalized field $\varphi(x)$:

$$\varphi(x) = \left(\frac{a^2 + x^2}{2a} \right)^{\frac{D-2}{2}} \phi(x) \quad (2.15)$$

The commutation relations of R_μ with φ are now

$$\begin{aligned} [\varphi(x), R_\mu] &= \frac{c}{a} \left\{ \frac{a^2 - x^2}{2} \partial_\mu + x_\mu (x^\rho \partial_\rho) \right\} \varphi(x) \\ &= \frac{c}{a} \left\{ \frac{a^2 + x^2}{2} + x^\rho (x_\mu \partial_\rho - x_\rho \partial_\mu) \right\} \varphi(x) \end{aligned} \quad (2.16)$$

Let us now introduce the well-known co-ordinates ξ_a ($a = 1, \dots, D, D+1$) :

$$\begin{aligned} \xi_\mu &= \frac{2 a x_\mu}{a^2 + x^2} \\ \xi_{D+1} &= \frac{a^2 - x^2}{a^2 + x^2} \end{aligned} \quad (2.17)$$

subject to the constraint

$$\xi_\mu \xi^\mu = \xi_{D+1}^2 + \xi_D^2 - (\xi_1^2 + \dots + \xi_5^2) = 1 \quad (2.18)$$

^{*}) In the case $D=4$ our 5 and 6 axes are interchanged as compared to those of Ref. 3).

In this new set of co-ordinates, the commutation relations involving the field φ gain greatly in simplicity :

$$[\varphi(\xi), L_{ab}] = \iota (\xi_a \partial_b - \xi_b \partial_a) \varphi(\xi) \quad (2.19)$$

and

$$[\varphi(\xi), S_a] = \iota \left\{ \xi^c (\xi_c \partial_a - \xi_a \partial_c) - \frac{D-2}{2} \xi_a \right\} \varphi(\xi) \quad (2.20)$$

Let us finally introduce an important identity which will play a fundamental role in future developments. If we define

$$l_{ab} = -\iota (\xi_a \partial_b - \xi_b \partial_a) \quad (2.21)$$

it is easy to prove that

$$-\left(\frac{a^2 + \lambda^2}{2a}\right)^2 \square = \left(\frac{a^2 + \lambda^2}{2a}\right)^{-\frac{D-2}{2}} \left\{ l^2 + \frac{D(D-2)}{4} \right\} \left(\frac{a^2 + \lambda^2}{2a}\right)^{\frac{D-2}{2}} \quad (2.22)$$

where

$$l^2 = \frac{1}{2} \sum_{a,b} l_{ab} l^{ab} \quad (2.23)$$

and \square is the d'Alembert operator in D dimensions.

Using Eq. (2.22), we see that the free field equation :

$$\square \phi = 0 \quad (2.24)$$

transforms into :

$$\left[l^2 + \frac{D(D-2)}{4} \right] \varphi = 0 \quad (2.25)$$

which exhibits explicitly the invariance of the d'Alembert equation under the $(L_{\mu\nu}, R_\mu)$ group ^{*)}.

If one writes Eq. (2.25) in the standard form :

$$[\ell^2 - \lambda(\lambda + D - 1)] \psi = 0 \quad (2.26)$$

where λ is a sort of $D+1$ dimensional spin, one sees that (2.25) is equivalent to the choice

$$\lambda = - \frac{D-2}{2} \quad (2.27)$$

In the next sections we shall find, on the basis of our approach to spontaneous breaking of conformal invariance, a new approximation scheme. In zero order approximation our "free equations" for the relevant fields will still be of the form (2.25), but with drastically changed values of λ .

*) Equations from (2.21) to (2.25) are contained (for $D=4$) in Ref. 4). For the sake of comparison, we notice that ℓ^2 as defined here is (-2) times the ℓ^2 of Ref. 4).

3. THE FUNDAMENTAL SHIFT

Let us now fix our attention on the conformal invariant Lagrangian :

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - g \phi^{\frac{2D}{D-2}} \quad (3.1)$$

As discussed in the Introduction, we are looking for a solution of the field theoretical problem in which the "vacuum" expectation value of the field $\phi(x)$ is non-vanishing

$$\langle 0 | \phi(x) | 0 \rangle = B(x) \quad (3.2)$$

We are led to consider non-vanishing values of $B(x)$ for two reasons : First of all, we wish to introduce in our general treatment a fundamental constant with dimensions of a length. It is our programme to have this constant appear in the expectation value $B(x)$ in Eq. (3.2) and not in the starting Lagrangian \mathcal{L} .

The second reason is related to the general form of the interaction term in the field theoretical Lagrangian (3.1) in which the field ϕ appears at the power $\phi^{(2D/D-2)}$. Apart from the special cases $D=4$ and $D=6$, we have to deal with non-integer powers of ϕ .

The classical Lagrangian (3.1) has thus in general a singular point at $\phi=0$. This means that perturbation theory, based on a small ϕ expansion does in general not work. If we thus look for a mathematical treatment, valid for any value of D , we resort to a different kind of expansion.

In this respect, an expansion for small values of the new field ϕ' defined by

$$\phi(x) = B(x) + \phi'(x) \quad (3.3a)$$

$$\langle 0 | \phi'(x) | 0 \rangle = 0 \quad (3.3b)$$

seems very advisable. It is well known that the value of $B(x)$ is constrained by the condition that the configuration $\phi'(x) \rightarrow 0$ is a configuration of dynamical equilibrium.

In the tree approximation one expresses the action

$$A = \int L d^p x$$

in terms of ϕ and one then requires that

$$\left[\frac{\delta A}{\delta \phi'(x)} \right]_{\phi' \rightarrow 0} = 0 \quad (3.4)$$

which of course leads to the classical Lagrange equation for the "classical field" $B(x)$

$$\frac{\partial}{\partial x_\mu} \frac{\delta L}{\delta \left(\frac{\partial B}{\partial x_\mu} \right)} - \frac{\delta L}{\delta B} = 0 \quad (3.5)$$

For the Lagrangian (3.1), Eq. (3.5) will take the form :

$$\square B(x) + 2g \frac{D}{D-2} B^{\frac{D+2}{D-2}}(x) = 0 \quad (3.6)$$

In trying to solve Eq. (3.6), we shall be guided by invariance considerations. The classical equation of motion (3.6) is invariant under the full conformal group. However, it is easy to see that it does not have any solution invariant under the full conformal group. Indeed, if we consider the constraints following from :

$$\langle 0 | [G_i, \phi(x)] | 0 \rangle = 0 \quad (3.7)$$

where G_i are all conformal generators, it is easy to see that Eqs. (3.7) do not admit any non-trivial solution.

In this situation we shall follow the very reasonable requirement that the $|0\rangle$ state should be invariant under the largest possible subgroup of the full conformal group. It would be even more reasonable, at least from a conventional point of view, that such an invariance subgroup would coincide with the Poincaré group formed by $M_{\mu\nu}$ and P_μ . Translation invariance of the $|0\rangle$ state implies :

$$\langle 0 | [\phi(x), P_\mu] | 0 \rangle = i \frac{\partial B}{\partial x_\mu} = 0 \quad (3.8)$$

from which one gets :

$$B(x) = \text{const.} \quad (3.9)$$

Now, apart from the trivial case $g=0$, Eq. (3.6) does not admit any constant solution.

So, in order not to give up our entire programme, we turn to $O(2,s)$ subgroup formed by $M_{\mu\nu}$ and R_μ . We recall that the operators R_μ [recall Eq. (2.6)] do indeed introduce a fundamental length.

It is easy to see that the invariance property of the $|0\rangle$ state under the $(L_{\mu\nu}, R_\mu)$ group fixes completely (apart from a multiplicative constant) the form of the function $B(x_\mu)$. First of all, Lorentz invariance tells us that $B(x_\mu)$ depends only on x_μ^2 . The identity

$$\langle 0 | [\phi(x), R_\mu] | 0 \rangle = 0 \quad (3.10)$$

leads to the following equation for $B(x^2)$

$$\frac{a^2 + x^2}{2} \frac{\partial B(x^2)}{\partial x_\mu} + \frac{1}{2} (D-2) x_\mu B(x^2) = 0 \quad (3.11)$$

The general Lorentz invariant solution of Eq. (3.11) can be written in the form :

$$B(x^2) = b \left(\frac{a^2 + x^2}{2a} \right)^{-\frac{(D-2)}{2}} \quad (3.12)$$

We come now to the result which plays the fundamental role in this investigation. The function $B(x^2)$ as given in Eq. (3.12) is a solution of Eq. (3.6). The dimensionless constant b will be uniquely determined in terms of the coupling constant g . In order to verify this important property we insert Eq. (3.12) into Eq. (3.6) and use the identity

$$\square \left(\frac{a^2 + x^2}{2a} \right)^{-\left(\frac{D-2}{2}\right)} = - \frac{(D-2)D}{4} \left(\frac{a^2 + x^2}{2a} \right)^{-\left(\frac{D+2}{2}\right)} \quad (3.13)$$

We indeed see that Eq. (3.6) reduces to the following equation for b

$$b = \frac{8g}{(D-2)^2} b^{\left(\frac{D+2}{D-2}\right)} \quad (3.14)$$

whose non-vanishing solution is

$$b = \left\{ \frac{8g}{(D-2)^2} \right\}^{-\left(\frac{D-2}{4}\right)} \quad (3.15)$$

Recalling Eq. (3.12), we thus find for $B(x)$

$$B(x) = \left\{ \frac{8g}{(D-2)^2} \left(\frac{a^2 + x^2}{2a} \right) \right\}^{-\left(\frac{D-2}{2}\right)} \quad (3.16)$$

In conclusion, we see that the classical equation (3.6) has the solution (3.16) corresponding to a "vacuum" invariant under the $O(s,2)$ group generated by $M_{\mu\nu}$ and R_μ , whereas it does not admit any non-vanishing Poincaré invariant solution.

The function $B(x)$ is singular on the surface

$$x^2 + a^2 = 0$$

which is invariant with respect to the transformations of the $O(s,2)$ subgroup. Since the expression for $B(x)$ as given by Eq. (3.6) is a solution of Eq. (3.6) for any value of a , we can avoid that the singularity takes place for real values of x_μ by the familiar trick of adding a small imaginary part to a .

Let us now recall that the classical equation (3.6) is not only invariant under the $(L_{\mu\nu}, R_\mu)$ subgroup, but under the full conformal group. This means that if one applied to $B(x)$, as given by Eq. (3.16), any finite conformal transformation, one still gets a solution.

Since the "broken" part of the conformal group contains the five generators S_μ , we expect a "general" solution depending on five free parameters. This is simply obtained by applying a dilatation (which amounts to changing the choice of the free parameter a) and a translation

$$x_\mu \rightarrow x_\mu + u_\mu$$

This leads to the generalized solution

$$B(x) = \left\{ \frac{\alpha^+(x+u)^2 + \alpha^-}{2} \right\}^{-\left(\frac{D-2}{2}\right)} \quad (3.17)$$

where α^+ and α^- are subject to the constraint :

$$\alpha^+ \alpha^- = \frac{8g}{(D-2)^2} \quad (3.18)$$

The general form (3.17) for $B(x)$ can be written in an explicitly $O(D,2)$ invariant form by introducing the well-known $D+2$ dimensional null vector

$$y_\mu = x_\mu, \quad y_{D+1} = \frac{1-x^2}{2}, \quad y_{D+2} = \frac{1+x^2}{2} \quad (3.19)$$

$$y_\nu y^\nu = -y_{D+2}^2 + y_{D+1}^2 + y_\mu y^\mu = 0 \quad (3.20)$$

Introducing the $D+2$ dimensional vector h_ν :

$$\begin{aligned} h_\mu &= \alpha^+ u_\mu \\ h_{D+1} &= -\frac{1}{2} (\alpha^+ - \alpha^- - \alpha^+ u^2) \\ h_{D+2} &= -\frac{1}{2} (\alpha^+ + \alpha^- + \alpha^+ u^2) \end{aligned} \quad (3.21)$$

subject to the constraint:

$$h_\mu h^\mu = - \frac{8g}{(D-2)^2} \quad (3.22)$$

Equation (3.17) takes the beautiful form ^{*)}

$$B(y) = \left\{ h_\mu y^\mu \right\}^{-\left(\frac{D-2}{2}\right)} \quad (3.23)$$

Equation (3.23), together with the condition (3.22), exhibits the $O(D,2)$ invariance of Eq. (3.6) and its spontaneous breaking. Indeed, the choice of the $D+2$ dimensional vector h_μ indicates the direction of this breaking. The zero state is invariant under the "little group" which leaves the vector h_μ invariant.

The fundamental constraint (3.22) indicates the structure of such a subgroup. Indeed, for the physical interesting case of positive coupling constant g , the modulus of h_μ is negative, leading to a $O(s,2)$ little group. If one wishes Lorentz invariance, one is led to our fundamental $(L_{\mu\nu}, R_\mu)$ subgroup. On the other hand, negative values of g give rise to a $O(D,1)$ little group. A Poincaré little group is only possible in the limiting case $g=0$.

The previous discussion can be summarized in the following Table :

Values of g	$g > 0$	$g = 0$	$g < 0$
Invariance subgroup of the "vacuum" state	$O(s,2)$ $M_{\mu\nu}, R_\mu$	Poincaré $M_{\mu\nu}, P_\mu$	$O(D,1)$ $M_{\mu\nu}, S_\mu$

From now on we shall consistently take $g > 0$ and therefore the invariance subgroup of the "vacuum" will be the $O(s,2)$ subgroup formed by $L_{\mu\nu}, R_\mu$.

*) This simple result for $B(y)$ can be simply seen in the six-dimensional formalism of Ref. 3) (for $D=4$). The field equation (3.6) can be written as

$$\square_\gamma B = 4g B^3$$

where B is subject to the homogeneity constraint $(y^i \partial_i) B = -B$. It is easy to check that Eqs. (3.22) and (3.23) satisfy the two previous conditions.

We could now proceed further and express our Lagrangian (3.1) in terms of the shifted field $\phi'(x)$ given in Eq. (3.3). However, it is clear that the use of a formalism which clearly exhibits our $O(s,2)$ symmetry will greatly improve our procedure.

For example if we introduce the "rationalized field" $\varphi(x)$ defined in Eq. (2.15), we obtain from Eqs. (3.3) and (3.12) :

$$\varphi(x) = b + \varphi'(x) \quad (3.24)$$

We thus see that the result of our treatment is a constant shift for the "rationalized field".

In the next section, we shall thus use a $O(s,2)$ covariant formalism in which $\varphi(x)$ will play the fundamental role. This will allow a neat simplification together with a deeper understanding of all our procedure.

4. THE SEMICLASSICAL APPROXIMATION

Let us proceed further in this semiclassical treatment of the conformal invariant field theory. We wish now to substitute in the Lagrangian (3.1) the "ansatz" (3.3)

$$\phi(x) = B(x) + \phi'(x) \quad (4.1)$$

and express our Lagrangian in terms of the new field $\phi'(x)$.

As pointed out in the last section, we shall be greatly helped by the use of a formalism which exhibits explicitly the $O(s,2)$ invariance of the $|0\rangle$ state.

We fix our attention on the action element :

$$dA = L d^D x \quad (4.2)$$

It is important to notice that although the action is invariant under the full conformal group, the volume element $d^D x$ is only Poincaré invariant. As a consequence, the Lagrangian density is also invariant only under the Poincaré group.

Since we wish to emphasize $O(s,2)$ and not Poincaré invariance, we shall find it convenient to introduce an $O(s,2)$ invariant volume element

$$dV = \frac{d^D x}{\left(\frac{a^2 + x^2}{2a}\right)^D} \quad (4.3)$$

If we write

$$dA = \mathcal{L} dV \quad (4.4)$$

we find that

$$\mathcal{L} = L \left(\frac{a^2 + x^2}{2a}\right)^D \quad (4.5)$$

is now $O(s,2)$ invariant.

We introduce the rationalized field ϕ defined in Eq. (2.15)

$$\varphi(x) = \left(\frac{a^2 + x^2}{2a} \right)^{\frac{D-2}{2}} \phi(x)$$

It is not hard to express \mathcal{L} in an explicitly $O(s,2)$ invariant form.

Starting from the Lagrangian :

$$L = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - g \phi^{\frac{2D}{D-2}}$$

and using Eq. (2.22), we obtain for \mathcal{L} (apart from irrelevant total derivative terms) the form :

$$\mathcal{L} = -\frac{1}{4} (\ell_{ab} \varphi) (\ell^{ab} \varphi) + \frac{D(D-2)}{g} \varphi^2 - g \varphi^{\frac{2D}{D-2}} \quad (4.6)$$

where we recall that [Eq. (2.21)]

$$\ell_{ab} = -i(\xi_a \partial_b - \xi_b \partial_a)$$

In a similar way the σ model Lagrangian becomes

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4} \left\{ (\ell_{ab} \varphi_\sigma) (\ell^{ab} \varphi_\sigma) + (\ell_{ab} \vec{\varphi}_\pi) (\ell^{ab} \vec{\varphi}_\pi) \right\} \\ & + \frac{D(D-2)}{g} \left(\varphi_\sigma^2 + \vec{\varphi}_\pi^2 \right) - g \left(\varphi_\sigma^2 + \vec{\varphi}_\pi^2 \right)^{\frac{D}{D-2}} \end{aligned} \quad (4.7)$$

We have already seen in the last section that the shift (4.1), with $B(x)$ given in Eq. (3.12), corresponds to a constant shift in the rationalized field φ_σ .

$$\varphi_\sigma = b + \varphi'_\sigma \quad (4.8)$$

Of course the shift (4.8) only affects the non-derivative terms of \mathcal{L} . Again the requirement that the linear term in φ'_σ vanishes leads to

$$g = \frac{1}{g} (D-2)^2 b^{-(4/D-2)} \quad (4.9)$$

Introducing Eq. (4.8) into Eq. (4.7), and expressing g in terms of b through Eq. (4.9), we get

$$\mathcal{L} = -\frac{1}{4} \left\{ (e_{ab} \dot{\varphi}_\sigma) (e^{ab} \dot{\varphi}_\sigma) + (e_{ab} \vec{\varphi}_\pi) (e^{ab} \vec{\varphi}_\pi) \right\} + \frac{b^2(D-2)}{8} \left\{ D(1+U) - (D-2)(1+U)^{\frac{D}{D-2}} \right\} \quad (4.10)$$

where

$$U = \frac{2 \dot{\varphi}_\sigma^2}{b} + \frac{\dot{\varphi}_\sigma'^2}{b^2} + \frac{\vec{\varphi}_\pi^2}{b^2} \quad (4.11)$$

Expanding Eq. (4.10) in powers of $1/b$ we get (neglecting a c number term in the Lagrangian) :

$$\mathcal{L} = -\frac{1}{2} \left\{ \frac{1}{2} (e_{ab} \dot{\varphi}_\sigma) (e^{ab} \dot{\varphi}_\sigma) - D \dot{\varphi}_\sigma'^2 \right\} - \frac{1}{4} (e_{ab} \vec{\varphi}_\pi) (e^{ab} \vec{\varphi}_\pi) - \frac{1}{b} \left(\frac{D}{2} \dot{\varphi}_\sigma' \vec{\varphi}_\pi^2 + \frac{D(D+2)}{6(D-2)} \dot{\varphi}_\sigma'^3 \right) + O\left(\frac{1}{b^2}\right) \quad (4.12)$$

Our semiclassical approximation is an expansion in $1/b$. The parameter b is related to the coupling constant by the relation (4.9). This shows that for the number of dimensions $D > 2$, the $1/b$ expansion is a weak coupling expansion; whereas for $D < 2$ it is a strong coupling expansion. We recall that the $D=2$ theory is a singular case which requires a separate treatment.

Let us now concentrate our attention on zero order terms in the $1/b$ expansion of the Lagrangian. We get the following equations for the sigma and pion fields :

$$e^2 \dot{\varphi}_\sigma' = D \dot{\varphi}_\sigma' \quad (4.13)$$

$$e^2 \vec{\varphi}_\pi = 0 \quad (4.14)$$

where

$$\ell^2 = \frac{1}{2} \ell_{ab} \ell^{ab} \quad (4.15)$$

And, in a more compact form, *)

$$\ell^2 \varphi = \lambda(\lambda + D - 1) \varphi \quad (4.16)$$

where

$$\begin{aligned} \lambda_\sigma &= 0 \\ \lambda_\pi &= 1 \end{aligned} \quad (4.17)$$

We see that the new independent equations for the pion and σ field correspond to simple integer values of the five-dimensional angular momentum λ . The simplicity of the result, together with the procedure used to obtain it, strongly suggests that the values (4.16) for λ_σ and λ_π have a profound meaning and represent the analogue of the Goldstone theorem in our framework. At the end of this section, we shall see that this is indeed the case.

Let us rewrite Eq. (4.16) in terms of the familiar Φ field. Using Eq. (2.22), we obtain

$$\square \Phi + \left(\lambda + \frac{D}{2} \right) \left(\lambda + \frac{D-2}{2} \right) \frac{4a^2}{(a^2 + x^2)^2} \Phi = 0 \quad (4.18)$$

from which, in particular

$$\square \phi'_\sigma + \frac{D(D+2)}{4} \frac{4a^2}{(a^2 + x^2)^2} \phi'_\sigma = 0 \quad (4.19)$$

$$\square \vec{\phi}_\pi + \frac{D(D-2)}{4} \frac{4a^2}{(a^2 + x^2)^2} \vec{\phi}_\pi = 0 \quad (4.20)$$

*) Recall that the ξ space has $D+1$ dimensions !

Our new "zero order" equations correspond to particles embedded in an external time dependent potential. We thus see that in our scheme the independent behaviour of the different particle fields does strongly differ from the free behaviour; this difference is independent of the value of the coupling constant.

The general properties of the solutions of Eqs. (4.19) and (4.20) will be discussed in detail in the next section. We wish now to show, as anticipated earlier, that Eqs. (4.19) and (4.20) have a deep group theoretical meaning and can be derived directly from the existence of those conservation laws which are spontaneously broken by our choice of the zero state.

We work in the familiar $D=4$ case and concentrate our attention on the conserved "currents" of the (π, σ) model. They are :

A) the axial current :

$$J_\mu^A = \left(\phi_\sigma \frac{\partial \phi_\pi}{\partial x_\mu} - \frac{\partial \phi_\sigma}{\partial x_\mu} \phi_\pi \right) \quad (4.21)$$

obeying the exact conservation law :

$$\frac{\partial J_\mu^A}{\partial x_\mu} = 0 \quad (4.22)$$

B) the (improved) energy momentum tensor

$$\begin{aligned} \theta_{\mu\nu} = & \partial_\mu \phi_\sigma \partial_\nu \phi_\sigma - \frac{1}{2} (\partial_\alpha \phi_\sigma) (\partial^\alpha \phi_\sigma) g_{\mu\nu} + \frac{1}{6} (g_{\mu\nu} \square - \partial_\mu \partial_\nu) \phi_\sigma^2 \\ & + g \phi_\sigma^4 g_{\mu\nu} + (\text{pion contribution}) \end{aligned} \quad (4.23)$$

which obeys the conditions

$$\theta_\mu^\mu = 0 \quad (4.24)$$

$$\frac{\partial \theta_{\mu\nu}}{\partial x_\nu} = 0 \quad (4.25)$$

Let us insert Eq. (4.1) into Eqs. (4.21) and (4.22) and compute j_{μ}^A and $\Theta_{\mu\nu}$ at the leading order in the $1/b$ expansion. In the case of the axial current, we get

$$j_{\mu}^A \cong \left(B(x) \frac{\partial \phi_{\pi}}{\partial x_{\mu}} - \frac{\partial B}{\partial x_{\mu}} \phi_{\pi} \right) \quad (4.26)$$

The current conservation relation leads to

$$B(x) \square \phi_{\pi} - \square B \phi_{\pi} = 0 \quad (4.27)$$

i.e.,

$$\square \phi_{\pi} - \frac{\square B}{B} \phi_{\pi} = \square \phi_{\pi} - \frac{8a^2}{(a^2 + \kappa^2)^2} \phi_{\pi} = 0 \quad (4.28)$$

which coincides with Eq. (4.20).

We now compute the leading contribution to $\Theta_{\mu\nu}$. Since the term of zero order in the field vanishes completely, the leading term will again be linear in ϕ'_{σ} . We get :

$$\begin{aligned} 3\Theta_{\mu\nu} = & -B \partial_{\mu} \partial_{\nu} \phi'_{\sigma} + g_{\mu\nu} B \square \phi'_{\sigma} \\ & + 2(\partial_{\mu} B \partial_{\nu} \phi'_{\sigma} + \partial_{\nu} B \partial_{\mu} \phi'_{\sigma}) - g_{\mu\nu} \partial_{\alpha} B \partial^{\alpha} \phi'_{\sigma} \\ & - \phi'_{\sigma} \partial_{\mu} \partial_{\nu} B - 2g_{\mu\nu} \phi'_{\sigma} \square B \end{aligned} \quad (4.29)$$

When we compute the trace, the complicated expression (4.29) simplifies and we obtain :

$$\Theta^{\mu}_{\mu} = B \square \phi'_{\sigma} - 3 \phi'_{\sigma} \square B \quad (4.30)$$

We thus see that the zero trace condition (4.24) leads to :

$$\square \phi'_{\sigma} - \frac{3 \square B}{B} \phi'_{\sigma} = \square \phi'_{\sigma} - \frac{24a^2}{(a^2 + \kappa^2)^2} \phi'_{\sigma} = 0 \quad (4.31)$$

which coincides with Eq. (4.19).

Similarly, it is easy to see that

$$\frac{\partial}{\partial \lambda_\nu} \mathcal{O}_{\mu\nu} = B_\mu \square \phi'_\sigma - \phi'_\sigma \square B_\mu \quad (4.32)$$

where

$$B_\mu = \frac{\partial \mathcal{B}}{\partial \lambda^\mu} \quad (4.32')$$

The divergenceless condition (4.25) does thus lead to :

$$\square \phi'_\sigma - \frac{\square B_\mu}{B_\mu} \phi'_\sigma = 0 \quad (4.33)$$

which again coincides with Eq. (4.19).

In conclusion, we see that the special values: $\lambda=0$ for the pion field and $\lambda=1$ for the σ field, follow from the same arguments which in the conventional approach force the Goldstone particles to be massless. The pion result is a consequence of spontaneously broken chiral invariance whereas the σ result follows from spontaneous breaking of conformal invariance.

5. THE ZERO ORDER FIELD EQUATIONS

Let us consider the zero order field equations derived in the last section. For sake of definiteness and simplicity, we shall work in the physical $D=4$ case and take $a=1$.

The field equations are

$$\ell^2 \varphi = \lambda(\lambda+3)\varphi \quad (5.1)$$

or, in terms of the usual field ϕ

$$\square \phi + (\lambda+1)(\lambda+2) \frac{4}{(1+x^2)^2} \phi = 0 \quad (5.2)$$

We recall that the value of the five-dimensional angular momentum takes simple integer values in the physical relevant cases :

free massless equation	$\lambda = -1$
zero order pion equation	$\lambda = 0$
zero order sigma equation	$\lambda = 1$.

We see that the net effect of the "phase transition" due to the charge of "vacuum state" is to modify λ from the original value -1 to values like $\lambda=0, 1$ which would naturally come out in the usual quantization of five-dimensional angular momentum.

Indeed $\lambda=0, 1$ allow the presence of "regular solutions" of the angular momentum problem. These solutions correspond, in our language, to modes whose "wave function" is regular both at $x^2 \rightarrow 0$ and $x^2 \rightarrow \infty$.

The "regular modes" (which are of course absent for $\lambda=-1$) are :

$$\begin{aligned} \lambda=0 \quad \varphi^0(\xi) &= 1 \\ \phi^0(x) &= \frac{2}{1+x^2} \end{aligned} \quad (5.3)$$

$$\begin{aligned} \lambda=1 \quad \varphi'_\mu(\xi) &= \xi_\mu \quad \mu=1, 2, \dots, 5 \\ \phi'_5(x) &= 2 \frac{1-x^2}{(1+x^2)^2} \end{aligned} \quad (5.4)$$

$$\phi'_i(x) = \frac{4 x_\mu}{(1+x^2)^2}$$

The presence of the regular modes is a new important feature due to the presence of the "potential term" in Eq. (5.2). This is simply seen for the pion regular mode. Indeed, the "wave function" $\phi^0(x)$ given in Eq. (5.3) has the following asymptotic behaviour

$$\frac{1}{\lambda} \phi^0(x) \begin{cases} \rightarrow 1 & \text{for } x^2 \rightarrow 0 \\ \rightarrow \frac{1}{x^2} & \text{for } x^2 \rightarrow \infty \end{cases} \quad (5.5)$$

We see that $\phi^0(x)$ tends in both limits to solutions of the free equation

$$\square \phi = 0$$

The presence of the "potential term" in Eq. (5.2) acts in such a way that $\phi^0(x)$ can "switch" from the regular solution at $x^2 \rightarrow 0$ to the regular solution at $x^2 \rightarrow \infty$.

Of course a complete set of solutions includes, in addition to the regular modes, solutions of Eq. (5.2) which are not vanishing at infinity or, possibly, which are singular at the origin.

Since Eq. (5.2) is not invariant under translations, the usual plane wave expansion of the general solution is not appropriate. On the other hand since Eq. (5.2) is Lorentz invariant, an expansion in four-dimensional spherical harmonics is very advisable.

We shall thus follow the technique developed in an earlier paper ⁸⁾, on the basis of a Euclidean formulation of field theory. In this case it will be more opportune to "rotate" the space components of our vectors by taking $x_i \rightarrow i x_i$.

It is to be said that, although the technique of Ref. 8) leads very quickly to the desired results, the final outcome, i.e., the two-point Green function, is completely independent of the particular quantization method which is being used.

Let us consider $\varphi(x) = \varphi(r, \alpha)$ in polar co-ordinates. We expand it in four-dimensional spherical harmonics

$$\varphi(r, \alpha) = \sum \varphi_{j\mu}(r) Y_{j\mu}(\alpha) \quad (5.6)$$

where the index μ labels the two four-dimensional angular quantum number n, m . The functions $Y_{j\mu}(\alpha)$ obey the completeness relation :

$$\sum_{j,\mu} Y_{j\mu}^+(\alpha) Y_{j\mu}(\alpha') = \delta^3(\alpha - \alpha') \quad (5.7)$$

The equation for $\varphi_{j\mu}(r)$ (which is of course independent of μ) is :

$$\left\{ \left(r \frac{d}{dr} \right)^2 - 2 \left(\frac{1-r^2}{1+r^2} \right) r \frac{d}{dr} - j(j+2) + \left(\frac{2r}{1+r^2} \right)^2 \lambda(\lambda+3) \right\} \varphi_j(r) = 0 \quad (5.8)$$

We choose the two solutions $f_j^\pm(r)$ of Eq. (5.8) in such a way that

$$f_j^+(r) = f_j^-\left(\frac{1}{r}\right) \quad (5.9)$$

and

$$\begin{aligned} f_j^+(r) &\rightarrow C r^{+j} & \text{for } r \rightarrow 0 \\ f_j^-(r) &\rightarrow C r^{-j} & \text{for } r \rightarrow \infty \end{aligned} \quad (5.10)$$

$f_j^+(r)$ and $f_j^-(r)$ are normalized through the Wronskian relation :

$$\left(r \frac{df^+}{dr} f^- - r \frac{df^-}{dr} f^+ \right) = \left(\frac{1+r^2}{2r} \right)^2 \quad (5.11)$$

We now introduce the quantization procedure of Ref. 8) by writing

$$\varphi_{j\mu}(r) = [a_{j\mu}^- f_j^-(r) + a_{j\mu}^+ f_j^+(r)] \quad (5.12)$$

where

$$[a_{j\mu}^-, a_{j'\mu'}^+] = \delta_{jj'} \delta_{\mu\mu'} \quad (5.13)$$

It is easy to verify that, as a consequence of our quantization procedure and of the Wronskian relation (5.11), the fundamental equal r commutator takes the correct value ^{*}):

$$\left[\varphi(r, \alpha_1), r \frac{d}{dr} \varphi(r, \alpha_2) \right] = \delta^3(\alpha_1 - \alpha_2) \left(\frac{1+r^2}{2r} \right)^2 \quad (5.14)$$

The explicit form of the functions $f_j^\pm(r)$ is :

$$f_j^\pm(r) = \frac{\Gamma(\lambda+2)}{2\sqrt{2}} \sqrt{\frac{\Gamma(j-\lambda)}{\Gamma(j+\lambda+3)}} \cdot \frac{1}{2\pi i} \oint dt \, t^{-(2+\lambda)} (1+t)^{\lambda+j+2} (r^2 - t)^{\lambda-j} \quad (5.15)$$

where the integration contour is extended to an anticlockwise circuit enclosing $t=0$. In the interesting cases $\lambda=-1, 0, +1$ we have :

$$\lambda = -1 \quad f_j^+ = \frac{1}{\sqrt{8(j+1)}} \left(\frac{1+r^2}{r} \right) r^{(j+1)}$$

$$\lambda = 0 \quad f_j^+ = \sqrt{\frac{j(j+2)}{8(j+1)}} \left\{ \frac{r^j}{j} + \frac{r^{(j+2)}}{j+2} \right\} \quad (5.16)$$

$$\lambda = 1 \quad f_j^+ = \sqrt{\frac{j(j+2)(j-1)(j+3)}{8(j+1)}} \cdot \frac{r}{1+r^2} \cdot \left\{ \frac{r^{(j-1)}}{(j-1)j} + 2 \frac{r^{(j+1)}}{j(j+2)} + \frac{r^{(j+3)}}{(j+2)(j+3)} \right\}$$

^{*}) If we define $\chi(r, \alpha) = (2r/1+r^2) \varphi(r, \alpha)$, we have $[\chi(r, \alpha_1) \chi(r, \alpha_2)] = \delta^3(\alpha_1 - \alpha_2)$, [as in Ref. 8)].

Looking at Eq. (5.15) or at the explicit expressions (5.16), we see that the functions f_j become singular for integer values of j smaller than or equal to λ . This phenomenon, which is of course absent in the "old case" $\lambda = -1$, is due to the presence, for $j \leq \lambda$, of the regular modes which exhibit boundary conditions which are quite different from the other modes.

One can, at least, give a provisory meaning to the solutions (5.15) and (5.16) by setting $j \rightarrow j+\epsilon$ (ϵ being a small parameter) in the definition of $f_j(r)$.

The asymptotic behaviour of f_j^+ for $r \rightarrow 0$ thus becomes :

$$f_j^+(r) \rightarrow r^{(j+\epsilon)} \quad \text{for } r \rightarrow 0 \quad (5.17)$$

This indicates that, in our procedure, we have attributed a dimensionality $1+\epsilon$ to our field Φ . This suggests that one should define a new "rationalized field" $\hat{\Phi}(x)$ defined as

$$\hat{\Phi}(x) = \left(\frac{1+r^2}{r} \right)^\epsilon \varphi(x) \quad (5.18)$$

which will now have with the operators L_{ab} the simple commutation relations

$$[\hat{\Phi}(x), L_{ab}] = \epsilon (\xi_a \partial_b - \xi_b \partial_a) \varphi(x) \quad (5.19)$$

($a, b = 1, 2, 3, 4, 5$).

We are now ready to construct the two-point Green function. We define the $O(3,2)$ invariant "vacuum" state by

$$a_\mu |0\rangle = 0, \quad \langle 0| a_\mu^\dagger = 0 \quad (5.20)$$

and define the Green function in terms of the zero expectation value of the R ordered product of two fields

$$g(x_1, x_2) = \langle 0| R[\hat{\Phi}(x_1), \hat{\Phi}(x_2)] |0\rangle \quad (5.21)$$

Because of the invariance properties of the vacuum, together with the commutation relations (5.19), we expect $g(x_1, x_2)$ to depend on the only $O(3,2)$ invariant which can be constructed in terms of the two vectors x_1 and x_2 :

$$Z = \xi_{1\mu} \xi_2^\mu = 1 - 2 \frac{(x_1 - x_2)^2}{(1+x_1^2)(1+x_2^2)} \quad (5.22)$$

Using the operator expansion (5.6) and (5.12) of ϕ , together with the commutation relations (5.13) and the definition (5.20) of the $|0\rangle$ state, we obtain for the Green function the general expression:

$$g(z) = -\frac{1}{8\pi^2} \left(\frac{\lambda - \varepsilon + 1}{\varepsilon} \right) \frac{1}{8\pi i} \oint \left(\frac{1-w^2}{2} \right)^{-(2+\lambda)} (w-z)^{\lambda-\varepsilon} dw \quad (5.23)$$

where the integral is extended to an anticlockwise circuit around $w=1$. In the cases of special interest, we have

$$\begin{aligned} \lambda = -1 \quad g(z) &= \frac{1}{8\pi^2} \frac{1}{1-z} \\ \lambda = 0 \quad g(z) &= \frac{1}{8\pi^2} \left\{ \frac{1}{\varepsilon} - \log(1-z) + \frac{1}{1-z} \right\} \\ \lambda = 1 \quad g(z) &= \frac{1}{8\pi^2} \left\{ \frac{3z}{\varepsilon} - 3 - 3z \log(1-z) + \frac{1}{1-z} \right\} \end{aligned} \quad (5.24)$$

It is easy to verify that the Green function $g(z)$ obeys the differential equation:

$$\left\{ \partial^2 - \lambda(\lambda+3) \right\} g(z) = -(\lambda+4) \lim_{\varepsilon \rightarrow 0} [\varepsilon g(z)] \quad (5.25)$$

which (for small values of ϵ) can be written as ^{*)} :

$$\left\{ \ell^2 - (\lambda - \epsilon)(\lambda - \epsilon + 3) \right\} g(z) = 0 \quad (5.26)$$

We see that, as expected, the Green functions have a strong deviation from the ones of free massless theory. This deviation, which is independent of the value of the coupling constant, leads to effects that - in a conventional approach - might be seen as due to strong interactions.

Looking at Eqs. (5.23) and (5.24), we see that the leading singularity at $z=1$ is the same as in the free case. This ensures that the zero expectation value of the equal time commutator of ϕ and $d\phi/dt$ will still retain the free value. On the other hand, it follows that the ultra-violet behaviour of the Green function has, at this stage, the same form as in the conventional theory.

Let us finally say a few words about the appearance of the small parameter ϵ . The presence of the effects connected with that parameter shows that the present stage of development is not completely consistent. We indeed expect that higher order effects will eliminate order by order the ϵ dependent terms. It is to be noted that a similar problem appears in the quantization of two-dimensional scalar field theory where, in our representation, the Green function would take the form

$$g(z) = \omega_{\text{int}} + \left\{ \frac{1}{\epsilon} - g(1-z) \right\}$$

The same situation is present in dual models, whose fundamental operator is a scalar field acting in a two-dimensional space time. In this case it is well known that if all constraints are taken into account correctly, all ϵ dependence disappears ⁹⁾.

^{*)} More precisely, Eq. (5.22) should be written :

$$\left\{ \ell^2 - (\lambda - \epsilon)(\lambda - \epsilon + 3) \right\} g(z) = \delta_s(\xi, -\xi_-)$$

where $\delta_s(\xi)$ is the hyperspherical δ function defined in Ref. 4).

At this point it is hard to give a clear prediction on how the solution of the ϵ problem will come about. A possible guess is that the final form of the Green functions might not differ so greatly from Eqs. (5.23) and (5.24). The main difference could be that ϵ will become a function $\epsilon(g^2)$, vanishing for $g^2 \rightarrow 0$. In other words, Eqs. (5.23) and (5.24) will become meaningful only when we shall introduce the correct anomalous dimensionality of the field $\phi(x)$.

Let us finally show that the lowest modes, responsible for the ϵ dependence of the Green function, are simply connected with the "spontaneously broken" constants of motion of our symmetry group. We separate out their contributions to the pion and sigma fields :

$$\varphi_{\pi}^{(0)} = q_{\pi} - \frac{i}{2} p_{\pi} \left[\epsilon g r + \frac{i}{4} \left(r^2 - \frac{1}{r^2} \right) \right] \quad (5.27)$$

$$\begin{aligned} \varphi_{\sigma}^{(0)} = & \frac{1-r^2}{1+r^2} \left\{ q_{\sigma,0} + i p_{\sigma,0} \frac{3}{2} \left[\epsilon g r - \frac{i}{12} \left(r^2 + \frac{1}{r^2} - 1 \right) \right] \right\} \\ & + \frac{2xi}{1+r^2} \left\{ q_{\sigma,i} - i p_{\sigma,i} \frac{3}{16} \left[\epsilon g r + \frac{r^2}{3} + \frac{r^4}{24} - \frac{r^{-2}}{3} - \frac{r^{-4}}{24} \right] \right\} \end{aligned} \quad (5.28)$$

where the operators q_{α} and p_{α} are appropriate symmetric and antisymmetric combinations of the creation and destruction operators. They obey the commutation relations

$$[q_{\alpha}, p_{\beta}] = i \delta_{\alpha\beta} \quad (5.29)$$

and, as a consequence

$$[\varphi_{\pi}(x), p_{\pi}] = i \quad (5.30)$$

$$[\varphi_\sigma(x), p_{\sigma 0}] = i \frac{1-x^2}{1+x^2} \quad (5.31a)$$

$$[\varphi_\sigma(x), p_{\sigma i}] = i \frac{2x_i}{1+x^2} \quad (5.31b)$$

Recalling the expressions (4.26) and (4.29), the operators j_μ^A and $\Theta_{\mu\nu}$ in terms of φ_π and φ_σ ; and performing the necessary surface integrations, we obtain for the axial charge Q_A and for the operators S_μ the following expressions; valid at the leading order in a $1/b$ expansion

$$Q_A \cong b p_\pi \quad (5.32)$$

$$S_5 \cong -D \cong -b p_{\sigma 0} \quad (5.33a)$$

$$S_i \cong -b p_{\sigma i} \quad (5.33b)$$

where [remember Eq. (4.9)]

$$b = \frac{1}{\sqrt{2g}} \quad (5.34)$$

The validity of Eqs. (5.32) and (5.33) can be simply checked by recalling Eqs. (5.30) and (5.31) and by observing that at the leading order in the $1/b$ expansion, we have the following commutation relations

$$[\varphi_\pi(x), Q_A] = i b \quad (5.35)$$

$$[\varphi_\sigma(x), S_\mu] = -i b \xi_\mu \quad (5.36)$$

Equation (5.36) follows simply from Eq. (2.20) [ξ_μ are defined in Eq. (2.17)].

The simple result expressed by Eqs. (5.32) and (5.33) shows clearly the physical meaning of the operators p_α and q_α . They are indeed directly connected with the "broken" operators Q_A and S_μ ; their role is thus to

re-establish the full conformal invariance which has been spoiled by our special choice of the "shift" $B(x)$. In particular they induce the transformation between the different "vacuum" states defined in different translational frames of reference.

As pointed out earlier, it is thus clear that an exact treatment of those operators is necessary. This will be, hopefully, one of the objects of a future investigation.

6. PHYSICAL INTERPRETATION

In the last sections we have developed in some detail a scheme of semi-classical approximation in the framework of conformally invariant Lagrangian field theory.

The basic idea is to investigate the possibility of spontaneous breaking of conformal invariance. The mathematical foundation of this approach seems to be reasonable and sound. It represents the generalization to the conformal group of methods which have been applied with success to the cases of chiral symmetry and gauge invariance.

On the other hand, the main feature of this approach, that of having an expectation value

$$\langle 0 | \phi(x) | 0 \rangle = B(x) \quad (6.1)$$

where $B(x)$ is not a constant is naturally raising some troublesome questions about the physical soundness of the whole approach.

Indeed from Eq. (6.1) it follows that the "vacuum" state $|0\rangle$ is not translation invariant. This circumstance can suggest that although the Lagrangian is perfectly translation invariant, the development of this theoretical approach might lead to physical amplitudes in which momentum conservation is absent. This would of course suggest to rule out the whole point of view advocated in this paper. It thus is necessary to discuss in detail the physical interpretation of the whole approach.

The customary definition of "observable quantities" in quantum field theory proceeds as follows. One starts by introducing a set of physical operators $j(x)$ representing the interaction of the quantized system with all possible external sources.

All observable quantities can then be expressed in terms of the zero expectation values

$$f(x_1, \dots, x_n) = \langle 0 | P j(x_1) \dots j(x_n) | 0 \rangle \quad (6.2)$$

or of their Fourier transforms

$$F(k_1 \dots k_n) = \int f(x_1 \dots x_n) e^{+i(k_1 x_1 + \dots + k_n x_n)} d^4x_1 \dots d^4x_n \quad (6.3)$$

Strongly interacting particles intervene as poles of the amplitude (6.3) in the different invariant kinematical variable. Their scattering amplitudes can be obtained through a suitable limiting procedure whose effect is to extract suitable residues from the one-particle poles.

In our case we can, tentatively, start in the same way: introduce the physical operators $j(x)$ and concentrate our attention on the $|0\rangle$ matrix elements (6.2) and (6.3). Now it is clear that, if we naïvely consider $f(x_i)$ or $F(k_i)$ as directly observable quantities, we run into serious trouble. Indeed since $|0\rangle$ is not translation invariant, $f(x_1, \dots, x_n)$ will not be invariant under the transformation $x_i \rightarrow x_i + h$ and, as a consequence, $F(k_1, \dots, k_n)$ will be different from zero also when the sum of all external momenta $\sum k_i$ is non-vanishing.

We wish, however, to point out that in our context, the mere definition (6.2) and (6.3) of observable quantities is unacceptable. This is due to the full translation invariance of the field theoretical Lagrangian which implies that, if $B(x)$ is a solution of the classical field theoretical equation, also $B(x+h)$ is an equally acceptable solution.

Of course, $B(x+h)$ can be written as :

$$\langle h | \phi(x) | h \rangle = B(x+h) \quad (6.4)$$

where the state

$$|h\rangle = e^{i(P_\mu h_\mu)} |0\rangle \quad (6.5)$$

is obtained by applying a translation operator on the original state $|0\rangle$.

Now it is clear that any state $|h\rangle$ corresponding to any value of the displacement vector h_μ is as good a candidate for a vacuum state as the original $|0\rangle$ state. This means that the amplitude

$$\langle h | P \delta(x_1) \dots \delta(x_n) | h \rangle = f(x_1+h \dots x_n+h) \quad (6.6)$$

$$\begin{aligned} \int \langle h | P \delta(x_1) \dots \delta(x_n) | h \rangle e^{i(k_1 x_1 + \dots + k_n x_n)} d^4 x_1 \dots d^4 x_n &= \\ = F(k_1 \dots k_n) e^{i h(k_1 + \dots + k_n)} &\quad (6.7) \end{aligned}$$

could be considered as good as a candidate for observable amplitudes as those defined in Eqs. (6.2) and (6.3).

On the other hand, if we believe that in nature there is no privileged translational frame of reference, no experiment will ever be able to distinguish the states $|h\rangle$ from each other. We thus see that what can be observed is a statistical average on all states $|h\rangle$. The observable amplitude will be

$$\begin{aligned} t(x_1 \dots x_n) &= \frac{1}{V} \int_V \langle h | P \delta(x_1) \dots \delta(x_n) | h \rangle d^4 h = \\ &= \frac{1}{V} \int_V f(x_1+h \dots x_n+h) d^4 h \end{aligned} \quad (6.8)$$

where the integration is extended to a very large volume V .

The fundamental averaging procedure on all possible vacuum states leads to apphysical amplitude $t(x_1, \dots, x_n)$ which is translational invariant. At the same time, in momentum space, the physical amplitude $T(k_1, \dots, k_n)$ has a momentum conservation delta function which selects the projection of the original amplitude $F(k_1, \dots, k_n)$ on the "momentum conservation shell".

One can give a more elegant form to Eq. (6.8) by introducing the "vacuum statistical matrix" :

$$\rho = \int |h\rangle \langle h| d^4 h \quad (6.9)$$

leading to

$$t(x_1, \dots, x_n) = \frac{\text{Trace} \{ \rho P[\delta(x_1) \dots \delta(x_n)] \}}{\text{Trace} \rho} \quad (6.10)$$

which reminds the definition of the Green function at finite temperature.

The physical meaning of the statistical matrix ρ can be illustrated by means of the following considerations. Let us consider the infinite multiplet which is obtained by applying on the original state $|0\rangle$ all possible conformal transformations. Within such a supermultiplet, one can find a complete set of eigenstates $|p_\mu\rangle$ of the momentum operators P_μ ^{*)}.

Let us analyze our $|0\rangle$ state in terms of the momentum eigenstates

$$|0\rangle = \int d^4p \, c(p) |p\rangle \quad (6.11)$$

and correspondingly

$$|h\rangle = \int d^4p \, c(p) e^{i(p \cdot h)} |p\rangle \quad (6.12)$$

Equation (6.12) shows that the modulus of the weight function $c(p)$ is independent of the choice of the translational frame of reference. Introducing Eq. (6.12) into Eq. (6.9), it is easy to find :

$$\rho = \int d^4h \, |h\rangle \langle h| = \int d^4p \, |c(p)|^2 |p\rangle \langle p| \quad (6.13)$$

*) The conformal supermultiplet will probably contain an eigenstate of P_μ corresponding to zero four-momentum. One might therefore ask why we have not followed the usual recipe and chosen that state as the "true vacuum". The reason is exhibited quite clearly in the framework of the one-dimensional model of Ref. 6). There it is shown that the lowest eigenstate of H is not normalizable, as a consequence expectation values of operators in that state are mathematically not defined. Only the eigenstates of compact operators like $R = \frac{1}{2}(aH + (1/a)K)$ are normalizable and can therefore be used to construct mathematically meaningful expectation values.

which gives for $t(x_1, \dots, x_n)$:

$$t(x_1, \dots, x_n) = \int d^4 p |C(p)|^2 \langle p | \prod j(x_i) \dots j(x_n) | p \rangle \quad (6.14)$$

Equation (6.13) expresses our "vacuum matrix" in the standard form of a statistical matrix, the matrix ρ does indeed commute with the four-momentum operator. This fact, shown in a more explicit form in Eq. (6.14), ensures that exact four-momentum conservation is indeed enforced.

Equations (6.11) and (6.13) show that our statistical matrix ρ measures the energy momentum spread of the state $|0\rangle$. The size of such a spread will of course depend on how much the operator

$$P_\mu + \frac{1}{a^2} K_\mu \quad (6.15)$$

differs from P_μ .

Such a spread will indeed be proportional to the "fundamental mass" $1/a$ and will tend to zero when $a \rightarrow \infty$. In the one-dimensional model, one finds that the ρ matrix has approximately the canonical form

$$\rho(H) \approx e^{-2aH} \quad (6.16)$$

where $1/2a$ plays the role of a "vacuum temperature".

Let us now discuss briefly some of the main new features brought forward by the introduction of a non-vanishing "vacuum temperature". It is clear that, until some actual estimates of physical processes using the new rules will be made, a complete understanding of their meaning will not be achieved. However, it is possible, already at this stage, to get at least a feeling for what is going on.

Our definitions (6.10), (6.13) and (6.14) of observable quantities involve in an essential way external sources leading to an off-mass shell physical amplitude. Since the "vacuum matrix" ρ is diagonal in momentum space, the Fourier transform of $t(x_1, \dots, x_n)$ will contain the standard delta function in the sum of the external momenta. At the same time, we expect that the location of the singularities in the different invariant kinematical variables will be in agreement with the general principles fixing the analytic properties of physical amplitudes.

If in particular we extract appropriate residues of the one-particle poles, we shall - as usual - obtain physical amplitudes containing incoming or outgoing physical hadrons. We shall see later that we can expect only poles related to zero mass particles.

On the other hand, rules differ in an essential manner from ordinary S matrix theory since the unitarity relations [which are present in the original amplitudes (6.2) and (6.3)] are expected to be spoiled by our averaging procedure. Of course, the fundamental trace ⁶⁾⁻¹⁰⁾ will satisfy general properties following from completeness, but these relations will be rather different from the usual equations (valid in the limit of a "zero vacuum temperature").

The origin of this situation is quite clear and is related to our choice of a fully massless Lagrangian. It is well known that in quantum electrodynamics, the presence of a massless photon creates a situation which is different from that of a standard scattering theory. First of all, the presence of forces of infinite range invalidates the asymptotic conditions which are needed for the definition of an S matrix. If, in the framework of some limiting procedure, one adopts the amplitudes given by Feynman rules, one is faced by the presence of infra-red divergences which do not allow to give a precise meaning to the scattering amplitude. It is well known that we obtain finite results only if we consider the observable inclusive amplitudes which take explicitly into account the production and absorption of any number of soft photons ^{*}).

In the case of electrodynamics, the problem of soft photon emission can be conveniently handled in the framework of a perturbation approach by means of only minor modifications to the standard rules. It has, anyway, been pointed out by T.D. Lee and Nauenberg ¹⁰⁾ that an appropriate ensemble averaging is required in a correct treatment of the infra-red problem.

^{*}) A possible philosophy is to concentrate our attention on inclusive reactions. The inclusive amplitude is represented by $F(k_1, \dots, k_r, -k_1, \dots, -k_r)$ where the function F is defined in Eq. (6.3). According to Eq. (6.7) the inclusive amplitude is independent of any displacement h_μ . So one more restrictive version of our recipe is to stick to the original definition (6.3), but to state that only inclusive amplitudes are observable.

In our case, within a very non-perturbative situation, production and absorption of non-observable soft particles becomes a fundamental phenomenon which is accounted by the presence of a non-zero vacuum temperature.

Let us finally make a few guesses on the possible singularities of our physical amplitude in the mass variables. On the basis of the form of the original massless Lagrangian, we can argue that the only poles are those corresponding to the presence of zero mass particles. This is indeed confirmed by the fact that the Green functions derived in Section 5 tend to the free form when the distance between particles becomes very large.

An important question is the form of the continuum due to many particle contributions. On the basis of our limited experience in the one-dimensional case, it is plausible to argue that such a continuum will present several bumps which could be interpreted as corresponding to possible resonance states. These bumps are due to the fact that the fundamental R operator has a discrete spectrum of eigenvalues. An analysis of these discrete states in terms of energy eigenfunctions will give rise to "resonant states" whose width will be measured in terms of the "fundamental mass" $1/a$. Much work is still required before we can have a clear picture of all the main physical implications of the present scheme.

7. CONCLUSIONS

The new point of view outlined in this paper consists essentially in obtaining a fundamental scale by means of an "abnormal" solution of a conformal invariant field theory. In this preliminary investigation we have been mainly concerned in testing the theoretical feasibility of this new programme and in studying its most general features.

It is clear that the scalar model which has been used is very far from giving even a rough approximation to what we could consider a satisfactory picture of the real world. It has been seen, however, that the general treatment can be simply extended to chiral multiplets of spinless particles. Inclusion of spin $\frac{1}{2}$ particles in the framework of a nucleon σ model is possible and has led to particularly beautiful results in the case of a supersymmetric Lagrangian ¹¹⁾.

The methods which have been used in this study of spontaneous breakdown of conformal invariance are very similar to those employed with success in the treatment of chiral theories. The new feature is that the zero expectation value of the field operator is not a constant; but a function of the coordinates x_μ , whose form is dictated by the "vacuum" invariance subgroup $(L_{\mu\nu}, R_\mu)$.

The question of the physical meaning of a non-translation invariant "vacuum" state has been discussed in detail in the last section. An interpretation in terms of an ensemble of statistically equivalent "vacuum states" has allowed to recover agreement with the fundamental requirement of energy momentum conservation. The key feature which emerges is that energy-momentum conservation is valid within a thermodynamical framework. The non-zero "vacuum temperature" accounts for phenomena related with emission and absorption of soft massless particles.

Looking at some of the features of the present approach, one is led to consider an interesting analogy with the independent particle model, which has been used with success both in atomic and in nuclear physics and whose development can be outlined as follows. First of all, one evaluates self-consistently a classical potential; one then computes the independent wave functions of the particles in that "external" potential; finally one takes into account the residual interaction among the particles. In this paper a similar programme has been carried out, in the framework of relativistic particle theory, up to the first two stages of development.

A very important point to be emphasized is that the size of the "potential term" in our dedependent particle equations is (at least in first approximation) independent of the value of the coupling constant. This means that we shall have strong deviations from free particle behaviour even for very weak coupling. The presence of an "abnormal vacuum" state generates effects which in a normal situation would be attributed to a strong interaction.

The fact that "weak forces" create strong effects is present in all spontaneous breaking mechanisms and in particular in unified gauge theories of weak electromagnetic interactions. In that case, in the presence of interactions of the electromagnetic size, the W intermediate boson acquires, through the Higgs mechanism, a large mass - whereas the photon remains massless. This might induce us to dream about a general theory of hadrons and leptons in which only a "weak-electromagnetic" interaction is present. The present mechanism could generate "strong interactions" for some particles (the hadrons), whereas some others (the leptons) will still behave as quasi-free.

At this point, it is probably wise to leave these bold anticipations aside, at least until many of the open questions at hand will be better understood. It is indeed still difficult to predict whether the present approach will develop into a full-fledged theory of hadrons, with some chance of explaining their most important empirical properties. On the other hand, the theoretical features which have been revealed in this first investigation, suggest that further attention should be dedicated to this proposed line of thought.

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R E F E R E N C E S

- 1) See, for example, "Dual Theory", edited by M. Jacob, Physics Reports reprint book, North Holland Publ. (1974).
- 2) For a general review, see :
E.S. Abers and B.W. Lee, "Gauge Theories", Physics Reports C9, 1 (1973).
- 3) See, for example :
G. Mack and A. Salam, Annals of Phys. 53, 174 (1969);
S. Ferrara, R. Gatto and A.F. Grillo, Springer Verlag (1973).
- 4) S. Adler, Phys.Rev. D6, 3445 (1972).
- 5) L. Michel and L.A. Radicati, Ann.Inst.H.Poincaré Vol.18, 185 (1973).
- 6) V. de Alfaro, S. Fubini and G. Furlan, "Conformal Invariance in Quantum Mechanics" - to be published.
- 7) L. Kadanoff and G. Baym, "Quantum Statistical Mechanics", W.A. Benjamin, N.Y. (1962).
- 8) S. Fubini, A.J. Hansen and R. Jackiw, Phys.Rev. 7D. (1973).
We refer to this paper for all details on four-dimensional spherical harmonics.
- 9) S. Fubini and G. Veneziano, Nuovo Cimento 67A, 29 (1970).
- 10) T.D. Lee and M. Nauenberg, Phys.Rev. B133, 1549 (1964).
- 11) V. de Alfaro and G. Furlan, "Spontaneously Broken Conformal Symmetry in the Nucleon σ Model" - to be published.