A new approach to design interval observers for linear systems

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***Abstract*—Interval observers are dynamic systems that provide upper and lower bounds of the true state trajectories of systems. In this work we introduce a technique to design interval observers for linear systems affected by state and measurement disturbances, based on the Internal Positive Representations (IPRs) of systems, that exploits the order preserving property of positive systems. The method can be applied to both continuous and discrete time systems.**

***Index Terms*—Uncertain systems; Linear system observers; Positive systems.**

I. Introduction

The properties of positive linear systems have been a subject of study in system theory for long time [11], [14]. Some of these properties, for example that the trajectories of positive systems are ordered with respect to the initial conditions and the forcing input, can be exploited in the problem of state estimation.

This paper proposes a framework for the state observation problem of linear systems in presence of bounded input/output uncertainties which is based on the use of positive systems to design *interval observers*, that is, a system of observers that provides a confident region that contains the trajectory of the observed system [13], [16], [17], [18]. A key tool to apply the positive properties to general (i.e. non necessarily positive) systems is the *internally positive realization* (IPR) [4], [5], [6], [12].

Some recent works on interval observers based on positive systems is reported in [2], [3], [20], where positive observed systems are considered. The case of stable linear systems is considered in [16], where the authors propose an approach based on a time-varying change of coordinates that transforms an autonomous system into a positive one. This approach is used to design a Luenberger observer with a positive error dynamics, on which an interval observer can be built. This idea has subsequently been extended to complex intervals in [7], and, using a time invariant change of coordinates, to a class of nonlinear systems in [18], to linear time-invariant and time-varying discrete-time systems in [8],[9], and to continuous-time time-varying systems in [10].

Essentially, the idea behind these approaches is to design a stable observer by means of an appropriate gain and then to find a coordinate change such that the resulting error dynamics is positive. In this paper we propose to reverse the approach: a positive observer of the system is initially built, and the observer gain is subsequently chosen so as to have a stable error dynamics. We show that this is always possible for observable systems. The design technique is very simple and uniform, it does not require a time-varying change of coordinates and it is straightforward to extend it to the case of discrete-time systems. The resulting interval observer has size $4n$.

In Section II, we recall some notions about positive systems, interval observers and positive representations. Section III introduces interval observers based on positive representations for systems with input/output disturbances. Section IV provides an example, and Section V concludes the paper.

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II. Background and preliminaries

Let $\mathbb{R}_+$ and $\mathbb{C}_+$ ($\mathbb{R}_-$, $\mathbb{C}_-$) denote the set of nonnegative (nonpositive resp.) reals and the set of complex numbers with nonnegative (nonpositive resp.) real parts. $\Re(z)$ and $\Im(z)$ respectively denote the real and imaginary part of a complex z. $\sigma(A)$ denotes the spectrum of a square matrix A. A is a Hurwitz matrix if $\Re(\sigma(A)) \subset (\mathbb{R}_- \setminus \{0\})$ and is a Schur matrix if $\sigma(A)$ is inside the unit disk in $\mathbb{C}$. Throughout this paper the inequalities and the min and max operators on vectors and matrices must be understood component-wise. $M \geq 0$ denotes a nonnegative matrix and $|M|$ the matrix where each entry is $|m_{ij}|$.

A. Positive systems and positive representations

Consider a continuous-time linear time invariant system Σ_L

$$\Sigma_L: \quad \begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \\ x(0) &= x_0, \end{aligned} \tag{1}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$, $y(t) \in \mathbb{R}^q$, and therefore $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{q\times n}$.

Definition 1. *The system* (1) *is said to be internally positive if, for any given nonnegative initial condition $x_0 \in \mathbb{R}_+$ and input function $u(t) \geq 0$, it is such that, $\forall t \geq 0$, $x(t) \in \mathbb{R}^n_+$ and $y(t) \in \mathbb{R}^q_+$.*

The positivity of system (1) is easy to check using well known results (see [14], [11]).

Definition 2. *A matrix M is said to be Metzler if all its off-diagonal elements are nonnegative.*

Theorem 1. *([14], p. 196.) The system* (1) *is internally positive if and only if A is Metzler, $B \geq 0$ and $C \geq 0$.*

Of course, (1) is asymptotically stable if and only if A is also a Hurwitz matrix. Other stability conditions for a positive system are proved in [1]. Positive systems have the notable property that they are ordered with respect to initial conditions and forcing inputs. The following result is well known in the literature.

Theorem 2. *Consider a system Σ_L* (1) *where A is Metzler and $B \geq 0$. Let $x_1(t)$ denote the state trajectory corresponding to the initial state x_{01} and input $u_1(t)$. Similarly, let $x_2(t)$ denote the trajectory corresponding to x_{02} and $u_2(t)$. If $x_{01} \leq x_{02}$ and $u_1(t) \leq u_2(t)$, then $x_1(t) \leq x_2(t)$, $\forall t \geq 0$.*

Given a matrix (or vector) M, the symbols M^+ and M^- denote its positive and negative parts, defined as

$$M^+ = \max(M, 0), \;\; M^- = \max(-M, 0). \tag{2}$$

Then, $M^+ \geq 0$, $M^- \geq 0$ and $M = M^+ - M^-$, $|M| = M^+ + M^-$. I_n denotes the identity matrix in $\mathbb{R}^n$. The following matrices will also be used: $\Delta_n = [I_n \;\; -I_n]$, $\bar{I}_n = [I_n \;\; I_n]^T$. Note that $\Delta_n \bar{I}_n = 0$. The following definitions were originally introduced in [6], [12].

Definition 3. *A* positive representation *of a vector $x \in \mathbb{R}^n$ is a positive vector $X \in \mathbb{R}^{2n}_+$ such that $x = \Delta_n X$. The* min-positive representation *of a vector $x \in \mathbb{R}^n$ is the positive vector $\pi(x) \in \mathbb{R}^{2n}_+$ defined as $\pi(x) = [(x^+)^T (x^-)^T]^T$. The* min-positive representation *of a matrix $M \in \mathbb{R}^{m\times n}$ is the positive matrix $\widetilde{M} \in \mathbb{R}^{2m\times 2n}_+$ defined as*

$$\widetilde{M} = \begin{bmatrix} M^+ & M^- \\ M^- & M^+ \end{bmatrix}. \tag{3}$$

Note that x^+ and x^- are orthogonal, and therefore $\|x\| = \|\pi(x)\|$. Moreover $x = \Delta_n \pi(x)$ and $\Delta_n \widetilde{M} = M \Delta_n$, for any $x \in \mathbb{R}^n$ and $M \in \mathbb{R}^{m\times n}$. From these $\Delta_n \widetilde{M} \pi(x) = Mx$, $\forall x \in \mathbb{R}^n$,

$M \in \mathbb{R}^{m\times n}$. For any given $v \in \mathbb{R}^n_+$, the vector $\pi(x) + \bar{I}_n v$ is a positive representation of $x \in \mathbb{R}^n$, in that $x = \Delta_n(\pi(x) + \bar{I}_n v)$ (recall that $\Delta_n \bar{I}_n = 0$).

Given a square matrix $M \in \mathbb{R}^{n\times n}$ we define its *Metzler representation* $[M] \in \mathbb{R}^{2n\times 2n}$ as

$$[M] = \begin{bmatrix} d^M + (M-d^M)^+ & (M-d^M)^- \\ (M-d^M)^- & d^M + (M-d^M)^+ \end{bmatrix}. \quad (4)$$

where d^M denotes the matrix having the same diagonal as M and 0 elsewhere. It is easy to check that $[M]$ is Metzler and that it enjoys the same property as the positive representation, $\Delta_n[M] = M\Delta_n$, thus $\Delta_n[M]\pi(x) = Mx$.

Note that the min-positive representation of vectors does not preserve the inequalities, in that $x_1 \le x_2$ does not imply that $\pi(x_1) \le \pi(x_2)$. Let, for a given $a \in \mathbb{R}^n$,

$$\phi_a(x) = \begin{bmatrix} x + a^- \\ a^- \end{bmatrix}, \quad (5)$$

where $a^- = \max(-a, 0)$ as in (2). It is easy to prove the following.

Lemma 3. *The function $\phi_a(x)$ defined in (5) is such that $\Delta_n\phi_a(x) = x$, $\forall x \in \mathbb{R}^n$, and for any three vectors a, x, b in $\mathbb{R}^n$ such that $a \le x \le b$ it follows $0 \le \phi_a(a) \le \phi_a(x) \le \phi_a(b)$,*

$$\|\phi_a(x) - \phi_a(a)\| = \|x - a\|, \quad \|\phi_a(b) - \phi_a(x)\| = \|b - x\|. \quad (6)$$

Remark 1. *Note that if $a \le x$, then $\phi_a(x)$ is a positive representation of x, i.e. $\phi_a(x) \in \mathbb{R}^{2n}_+$ and $x = \Delta_n\phi_a(x)$. Moreover, $\phi_a(a) = \pi(a)$ (min-positive representation).*

This can be extended to matrices of the same size by defining $\tilde{\phi}_A(M)$ as

$$\tilde{\phi}_A(M) = \begin{bmatrix} M + A^- & A^- \\ A^- & M + A^- \end{bmatrix}, \quad (7)$$

If $A \le M$, $\tilde{\phi}_A(M)$ is nonnegative. Moreover, if $\Delta_n X = x$ then $\Delta_n\tilde{\phi}_A(M)X = Mx$.

Lemma 4. *$\tilde{\phi}_A(M)$ defined in (7) is such that for any three matrices A_1, M, A_2 in $\mathbb{R}^{n\times m}$ such that $A_1 \le M \le A_2$ it follows $0 \le \tilde{\phi}_{A_1}(A_1) \le \tilde{\phi}_{A_1}(M) \le \tilde{\phi}_{A_1}(A_2)$.*

Vectors inequality on positive representations may be preserved through coordinate changes. Given a nonsingular matrix $T \in \mathbb{R}^{n\times n}$ and a vector a in $\mathbb{R}^n$, and defining $U = T^{-1}$, let us define the function $\psi_{T,a} : \mathbb{R}^n \to \mathbb{R}^{2n}$:

$$\psi_{T,a}(x) = \widetilde{U}\phi_a(x) = \begin{bmatrix} U^+ & U^- \\ U^- & U^+ \end{bmatrix} \begin{bmatrix} x + a^- \\ a^- \end{bmatrix} \quad (8)$$

Lemma 5. *The function $\psi_{T,a}(x)$ defined in (8) is such that*

$$\Delta_n\widetilde{T}\,\psi_{T,a}(x) = x, \quad \forall x \in \mathbb{R}^n, \quad (9)$$

and, for any three vectors a, x, b in $\mathbb{R}^n$ such that $a \le x \le b$ we have $0 \le \psi_{T,a}(a) \le \psi_{T,a}(x) \le \psi_{T,a}(b)$.

Internally Positive Representations (IPRs) of systems have been defined and investigated in [6], [12] for discrete-time systems, and in [4], [5] for continuous-time systems. An IPR of system Σ_L (1) is an internally positive system, endowed with four transformations T^f_X, T^b_X, T_U and T_Y, that provide the same state and output trajectories as the original, non necessarily positive, system. Denoting $(x(t), y(t))$ the state and output trajectories of system (1), an IPR of (1) is such that when its positive initial state is computed as $X(0) = T^f_X(x(0))$, and its positive input as $U(t) = T_U(u(t))$, then the state and output trajectories $(X(t), Y(t))$ of the IPR are positive and such that $x(t) = T^b_X\big(X(t)\big)$ and $y(t) = T_Y\big(Y(t)\big)$, $t \ge 0$ (see [4] for more details).

In general, any linear system admits infinite IPRs (see [4]). In this paper we use the V-IPR defined in Theorem 4 of [4]:

$$\mathcal{I}: \quad \begin{aligned} \dot{X}(t) &= [A]\,X(t) + \widetilde{B}U(t) \\ Y(t) &= \widetilde{C}X(t) \\ T^f_X &= \pi(x),\ T_U = \pi(u),\ T^b_X = \Delta_n X,\ T_Y = \Delta_q Y, \end{aligned} \quad (10)$$

where $X(t) \in \mathbb{R}^{2n}_+$, $U(t) \in \mathbb{R}^{2p}_+$, $Y(t) \in \mathbb{R}^{2q}_+$. $[A] \in \mathbb{R}^{2n\times 2n}$ is the Metzler representation of A defined by (4), $\widetilde{B}$ and $\widetilde{C}$ are the positive representations of B and C defined by (3).

Theorem 6. [4] *System $\mathcal{I}$ in (10) is an IPR of Σ_L in (1).*

Due to the larger state space, $\mathcal{I}$ has more natural modes than Σ_L and the stability of Σ_L does not imply the stability of $\mathcal{I}$. In other words, when A is Hurwitz, $[A]$ is not necessarily Hurwitz. We summarize the results that are useful in the stability analysis of the IPR $\mathcal{I}$ (see [5]).

Lemma 7. [4] *The spectrum of the Metzler representation of a matrix A is $\sigma([A]) = \sigma(A) \cup \sigma(d^A + |A - d^A|)$.*

The next Lemma links the stability of $\mathcal{I}$ to the position of the eigenvalues of A via a coordinate change that transforms A in the Real Jordan Form, the usual Jordan block form where the complex eigenvalues are represented through blocks with real entries (see [12] for details).

Lemma 8. [4] *Let $\mathcal{S} \subset \mathbb{C}_-$ be defined as follows*

$$\mathcal{S} = \big\{z \in \mathbb{C} :\ \Re(z) + |\Im(z)| < 0\big\}. \quad (11)$$

Then, if a square matrix A is in the Real Jordan Form, its Metzler representation $[A]$ is Hurwitz if and only if $\sigma(A) \subset \mathcal{S}$.

Since there is always a coordinate change for Σ_L that transforms A in the Real Jordan Form, the main limitation for the stability of $\mathcal{I}$ is the position of its eigenvalues. Notice that $\mathbb{R}_- \subset \mathcal{S}$, and therefore Jordan matrices with only negative real eigenvalues always satisfy the stability condition of Lemma 8. If A is not in Real Jordan Form, the condition of Lemma 8 is neither necessary nor sufficient for $[A]$ being Hurwitz.

B. Interval observers

Informally, interval observers are dynamical systems that provide an upper and lower bound for the state trajectory of the observed system under disturbance inputs and/or parametric uncertainties starting from a suitable initial condition.

Definition 4. *Consider system Σ_ν*

$$\Sigma_\nu: \quad \begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + \nu^s(t) \\ y(t) &= Cx(t) + \nu^m(t) \\ x(0) &= x_0, \end{aligned} \quad (12)$$

with $x(t), \nu^s(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$, $y(t), \nu^m(t) \in \mathbb{R}^q$, both $u(t)$ and $\nu^s(t)$ piecewise continuous to guarantee the existence of the solution $x(t)$ in $[0, \infty)$. Assume that there are known bounds $\bar{\nu}^s(t) = (\nu^s_1(t), \nu^s_2(t)) \in \mathbb{R}^{2n}$ and $\bar{\nu}^m(t) = (\nu^m_1(t), \nu^m_2(t)) \in \mathbb{R}^{2q}$ on the unknown disturbances such that, $\forall t \ge 0$, $\nu^s(t) \in [\nu^s_1(t), \nu^s_2(t)]$, $\nu^m(t) \in [\nu^m_1(t), \nu^m_2(t)]$. Moreover, two bounds on the initial condition $x_{01} \le x_0 \le x_{02}$ are known. Then, a dynamical system

$$\dot{Z}(t) = F(Z(t), u(t), y(t), \bar{\nu}^s(t), \bar{\nu}^m(t)), \quad (13)$$

with $Z(0) = G(x_{01}, x_{02})$, and outputs

$$\underline{x}(t) = H_1\big(t, Z(t)\big), \qquad \overline{x}(t) = H_2\big(t, Z(t)\big), \quad (14)$$

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is called an interval observer *for* Σ_ν *if*

1) *the system* (13) *is Input-State Stable (ISS, see [19]);*
2) *for any* $x(0)$ *such that* $x_{01} \le x(0) \le x_{02}$*, the solutions* $x(t)$ *and* $Z(t)$ *of* (12), (13) *satisfy the inequalities* $\underline{x}(t) \le x(t) \le \overline{x}(t)$ *for all* $t \ge 0$*;*
3) *if* $\|\nu_2^s(t)-\nu_1^s(t)\|$ *and* $\|\nu_2^m(t)-\nu_1^m(t)\|$ *are uniformly bounded then also* $\|\overline{x}(t)-\underline{x}(t)\|$ *is uniformly bounded, and if* $\nu_1^s(t) = \nu_2^s(t)$, $\nu_1^m(t) = \nu_2^m(t)$ *for all* $t \ge 0$ *then* $\|\overline{x}(t)-\underline{x}(t)\| \to 0$.

Similar definitions have been proposed elsewhere (see for example [16]). Here we require that the interval observer behaves like an ordinary observer when the disturbance is known or absent, and that if the disturbance is bounded the width of the estimation interval is bounded, even in the case $\|x(t)\| \to \infty$.

III. IPR-BASED INTERVAL OBSERVERS

Before describing the proposed interval observers, we present a *positive* Luenberger observer for system Σ_ν.

A. An IPR-based positive observer

Consider the Luenberger observer of system Σ_ν in (12)

$$\Sigma_O : \quad \begin{aligned} \dot{\hat{x}}(t) =& (A-KC)\,\hat{x}(t) + Bu(t) + Ky(t) \\ \hat{x}(0) =& \hat{x}_0, \end{aligned} \tag{15}$$

From this, defining the matrix $A_K = A - KC$, the dynamics of the observation error $\epsilon(t) = x(t) - \hat{x}(t)$ is given by

$$\dot{\epsilon}(t) = A_K\,\epsilon(t) + \nu^s(t) - K\nu^m(t). \tag{16}$$

If K is chosen such that A_K is Hurwitz, when $\nu^s(t) = \nu^m(t) = 0$, $\epsilon(t)$ exponentially goes to zero, and when $\nu^s(t)$ and $\nu^m(t)$ are uniformly bounded, $\epsilon(t)$ is uniformly bounded too. The following Theorem presents a positive observer for Σ_ν by simply considering the V-IPR of Σ_O with an appropriate choice of K.

Theorem 9. *Consider* Σ_ν *as in* (12)*. Given* $K \in \mathbb{R}^{n\times q}$*, let* $A_K = A - KC$*. If* K *is such that* $[A_K]$ *is Hurwitz, then the system*

$$\Omega : \quad \begin{aligned} \dot{Z}(t) =& [A_K]Z(t) + \widetilde{B}\pi(u(t)) + \widetilde{K}\pi(y(t)) \\ Z(0) =& \pi(\hat{x}_0) \in \mathbb{R}_+^{2n} \\ \hat{x}(t) =& \Delta_n Z(t), \end{aligned} \tag{17}$$

where $\widetilde{B}$, $\widetilde{K}$ *are the positive representations of* B,K *as in* (3)*, is such that the following propositions hold* $\forall t \ge 0$*:*

- $Z(t) \ge 0$*;*
- *if* $\nu^s(t) = \nu^m(t) = 0$ *then* $\hat{x}(t)$ *exponentially converges to* $x(t)$*. Moreover if* $Z(0) = \pi(x(0))$*, then* $\hat{x}(t) = x(t)$*;*
- *if* $x(t)$*,* $\nu^s(t)$ *and* $\nu^m(t)$ *are uniformly bounded then* $Z(t)$ *is uniformly bounded.*

Proof. The positivity of $Z(t)$ follows from the fact that $[A_K]$ is Metzler, $\widetilde{B}, \widetilde{K} \ge 0$, the inputs are positive and $Z(0) \ge 0$. The assertion $\hat{x}(t) \to x(t)$ exponentially when $\nu^s(t) = \nu^m(t) = 0$ follows from the fact that (17) is the IPR of the Luenberger observer Σ_O. To prove this, let us compute $\dot{\hat{x}}(t) = \Delta_n \dot{Z}(t)$.

$$\begin{aligned} \dot{\hat{x}}(t) =& \Delta_n \dot{Z}(t) \\ =& \Delta_n[A_K]Z(t) + \Delta_n\widetilde{B}\,\pi(u(t)) + \Delta_n\widetilde{K}\,\pi(y(t)) \\ =& A_K\Delta_n Z(t) + Bu(t) + Ky(t) \\ =& A_K\hat{x}(t) + Bu(t) + Ky(t), \end{aligned} \tag{18}$$

which is the Luenberger observer (15). Since $[A_K]$ is Hurwitz, the boundedness of $x(t)$, and hence of $\pi(y(t))$, and of $\nu^s(t)$, $\nu^m(t)$ implies that $Z(t)$ is bounded too. □

Remark 2. *Notice that* Ω *defined in* (17)*, despite the apparent similarity to a Luenberger observer, makes use of the nonlinear function* $\pi(.)$*. Consequently, the ordering of trajectories (see Theorem 2) holds with respect to* $\pi(u(t))$ *and not to* $u(t)$*.*

Remark 3. *The main assumption in Theorem 9 is that the observer gain* K *is chosen such that* $[A_K]$ *is Hurwitz. Thanks to Lemma 7,* $[A_K]$ *Hurwitz is equivalent to have both* A_K *and* $d^{A_K} + |A_K - d^{A_K}|$ *Hurwitz. Recalling that* $A_K = A - KC$*, if the pair* (A, C) *is detectable, then* K *can be designed such that* A_K *is Hurwitz. However,* A_K *Hurwitz does not imply that* $d^{A_K} + |A_K - d^{A_K}|$ *is Hurwitz too. This issue is investigated in the Appendix.*

B. An IPR-based interval observer in original coordinates

A couple of positive observers built using an IPR like the one in Theorem 9 can be used to design an interval observer. The following result, which is well known and simple to prove, suggests that an interval observer can be built by using a couple of positive observers of the type (17).

Lemma 10. *Let* $x(t) \in \mathbb{R}^n$ *be a vector function, and let* $X(t) \in \mathbb{R}_+^{2n}$ *be a positive representation of* $x(t)$*, i.e.* $x(t) = \Delta_n X(t)$*. Considering positive vector functions* $\underline{X}(t) = [\underline{X}_1(t)^T, \underline{X}_2(t)^T]^T \in \mathbb{R}_+^{2n}$ *and* $\overline{X}(t) = [\overline{X}_1(t)^T, \overline{X}_2(t)^T]^T \in \mathbb{R}_+^{2n}$*. If for all* $t \ge 0$*,* $\underline{X}(t) \le X(t) \le \overline{X}(t)$*, then*

$$\underline{X}_1(t) - \overline{X}_2(t) \;\le\; x(t) \;\le\; \overline{X}_1(t) - \underline{X}_2(t). \tag{19}$$

The proposed interval observer is made of two subsystems that resemble the V-IPR of the Luenberger observer presented in Theorem 9. For the sake of concision it is useful to aggregate the state and measurement disturbances $\nu^s(t)$, $\nu^m(t)$ into the equivalent disturbance

$$w(t) = \nu^s(t) - K\nu^m(t). \tag{20}$$

Given an observer gain K a bound $\overline{w}(t) = [w_1(t), w_2(t)]$ for $w(t)$ can be easily derived from the bounds $\bar{\nu}^s(t)$, $\bar{\nu}^m(t)$.

The main difference between the interval observer and the Luenberger observer of Theorem 9 is that $\phi_{x_{o1}}(.)$ and $\phi_{w_1}(.)$ replace the min-positive representation of, respectively, the initial conditions and the disturbances, in order to preserve inequalities (see Lemma 3).

Theorem 11. *Given the system* Σ_ν *in* (12)*,* $K \in \mathbb{R}^{n\times q}$ *such that* $[A_K]$ *is Hurwitz, where* $A_K = A - KC$*, and a bound* $\overline{w}(t)$ *on the equivalent state disturbance, the dynamical system*

$$\begin{aligned} \dot{\underline{Z}}(t) =& [A_K]\underline{Z}(t) + \widetilde{B}\,\pi(u(t)) + \widetilde{K}\,\pi(y(t)) + \phi_{w_1}(w_1(t)) \\ \dot{\overline{Z}}(t) =& [A_K]\overline{Z}(t) + \widetilde{B}\,\pi(u(t)) + \widetilde{K}\,\pi(y(t)) + \phi_{w_1}(w_2(t)) \\ \underline{Z}(0) \;=&\; \phi_{x_{01}}(x_{01}), \quad \overline{Z}(0) \;=\; \phi_{x_{01}}(x_{02}), \end{aligned} \tag{21}$$

together with the functions

$$\begin{aligned} \underline{x}(t) \;=&\; [I_n\, 0_{n\times n}]\underline{Z}(t) - [0_{n\times n}\, I_n]\overline{Z}(t) \\ \overline{x}(t) \;=&\; [I_n\, 0_{n\times n}]\overline{Z}(t) - [0_{n\times n}\, I_n]\underline{Z}(t) \end{aligned} \tag{22}$$

is an interval observer for Σ_ν *according to Definition 4, and in particular* $\underline{x}(t) \le x(t) \le \overline{x}(t)$*.*

Proof. Note that the interval observer (21) is made of two positive subsystem, so that $\underline{Z}(t)$, $\overline{Z}(t) \in \mathbb{R}_+^{2n}$, and the state of the observer is $Z(t) = [\underline{Z}(t)^T, \overline{Z}(t)^T]^T \in \mathbb{R}_+^{4n}$. Since $[A_K]$ is Hurwitz by assumption, then (21) is ISS, and $Z(t)$ is uniformly bounded if $u(t)$, $y(t)$ and $\overline{w}(t)$ are uniformly bounded. To prove $\underline{x}(t) \le x(t) \le \overline{x}(t)$ the first step is to show that

$$\dot{\zeta}(t) \;=\; [A_K]\zeta(t) + \widetilde{B}\,\pi(u(t)) + \widetilde{K}\,\pi(y(t)) + \phi_{w_1}(w(t)) \tag{23}$$

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with $\zeta(0) = \phi_{x_{01}}(x_0) \in \mathbb{R}_+^{2n}$ is an IPR of Σ_ν (12), so that $x(t) = \Delta_n \zeta(t)$. This is easily obtained using the same approach as in Theorem 9. The second step is to infer that the inequality $\underline{Z}(t) \le \zeta(t) \le \overline{Z}(t)$ holds for all $t \ge 0$. This is a straightforward consequence of the ordering of trajectories induced by the initial conditions $\underline{Z}(0) \le \zeta(0) \le \overline{Z}(0)$ and the forcing terms, $\phi_{w_1}(w_1(t)) \le \phi_{w_1}(w(t)) \le \phi_{w_1}(w_2(t))$, that results from Lemma 3. At this point the thesis directly follows from Lemma 10 and Theorem 2. When $w_1(t) = w_2(t)$ for all $t \ge 0$, then the two subsystems in (21) coincide, except for the initial state (note that this means that $w(t)$ is known, and $w(t) = w_1(t) = w_2(t)$, so that $\phi_{w_1}(w_1(t)) = \phi_w(w(t)) = \pi(w(t))$, see Remark 1). Thus, both subsystems in (21) coincide with the positive observer (17), but have different initialization. However, thanks to the Hurwitz property of $[A_K]$ we have $\|\overline{Z}(t) - \underline{Z}(t)\| \to 0$ and from this $\|\overline{x}(t) - \underline{x}(t)\| \to 0$. It remains to prove that $\|w_2(t) - w_1(t)\|$ bounded implies $\|\overline{x}(t) - \underline{x}(t)\|$ bounded. Let $\xi(t) = \overline{x}(t) - \underline{x}(t)$. From (22) we easily derive the identity $\xi(t) = [I_n\ I_n]\big(\overline{Z}(t) - \underline{Z}(t)\big)$. From this

$$\begin{aligned}\dot{\xi}(t) &= [I_n\ I_n][A_K](\overline{Z}(t) - \underline{Z}(t)) + w_2(t) - w_1(t) \\ &= \left(d^{A_K} + |A_K - d^{A_K}|\right)\xi(t) + w_2(t) - w_1(t). \qquad (24)\end{aligned}$$

The matrix $d^{A_K} + |A_K - d^{A_K}|$ is Hurwitz, because $\sigma(d^{A_K} + |A_K - d^{A_K}|) \subseteq \sigma([A_K])$ (Lemma 7), and $[A_K]$ is Hurwitz by assumption. Then the system (24) is ISS, and therefore $\|w_2(t) - w_1(t)\|$ uniformly bounded implies that $\|\overline{x}(t) - \underline{x}(t)\|$ is uniformly bounded too. □

C. An interval observer with a coordinate change

In Remark 3 and in Appendix we discussed the non-trivial problem of finding K that makes $[A_K]$ Hurwitz. When this is not possible, we cannot find an interval observer of the type (21). However, we can exploit the result of Lemma 8, which provides an easy condition for the stability of the IPR in the case of matrices in Real Jordan Form. A straightforward solution to the problem of finding an IPR-based interval observer can be therefore pursued in three steps:

1) find K assigning to $A_K = A - KC$ a set of eigenvalues in $\mathcal{S}$ (11);
2) find the change of coordinates that transforms A_K in the Real Jordan Form J_K;
3) design the interval observer of the transformed system and translate the intervals back to the original coordinates.

Notice that, for an observable pair (A, C), step 1 is a standard pole placement. For step 3 we make use of Lemma 5.

Theorem 12. *Given the system Σ_ν in (12), $K \in \mathbb{R}^{n\times q}$ such that $\sigma(A_K) \subset \mathcal{S}$ (defined in Lemma 8), a bound $w_1(t) \le w(t) \le w_2(t)$ on the equivalent state disturbance, $T \in \mathbb{R}^{n\times n}$ the nonsingular matrix that transforms A_K in real Jordan form J_K (i.e., $J_K = T^{-1}A_K T$), the dynamical system*

$$\begin{aligned}\dot{\underline{Z}}(t) =& [J_K]\underline{Z}(t) + \widetilde{B}_T\pi(u(t)) + \widetilde{K}_T\,\pi(y(t)) \\ &+ \psi_{T,w_1}(w_1(t)) \\ \dot{\overline{Z}}(t) =& [J_K]\overline{Z}(t) + \widetilde{B}_T\pi(u(t)) + \widetilde{K}_T\,\pi(y(t)) \qquad (25) \\ &+ \psi_{T,w_1}(w_2(t)) \\ \underline{Z}(0) =& \psi_{T,x_{01}}(x_{01}), \quad \overline{Z}(0) = \psi_{T,x_{01}}(x_{02}),\end{aligned}$$

together with the functions

$$\begin{aligned}\underline{x}(t) &= [I_n\, 0_n]\widetilde{T}\underline{Z}(t) - [0_n\, I_n]\widetilde{T}\overline{Z}(t), \\ \overline{x}(t) &= [I_n\, 0_n]\widetilde{T}\overline{Z}(t) - [0_n\, I_n]\widetilde{T}\underline{Z}(t),\end{aligned} \qquad (26)$$

where $[J_K]$ is the Metzler representation of J_K, $\widetilde{B}_T$ and $\widetilde{K}_T$ are the min-positive representation of, respectively, $B_T = T^{-1}B$ and $K_T = T^{-1}K$, is an interval observer for Σ_ν according to Definition 4.

Proof. The observer is ISS because $\sigma(A_K) = \sigma(J_K)$ and, thanks to Lemma 8, $\sigma(J_K) \subset \mathcal{S}$ implies that $[J_K]$ is Hurwitz. To prove the trajectory inclusion $\underline{x}(t) \le x(t) \le \overline{x}(t)$, the first step is to show that the system

$$\dot{\zeta}(t) = [J_K]\zeta(t) + \widetilde{B}_T\,\pi(u(t)) + \widetilde{K}_T\,\pi(y(t)) + \psi_{T,w_1}(w(t)),$$

with $\zeta(0) = \psi_{T,x_{01}}(x_0)$ is such that $x(t) = T\Delta_n\zeta(t)$. This is easy to prove using the same approach as in Theorem 9. Notice that this implies also $x(t) = \Delta_n\widetilde{T}\zeta(t)$. The second step is to infer that the inequality $\underline{Z}(t) \le \zeta(t) \le \overline{Z}(t)$ holds for all $t \ge 0$, which is a consequence of the ordering of trajectories induced by the initial conditions and the forcing terms, according to Theorem 2. The ordering of forcing terms is a consequence of Lemma 5. Since $\widetilde{T}$ is positive, from $\underline{Z}(t) \le \zeta(t) \le \overline{Z}(t)$ we have $\widetilde{T}\underline{Z}(t) \le \widetilde{T}\zeta(t) \le \widetilde{T}\overline{Z}(t)$. Recalling that $x(t) = \Delta_n\widetilde{T}\zeta(t)$ the thesis directly follows from Lemma 10. The boundedness of the interval amplitude is an implication of the Hurwitz property of $[J_K]$. □

Remark 4. *The observability of (A, C) ensures that it is possible to assign $\sigma(A_K)$ in $\mathcal{S}$. Of course, such hypothesis can be relaxed, by only assuming that the eigenvalues of A associated to unobservable modes, if any, belong to $\mathcal{S}$. Otherwise, a time-varying change of coordinates can still be used for any detectable pair (A,C) [7], [16].*

Remark 5. *The results presented here for continuous-time systems extend immediately to discrete-time systems using the corresponding IPR* [6], [12]. *The only difference is that the min-positive representation $\widetilde{A_K}$ replaces $[A_K]$ in* (21) *and $\widetilde{J_K}$ replaces $[J_K]$ in* (25). *The stability region for the IPR in Real Jordan Form is* $\mathcal{P}_4 = \{z \in \mathbb{C} :\ \Re(z) + |\Im(z)| < 1\}$ [12].

Remark 6. *The approach can be extended to uncertain systems where disturbances have a structured form, for example $B(t,x)u(t)$, with the unknown $B(t,x)$ satisfying $B_1(t) \le B(t,x) \le B_2(t)$. In this case, using Lemma 4, the corresponding bounds on the positive representation are*

$$\tilde{\phi}_{B_1}(B_1)\pi(u) \ \le\ \tilde{\phi}_{B_1}(B(t,x))\pi(u) \le\ \tilde{\phi}_{B_1}(B_2)\pi(u), \qquad (27)$$

that can be introduced in the forcing term of the observer (25). *Another case which is easy to deal with in the IPR approach is that of nonlinear systems containing terms of the form $|x|$, see [15].*

IV. Example

In this section we illustrate the approach described in Section III-C with a simple example. Consider system Σ_ν in (12) with

$$A = \begin{bmatrix} -1.5 & -1 & -0.5 \\ 1.5 & 1 & -0.5 \\ 1 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$
$$C = \begin{bmatrix} -1 & -2 & 1 \\ 1 & 0 & 2 \end{bmatrix}, \qquad (28)$$

and $u(t) = 3\sin(t) - 2\sin(3t)$. The matrix A has eigenvalues -0.5 and $\pm j$ and it is therefore not exponentially stable. Since (A, C) is observable, we choose $A_K = A - KC$ such that $\sigma(A_K) = L = \{-1.5, -1.8, -2.1\}$. The interval observer is built using a coordinate change $z = T^{-1}x$, where T is such that $T^{-1}A_K T = J_K = \mathrm{diag}(\sigma(A_K))$. After noticing that $\sigma(A_K) = L \subset \mathbb{R}_-$ we conclude that $[J_K]$, the Metzler representation of J_K, is Hurwitz according to Lemma 8. This is easily confirmed by the explicit computation of $[J_K]$,

$$[J_K] = \begin{bmatrix} J_K & 0_{3\times3} \\ 0_{3\times3} & J_K \end{bmatrix}. \qquad (29)$$

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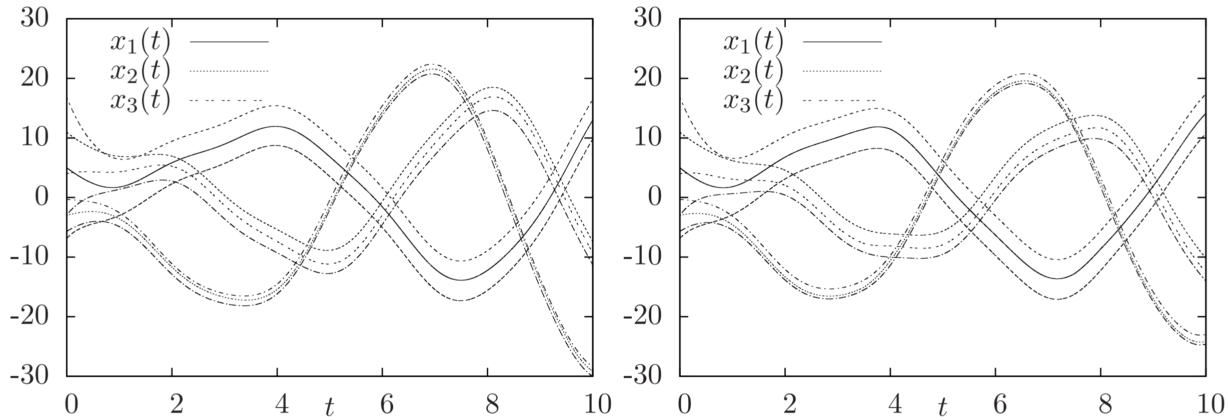

Fig. 1. Interval observer for the state variables $x_1(t)$, $x_2(t)$, $x_3(t)$ for two realizations of the disturbance $w_a(t)$, $w_b(t)$ having the same bounds. Interval estimates computed from the interval observer described in Section III-C for the state variables $x_1(t)$, $x_2(t)$, $x_3(t)$. Two distinct realizations of the bounded disturbance $w(t)$ are reported: $w(t) = w_a(t)$ (left) and $w(t) = w_b(t)$ (right).

from which it follows that $[J_K]$ has the same spectrum as A_K and J_K, *i.e.* $\sigma([J_K]) = \sigma(A_K) = L$. Let us consider two different disturbances $w(t)$, namely $w_a(t) = B_w \sin(t)$,

$$w_b(t) = \begin{cases} B_w & \text{if } \mathrm{mod}(t,4) > 3.5 \\ -B_w & \text{if } \mathrm{mod}(t,4) \leq 3.5, \end{cases} \tag{30}$$

with $B_w = \frac{1}{2}[1,\, 1,\, 1]^T$. A uniform bound for these disturbances is $|w(t)| \leq B_w$. An interval observer with coordinate change can now be designed starting from the following bound on the initial conditions (in the original coordinates), $x_0^- = [4, -4, 3]^T \leq x_0 = [5, -3, 4]^T \leq x_0^+ = [6, -2, 5]^T$. The resulting estimation intervals on all the state variables for the two cases are shown in Fig. 1. It may be noticed that in the example the interval width is not uniform across state variables, but it is independent of the disturbance realization. It must also be remarked that the size of the estimation interval, when there is a coordinate change, depends on both the size of $w(t)$ and the coordinate change T. Thus for a different choice of K the approximation could be worse (or better) than the one shown in this example even in presence of the same disturbance.

V. Conclusions

We have shown that an interval observer for continuous-time and discrete-time linear systems can be designed using simple pole placement algorithms and a fixed coordinate change. The resulting algorithm has size $4n$ and it is guaranteed to be stable if the pair (A, C) is observable. With respect to [7], [10], [16], [18], whose algorithms have dimensions $2n$, the proposed method is simpler in that it does not require a time-varying change of coordinates nor matrix equations to be solved. The explicit representation of positive and negative parts of state and uncertainties allows to express in a direct way bounds for the case of multiplicative disturbances, which can be useful to extend the interval observer approach to other classes of systems, like nonlinear and time-delay systems. Further issues to investigate include the optimization of the coordinate change to minimize the width of the estimation interval.

Appendix

In this Appendix we discuss the problem of finding a matrix K such that the Metzler representation of $A_K = A - KC$ is Hurwitz. First of all we show that there are pairs (A, C), with A unstable, for which $[A_K]$ can be Hurwitz.

Example 1. *Consider Σ_L in* (1) *with*

$$A = \begin{bmatrix} 0.5 & 0 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}. \tag{31}$$

A is obviously not Hurwitz. Choosing $K = [1\ 1/2]^T$ we have $\sigma(A_K) = \sigma([A_K]) = \{-0.134, -1.866\}$ ($[A_K]$ has two negative eigenvalues of multiplicity 2*). Thus Theorem 9 enables us to conclude that* (17) *provides a stable positive observer of an unstable non-positive system.*

Despite this, it may be that A_K is Hurwitz and $[A_K]$ is not. In such cases, it holds true that Ω (17) is an exponential observer for system Σ_ν, but Ω is not ISS.

The problem of finding K that makes $[A_K]$ Hurwitz is not trivial, and in particular for some pair (A, C) it may be impossible to find K to make $[A_K]$ Hurwitz. Theorem 2.1 of [2] is a useful tool to investigate this issue. It leads to the following result that we state for the case of scalar $y(t)$, but whose extension to the multi-output case is straightforward.

Lemma 13. *If, given $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ and $C = (c_i) \in \mathbb{R}^n$, there exist $K = (k_i) \in \mathbb{R}^n$, and strictly positive $\lambda = (\lambda_i) \in (\mathbb{R}_+ \setminus \{0\})^n$ such that $A_K = A - KC$ is Hurwitz and*

$$a_{ii} \ < \ k_i c_i - \sum_{j=1,\, j \neq i}^{n} |a_{ij} - k_i c_j| \frac{\lambda_j}{\lambda_i} \tag{32}$$

holds for $i = 1, \ldots, n$, then $[A_K] = [A - KC]$ is Hurwitz.

Proof. From Lemma 7, $[A_K]$ is Hurwitz if and only if both A_K and $d^{A_K} + |A_K - d^{A_K}|$ are Hurwitz. A_K is Hurwitz by hypothesis. Using the fourth stability condition of Theorem 2.1 of [2] we have $(d^{A_K} + |A_K - d^{A_K}|)\lambda \ < \ 0$ for $\lambda > 0$, which is equivalent to (32). □

Corollary 14. *If, given A and C as before, there exist K and strictly positive λ such that $A_K = A - KC$ is Metzler and condition* (32) *holds for $i = 1, \ldots, n$, then $[A_K] = [A - KC]$ is Hurwitz.*

Proof. If A_K is Metzler, Theorem 2.1 of [2] can be applied. Condition (32) implies that

$$A_K \lambda = (d^{A_K} + (A_K - d^{A_K}))\lambda \ < \ (d^{A_K} + |A_K - d^{A_K}|)\lambda \ < \ 0,$$

or, equivalently, that A_K is Hurwitz. □

Notice that (32) taken for $i = 1, \ldots, n$ is a system of equations in which each k_i occurs in just one equation. Corollary 14 restricts the search to matrices A_K that are Metzler, but this condition leads to a system of inequalities for A_K that is easier to verify than the condition of being Hurwitz. From the previous discussion it follows that it is sometimes impossible to find K such that $[A_K]$ is Hurwitz. For instance, from Lemma 13 it is clear that no K can make $[A_K]$ Hurwitz when A is diagonal and contains *only* positive real entries.

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