A New Approach to Hilbert's Third Problem

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1. INTRODUCTION. If two polygons have the same area, it is always possible to decompose one of them into a finite number of polygons that can be rearranged to form the second polygon. This is the well-known Bolyai-Gerwien theorem [**3**, pp. 49–56]. One might ask whether this is true in space for polyhedra. In fact, F. Bolyai and Gauss had already asked this around 1844. Hilbert raised this question again: it was the third of his celebrated list of twenty-three problems in 1900 [**11**]. The negative answer was given by Max Dehn in 1902 [**6**].

Let $F, F_1, \dots, F_k$ be polyhedra. By writing $F = F_1 + \cdots + F_k$ we mean that the interiors of the polyhedra $F_1, \dots, F_k$ are pairwise disjoint and $F = F_1 \cup \cdots \cup F_k$. Polyhedra F and G are *equidecomposable* if the polyhedron F can be suitably decomposed into a finite number of pieces that can be reassembled to give the polyhedron G. To be precise, there exist polyhedra $F_1, \dots, F_k$ and $G_1, \dots, G_k$ such that F_i and G_i are congruent and

$$F = F_1 + \cdots + F_k, \quad G = G_1 + \cdots + G_k.$$

Dehn showed that a regular tetrahedron is not equidecomposable with a cube of the same volume. This "bad news" also means that even though we can define the area of polygons using elementary methods and no calculus, the limit process cannot be avoided when defining the volume of polyhedra.

In a way, Dehn's work has led several lives. Dehn's own exposition was hard to understand. In 1903 Kagan published a paper in which Dehn's argument was considerably refined and presented in a more readable fashion. In the 1950s a number of interesting results in the theory of equidecomposability were obtained by the Swiss geometer Hadwiger and his students. Their work allows one to take a new look at the work of Dehn and to obtain Dehn's basic result by using transparent ideas in a modern treatment. The only shortcoming in this treatment is the application of the axiom of choice through the use of a Hamel basis. Finally, a reworked version of Hadwiger's proof was given by Boltianskii in [**3**], in which consideration of the whole real line is replaced with finitely generated subspaces of $\mathbb{R}$ (over the rationals). This allows one to avoid the use of the axiom of choice.

There are many open problems still to be solved concerning the generalized third problem of Hilbert. Sydler's result (Theorem 4), to cite just one example, is unknown for the n-dimensional Euclidean space $\mathbb{E}^n$ ($n \geq 5$), for the sphere $\mathbb{S}^n$ ($n \geq 3$), and for the hyperbolic space $\mathbb{H}^n$ ($n \geq 3$).

There is a natural idea that seemingly settles Hilbert's third problem immediately. This makes the history of the problem even more interesting. Using this idea in 1896(!) R. Bricard published a paper [**2**] in which he claimed to have proved the non-equidecomposability of the regular tetrahedron and the cube. Bricard formulated the following theorem:

Theorem 1. *If polyhedra $A^{(1)}$ and $A^{(2)}$ with dihedral angles $\alpha_1, \dots, \alpha_s$ and $\beta_1, \dots, \beta_r$, respectively, are equidecomposable, then there exist positive integers m_i*

$(i = 1, \dots, s)$ and n_j $(j = 1, \dots, r)$ such that

$$m_1\alpha_1 + \cdots + m_s\alpha_s = n_1\beta_1 + \cdots + n_r\beta_r + p\pi, \tag{1}$$

where p is an integer.

We call this assertion *Bricard's condition*. Now assume by contradiction that the regular tetrahedron with dihedral angles α is equidecomposable with the cube. Then positive integers m and n exist such that $m\alpha = n(\pi/2)$. But this contradicts the fact that α/π is irrational (see Lemma 2). Therefore Bricard's condition immediately resolves Hilbert's third problem.

Unfortunately there was a gap in Bricard's proof of Theorem 1. Nevertheless, it turned out to be a true statement. Although in 1902 Dehn succeeded in proving Theorem 1, the proof takes a roundabout approach by way of Dehn's own solution to Hilbert's third problem. For this reason we cannot use Bricard's condition to solve Hilbert's problem. Or can we?

Surprisingly, no *direct* proof of Bricard's condition exists. The simplest published proof we have found (see [**3**, pp. 121–124]) is three pages long and uses the Dehn-Hadwiger theorem (Theorem 3). That is, Bricard's condition is proved as a *consequence* of the solution of Hilbert's third problem. In this article we give a short direct proof of Bricard's condition that was overlooked for a century. Therefore it provides a new solution to Hilbert's problem. Our proof is completely elementary. Since it uses no linear algebra, it could even be presented in a high-school math club.

The Dehn-Hadwiger theorem settles Hilbert's third problem. We mentioned earlier that it can be used to establish Bricard's condition, too. On the other hand, Bricard's condition settles Hilbert's third problem, but it does not imply the Dehn-Hadwiger theorem. We introduce a new condition, which we call the "modified Bricard's condition," that addresses this issue: the Dehn-Hadwiger theorem follows from it in one line.

It would be natural to prove Bricard's condition first and then simply conclude the negative answer to Hilbert's third problem. Instead, we start this discussion by solving Hilbert's problem first. We do so because we want to show that Hilbert's third problem has a complete, two-page solution. The actual proof takes only one page; on the other page we provide the proofs of two well-known lemmas.

In section 4 we also describe Dehn's method. The interested reader can find the details, for example, in *Proofs from THE BOOK* [**1**] or in [**3**].

Notation. The symbol $\mathbb{Z}$ signifies the set of all integers, $\mathbb{N}$ the set of positive integers. If L is the union of finitely many line segments, $l(L)$ denotes the total length of L.

2. A NEW SOLUTION TO HILBERT'S THIRD PROBLEM.

Our proof is based on an idea that we call the method of "integer measures": Given positive *real* weights $a_1, \dots, a_N$ we can always replace them with positive *integer* weights $A_1, \dots, A_N$ in such a way that equalities between the weights are preserved, in the following sense: if I_1 and I_2 are subsets of $\{1, \dots, N\}$ such that

$$\sum_{i\in I_1} a_i = \sum_{j\in I_2} a_j, \tag{2}$$

then

$$\sum_{i\in I_1} A_i = \sum_{j\in I_2} A_j. \tag{3}$$

(In fact, positive integer weights also exist such that (3) holds if and only if (2) holds.) The foregoing statement is a consequence of Lemma 1. Since Lemma 1 is proved by the pigeonhole principle, in some sense our solution to Hilbert's third problem is based on the pigeonhole principle.

Theorem 2. *A regular tetrahedron is not equidecomposable with a cube of the same volume.*

Proof. We argue by contradiction. Assume that the regular tetrahedron T and the cube C are equidecomposable:

$$T = P_1^{(1)} + \cdots + P_k^{(1)} \tag{4}$$

$$C = P_1^{(2)} + \cdots + P_k^{(2)}, \tag{5}$$

where $P_i^{(1)}$ and $P_i^{(2)}$ are congruent polyhedra for each i.

In the decompositions (4) and (5) consider all vertices and all possible intersections of the edges of the $P_i^{(j)}$ polyhedra. This set of points divides the edges into one or more open line segments that we call links. Let $L_1, \ldots, L_N$ be all the links (coming from both (4) and (5)), and let $l(L_i)$ denote the length of L_i. By Lemma 1 we can find *positive* integers $q, p_1, \ldots, p_N$ such that

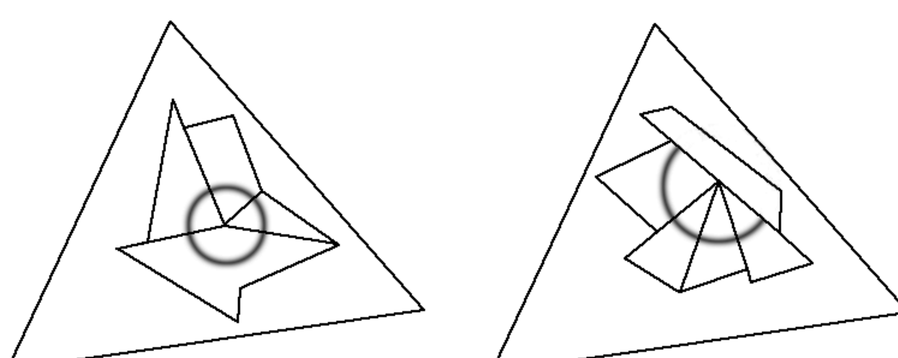
$$\left| l(L_i) - \frac{p_i}{q} \right| < \frac{1}{2Nq} \quad (i = 1, \ldots, N). \tag{6}$$

We call $m(L_i) := p_i$ the "integer measure" of the link L_i.

Consider now

$$\Sigma_1 := \sum \sum m(L_i)\alpha_j, \tag{7}$$

where the summation extends over all links L_i in the decomposition (4) and all dihedral angles adjoining the link L_i. Notice that when we add the dihedral angles adjoining a link L_i we usually get 2π (see Figure 1(a)). However, we get π when L_i lies in the interior of a face of a polyhedron $P_j^{(1)}$ (see Figure 1(b)) or in the interior of a face of T. Finally, when L_i lies on an edge of T, we get the dihedral angle $\alpha = \arccos(1/3)$ of T.



(a) (b)

Figure 1.

Therefore,

$$\Sigma_1 = m_1\alpha + p_1\pi$$

with m_1 in $\mathbb{N}$ and p_1 in $\mathbb{N} \cup \{0\}$. In exactly the same way we define Σ_2 for the decomposition (5) and conclude that

$$\Sigma_2 = m_2\frac{\pi}{2} + p_2\pi,$$

where m_2 belongs to $\mathbb{N}$ and p_2 to $\mathbb{N} \cup \{0\}$.

Let e be any edge of a polyhedron $P_i^{(1)}$ in (4), and let α_j be the dihedral angle of $P_i^{(1)}$ that adjoins e. In Σ_1 the coefficient of this α_j angle is simply $\sum m(L_u)$, where we sum over all links forming e. Let e' denote the edge of $P_i^{(2)}$ that corresponds to e, and let α'_j be the dihedral angle adjoining e' (so $\alpha_j = \alpha'_j$). In Σ_2 the coefficient of this α'_j angle is $\sum m(L_v)$, where we sum over all links forming e'. Using $\sum l(L_u) = l(e) = l(e') = \sum l(L_v)$ and (6), we find that

$$\begin{aligned}\left|\sum m(L_v) - \sum m(L_u)\right| &= q\left|\sum\Big(\frac{m(L_v)}{q} - l(L_v)\Big) - \sum\Big(\frac{m(L_u)}{q} - l(L_u)\Big)\right| \\ &< 2Nq\frac{1}{2Nq} = 1, \end{aligned} \tag{8}$$

which implies that $\sum m(L_u) = \sum m(L_v)$. In other words, the coefficients of the corresponding dihedral angles in Σ_1 and Σ_2 are equal. Hence $\Sigma_1 = \Sigma_2$, and α/π is rational. This contradicts the fact that α/π is irrational (see Lemma 2). ■

For the sake of completeness we also give a proof of the following two lemmas, which are well-known results.

Lemma 1. *Let $a_1, \dots, a_n$ be real numbers. For each $\varepsilon > 0$ it is possible to approximate $a_1, \dots, a_n$ simultaneously by rational numbers $p_1/q, \dots, p_n/q$, in the sense that*

$$\left|a_i - \frac{p_i}{q}\right| < \frac{\varepsilon}{q} \quad (i = 1, \dots, n). \tag{9}$$

In addition, if all a_i are positive, then the p_i can be chosen to be positive.

Proof. Let M be a positive integer satisfying $1/M < \varepsilon$. Consider the following $M^n + 1$ points of the unit cube $C := [0, 1]^n$ in $\mathbb{R}^n$:

$$Q_l := (\{la_1\}, \{la_2\}, \dots, \{la_n\}) \quad (l = 0, 1, \dots, M^n),$$

where $\{x\}$ denotes the fractional part of x. We can decompose C into M^n congruent small cubes whose edges have length $1/M$. Since there are $M^n + 1$ points Q_l in C, there must be a small cube that contains at least two of these points, say, Q_u and Q_v ($u \neq v$). But then for $i = 1, \dots, n$ we have

$$|\{ua_i\} - \{va_i\}| \leq \frac{1}{M}.$$

Thus

$$|(u-v)a_i - p_i| \le \frac{1}{M} \quad (i = 1, \ldots, n)$$

for integers p_i. Letting $q := |u - v|$ the statement follows.

If all a_i are positive, then we find the rational numbers satisfying (9), where ε is replaced with $\min(\varepsilon, a_1, \ldots, a_n)$. Clearly now all p_i must be positive. ∎

Lemma 2. *If* $\cos\alpha = 1/3$, *then* α/π *is irrational.*

Proof. Let n be a positive integer. Using the identity

$$\cos(n+1)\omega = 2\cos\omega\cos n\omega - \cos(n-1)\omega,$$

we can see by induction that $\cos n\omega$ is a polynomial of $\cos\omega$, i.e., with some polynomial T_n we have $\cos(n\omega) = T_n(\cos\omega)$. We note that $T_n(x)$ is of degree n, it has integer coefficients, and its leading coefficient is 2^{n-1}.

Seeking a contradiction, we suppose that $\alpha = p\pi/q$ for integers p and q $(q \ge 1)$. From the definition of $T_n(x)$ we observe that

$$T_q\left(\frac{1}{3}\right) = T_q\left(\cos\left(\frac{p}{q}\pi\right)\right) = \cos(p\pi) = \pm 1,$$

so $1/3$ is a root of a polynomial with integer coefficients. For this to be the case 3 has to divide the leading coefficient 2^{n-1}, which is a contradiction. ∎

Remarks. It was important in the proof of Theorem 2 that the p_i in (6) be *positive* integers. (Otherwise we do not get a contradiction.) Lemma 1 ensures that p_i can be chosen positive. Alternatively, instead of (6) we could require that

$$\left|l(L_i) - \frac{p_i}{q}\right| < \frac{\min(1, l(L_1), \ldots, l(L_N))}{2Nq} \quad (i = 1, \ldots, N).$$

This would force p_i to be positive.

The double sum in (7) can be interpreted geometrically as choosing $m(L_i)$ points, "basic points," on the link L_i in (4) and adding the dihedral angles at these basic points. We can choose basic points on the links of the decomposition (5), as well. However it may not be possible to select basic points in the two decompositions in such a way that they correspond to each other under the isometries that take the polyhedra $P_j^{(1)}$ to the polyhedra $P_j^{(2)}$. On the other hand, we can choose them in such a way that the *number of basic points* is the same on the corresponding edges of $P_j^{(1)}$ and $P_j^{(2)}$.

3. PROOF OF BRICARD'S CONDITION.

Assume that the polyhedra $A^{(1)}$ and $A^{(2)}$ are equidecomposable:

$$A^{(1)} = P_1^{(1)} + \cdots + P_k^{(1)}, \tag{10}$$

$$A^{(2)} = P_1^{(2)} + \cdots + P_k^{(2)}, \tag{11}$$

where $P_i^{(1)}$ and $P_i^{(2)}$ are congruent polyhedra for each i. Bricard published the following *false proof* of Theorem 1.

Consider all links in the decompositions (10) and (11). Let e be an edge of a polyhedron $P_j^{(1)}$ in (10), and let e' to be the corresponding edge in (11). Note that the links forming e and e' may not correspond to each other under an isometry that takes $P_j^{(1)}$ to $P_j^{(2)}$. (In fact, the number of links forming e and e' may be different.)

Assume that by breaking the links up into smaller links we are able to achieve an invariant decomposition into links, that is, the links forming an edge e and the corresponding edge e' corresponds to each other. Let Σ_1 denote the sum of dihedral angles at all links in the decomposition of $A^{(1)}$. (If an edge e is decomposed into g links, then the dihedral angle α that adjoins e will appear g times in the sum Σ_1.) We define Σ_2 for the links in (11) in a similar manner.

When adding the dihedral angles adjoining a link, we get 2π, π, α_i, $\alpha_i - \pi$, β_j, or $\beta_j - \pi$ depending on where the link is. (For more details, see the correct proof that we will give shortly.) This observation leads to:

$$\Sigma_1 = m_1\alpha_1 + \cdots + m_s\alpha_s + p_1\pi,$$
$$\Sigma_2 = n_1\beta_1 + \cdots + n_r\beta_r + p_2\pi,$$

with the m_i and n_j in $\mathbb{N}$ and integers p_1 and p_2. The fact that the links correspond to each other implies that the two sums are equal: $\Sigma_1 = \Sigma_2$.

Thus the foregoing argument contains a proof of formula (1) provided that there exists an invariant decomposition into links. Bricard assumed that such decomposition always exists, but this is false. (For a counterexample, see [**3**, p. 120].) Boltianskii concludes [**3**, p. 121]: "Thus the proof proposed by Bricard contains a gap that cannot be filled."

We give the following simple direct proof of Bricard's condition:

Proof. Let $L_1, \ldots, L_N$ list all links in the decompositions (10) and (11). By Lemma 1 we can find positive integers $q, p_1, \ldots, p_N$ such that

$$\left| l(L_i) - \frac{p_i}{q} \right| < \frac{1}{2Nq} \quad (i = 1, \ldots, N).$$

Let

$$\Sigma_1 := \sum \sum m(L_i)\alpha_j,$$

where the summation extends over all links L_i in the decomposition (10) and all dihedral angles adjoining the link L_i. (Recall that $m(L_i) := p_i$ signifies the integer measure of L_i.)

Let s_i denote the sum of the dihedral angles adjoining a link L_i. Define k_i to be 1 when L_i lies in the interior of a face of a polyhedron $P_j^{(1)}$ or $A^{(1)}$, otherwise define k_i to be 0. Notice that if L_i lies on an edge of $A^{(1)}$, then s_i is of the form $s_i = \alpha_l - k_i\pi$, otherwise $s_i = 2\pi - k_i\pi$. Therefore, there exist positive integers $m_1, \ldots, m_s$ and an integer p_1 such that

$$\Sigma_1 = m_1\alpha_1 + \cdots + m_s\alpha_s + p_1\pi.$$

If we define Σ_2 in a similar manner for the decomposition (11), then we arrive by the same argument at:

$$\Sigma_2 = n_1\beta_1 + \cdots + n_r\beta_r + p_2\pi,$$

with $n_1, \dots, n_r$ in $\mathbb{N}$ and p_2 in $\mathbb{Z}$. Exactly as in our solution of Hilbert's third problem we see now that $\Sigma_1 = \Sigma_2$, whence Bricard's condition follows. ■

4. DEHN'S METHOD. After simplifications, Dehn's method of attacking Hilbert's third problem can be sketched as follows. If $M = \{x_1, \dots, x_n\}$ is a finite set of real numbers, we use $V(M)$ in the ensuing discussion to signify the vector space that is generated by M over the rational numbers $\mathbb{Q}$:

$$V(M) = \left\{ \sum_{i=1}^{n} q_i x_i \,:\, q_i \in \mathbb{Q} \right\}.$$

Linear functionals on $V(M)$ are also called *additive functions*. In fact, a function $f : V(M) \to \mathbb{R}$ is additive if and only if $f(x+y) = f(x) + f(y)$ for all x and y in $V(M)$. (This property implies that $f(wx) = wf(x)$ whenever x is in $V(M)$ and w is rational.)

Now let A be a polyhedron, let $\alpha_1, \alpha_2, \dots, \alpha_s$ be the dihedral angles of A, and let $l_1, l_2, \dots, l_s$ be the lengths of the corresponding edges of A. If $f : V(\{\alpha_1, \dots, \alpha_s, \pi\}) \to \mathbb{R}$ is an additive function satisfying $f(\pi) = 0$, we denote the sum

$$l_1 f(\alpha_1) + l_2 f(\alpha_2) + \cdots + l_s f(\alpha_s)$$

by $f(A)$ and call it the *Dehn invariant at f* of the polyhedron A. (We remark that from an algebraic point of view the Dehn invariant can be defined by tensor products: $\sum l_i \otimes_{\mathbb{Q}} \alpha_i$ in $\mathbb{R} \otimes_{\mathbb{Q}} (\mathbb{R}/(\pi\mathbb{Q}))$.)

The following theorem is called the Dehn-Hadwiger theorem:

Theorem 3 (Dehn-Hadwiger). *Let $A^{(1)}$ and $A^{(2)}$ be polyhedra, and let M be the set containing the number π and all dihedral angles of $A^{(1)}$ and $A^{(2)}$. If $f : V(M) \to \mathbb{R}$ is an additive function such that $f(\pi) = 0$ and $f(A^{(1)}) \neq f(A^{(2)})$, then $A^{(1)}$ and $A^{(2)}$ are not equidecomposable.*

The proof of this theorem is based on the ingenious idea that the Dehn invariants are preserved under decompositions of polyhedra. More precisely, the following property of Dehn invariants holds: Let $P = P_1 + \cdots + P_k$ be a decomposition of a polyhedron P into polyhedra $P_1, \dots, P_k$. If M' is a (finite) set that contains all dihedral angles of $P, P_1, \dots, P_k$ together with the number π and $f : V(M') \to \mathbb{R}$ is an additive function satisfying $f(\pi) = 0$, then

$$f(P) = f(P_1) + \cdots + f(P_k). \tag{12}$$

Assume that $A^{(1)}$ and $A^{(2)}$ are equidecomposable and that (10) and (11) hold. Let $f : V(M) \to \mathbb{R}$ be an additive function satisfying $f(\pi) = 0$. We emphasize that in Theorem 3 the domain of f is $V(M)$. However, in its proof we must use an additive function that is defined on a larger vector space $V(M')$. Namely, let M' be the set containing the number π and all the dihedral angles of $P_i^{(1)}$ and $P_j^{(2)}$ $(i, j = 1, \dots, k)$. Now recall the theorem that any linear functional $f : V(M) \to \mathbb{R}$ can be extended to the larger vector space $V(M')$. Since $P_i^{(1)}$ and $P_i^{(2)}$ are congruent, we have $f(P_i^{(1)}) = f(P_i^{(2)})$. This and property (12) imply $f(A^{(1)}) = f(A^{(2)})$, establishing the Dehn-Hadwiger theorem.

All that is required for the proof of Theorem 2 is to construct an additive function f such that $f(\pi) = 0$ and $f(T) \neq f(C)$, where T is the regular tetrahedron with dihedral angle α and edge length l, and C is the cube with the same volume as T.

Set $M = \{\alpha, \pi/2, \pi\}$ and notice that $V(M) = V(\{\alpha, \pi\})$. We define f on $V(M)$ by $f(r_1\alpha + r_2\pi) = r_1$, where r_1 and r_2 are arbitrary rational numbers. Then f is a well-defined function, since α/π is irrational (Lemma 2): each x in $V(M)$ can be expressed uniquely in the form $x = r_1\alpha + r_2\pi$. Clearly, f is an additive function on $V(M)$. From $f(\pi/2) = 0$ we have $f(C) = 0$. On the other hand, $f(T) = 6lf(\alpha) = 6l \neq 0$. From $f(T) \neq f(C)$ we conclude that the regular tetrahedron is not equidecomposable with the cube.

5. NECESSARY AND SUFFICIENT CONDITIONS. The theory of equidecomposability was enriched by a remarkable result in 1965, when Sydler proved that Dehn's necessary condition is also a sufficient condition for the equidecomposability of two polyhedra. This result is called Sydler's theorem. It is a deep theorem whose proof the reader can find in [**3**, pp. 142–166].

Theorem 4 (Dehn-Sydler). *Let $A^{(1)}$ and $A^{(2)}$ be polyhedra of equal volume, and let M be the set containing all dihedral angles of $A^{(1)}$ and $A^{(2)}$ and the number π. Then $A^{(1)}$ and $A^{(2)}$ are equidecomposable if and only if $f(A^{(1)}) = f(A^{(2)})$ holds for each additive function $f : V(M) \to \mathbb{R}$ satisfying $f(\pi) = 0$.*

But what can we say about Bricard's condition? It turns out that, if we suitably modify Bricard's condition, we can get a different necessary and sufficient condition for equidecomposability. With the help of Sydler's theorem we will prove the following:

Theorem 5. *Let $A^{(1)}$ and $A^{(2)}$ be polyhedra of equal volume and with dihedral angles $\alpha_1, \dots, \alpha_s$ and $\beta_1, \dots, \beta_r$, respectively, and let $l_1, \dots, l_s$ and $l'_1, \dots, l'_r$ be the lengths of the edges of $A^{(1)}$ and $A^{(2)}$ that are adjacent to those angles. Then $A^{(1)}$ and $A^{(2)}$ are equidecomposable if and only if* (a) *there exist $s + r + 1$ sequences of rational numbers $\{m_{1,k}\}_{k=1}^{\infty}, \dots, \{m_{s,k}\}_{k=1}^{\infty}, \{n_{1,k}\}_{k=1}^{\infty}, \dots, \{n_{r,k}\}_{k=1}^{\infty}, \{p_k\}_{k=1}^{\infty}$ such that*

$$m_{i,k} \to l_i, \quad n_{j,k} \to l'_j \quad (k \to \infty) \tag{13}$$

for each i in $\{1, \dots, s\}$ and each j in $\{1, \dots, r\}$ and (b) *the equation*

$$m_{1,k}\alpha_1 + \cdots + m_{s,k}\alpha_s = n_{1,k}\beta_1 + \cdots + n_{r,k}\beta_r + p_k\pi \tag{14}$$

holds for $k = 1, 2, 3, \dots$.

(We refer to the combined conditions (13) and (14) in this theorem as the *modified Bricard condition*.)

Proof. Suppose first that $A^{(1)}$ and $A^{(2)}$ are equidecomposable. The proof of the modified Bricard condition is basically the same as the proof of Bricard condition with a slight alteration.

Let $1 > \varepsilon_1 > \varepsilon_2 > \cdots$ be any sequence converging to zero, let $L_1, \dots, L_N$ be the links in (10) and (11), and let k be a fixed positive integer. By Lemma 1 we can find rational numbers $p_1^{(k)}/q^{(k)}, \dots, p_N^{(k)}/q^{(k)}$ with positive numerators and denominator such that

$$\left| l(L_i) - \frac{p_i^{(k)}}{q^{(k)}} \right| < \frac{\varepsilon_k}{2Nq^{(k)}} \quad (i = 1, \dots, N). \tag{15}$$

Then $m(L_i) = p_i^{(k)}$ is the integer measure of L_i. Following the proof of Theorem 1 verbatim we arrive at equation (1), keeping in mind that now in (1) the integers m_i and

n_j and p all depend on k ($m(L_i)$ also depends on k). We define

$$m_{i,k} = \frac{m_i}{q^{(k)}} \quad (i = 1, \ldots, s), \qquad n_{j,k} = \frac{n_j}{q^{(k)}} \quad (j = 1, \ldots, r), \qquad p_k = \frac{p}{q^{(k)}}.$$

For each positive integer k relation (1) takes the form

$$m_{1,k}\alpha_1 + \cdots + m_{s,k}\alpha_s = n_{1,k}\beta_1 + \cdots + n_{r,k}\beta_r + p_k\pi.$$

Let the dihedral angle α_1 adjoin the edge e_1 of the polyhedron $A^{(1)}$ (so $l(e_1) = l_1$). From the way we have established (1) it follows that $m_1 = \sum m(L_u)$, where the sum extends over all links L_u forming e_1. If in $\sum l(L_u)$ we sum over the same links, we conclude on the basis of the triangle inequality and (15) that

$$\left| l_1 - \frac{m_1}{q^{(k)}} \right| = \left| \sum l(L_u) - \frac{\sum m(L_u)}{q^{(k)}} \right| < N \frac{\varepsilon_k}{2Nq^{(k)}} < \varepsilon_k.$$

By letting $k \to \infty$ we see that $m_{1,k} = m_1/q^{(k)} \to l_1$. This argument works for any edge of $A^{(1)}$ or $A^{(2)}$, which proves (13). Accordingly, the modified Bricard condition holds.

To prove the other direction of Theorem 5, suppose that the modified Bricard condition is satisfied. We would like to show that $A^{(1)}$ and $A^{(2)}$ are equidecomposable.

By Theorem 4, it is enough to show that $f(A^{(1)}) = f(A^{(2)})$ holds for any additive function $f : V(M) \to \mathbb{R}$ satisfying $f(\pi) = 0$, where M is the set consisting of the number π and the dihedral angles of $A^{(1)}$ and $A^{(2)}$. From (14) we get

$$f(m_{1,k}\alpha_1 + \cdots + m_{s,k}\alpha_s) = f(n_{1,k}\beta_1 + \cdots + n_{r,k}\beta_r + p_k\pi),$$

so by the linearity of f

$$m_{1,k} f(\alpha_1) + \cdots + m_{s,k} f(\alpha_s) = n_{1,k} f(\beta_1) + \cdots + n_{r,k} f(\beta_r) + p_k f(\pi). \tag{16}$$

Here $f(\pi) = 0$ and $m_{i,k} \to l_i$ and $n_{j,k} \to l'_j$ as $k \to \infty$, whence $f(A^{(1)}) = f(A^{(2)})$. This completes the proof of Theorem 5. ∎

Remark. The necessity part of Theorem 5 is the following statement: if $A^{(1)}$ and $A^{(2)}$ are equidecomposable, then the modified Bricard condition is satisfied. Notice that this statement implies the Dehn-Hadwiger theorem in one line (see (16)).

6. DECOMPOSITIONS OF RECTANGLES. The method of integer measures can be applied successfully to certain other problems whose solutions involve additive functions. We provide an example.

Suppose that the rectangle R can be decomposed into finitely many rectangles with disjoint interiors and that these rectangles can be *translated* in such a way that we get a decomposition of another rectangle R'. In this event we say that the rectangles R and R' can be decomposed into each other by translations of rectangles.

Theorem 6. *Let R and R' be rectangles in parallel positions with dimensions $a \times b$ and $c \times d$, respectively. It is possible to decompose R into R' by translations of rectangles if and only if $ab = cd$ and a/c is rational.*

Proof. If $ab = cd$ and a/c is rational, then $a/c = d/b = u/v$, where u and v are positive integers, so $a/u = c/v$ and $b/v = d/u$. Therefore both rectangles can be decomposed into uv rectangles with dimensions $(a/u) \times (b/v)$. Thus the two rectangles R and R' can be decomposed into each other by translations of rectangles.

To prove the other direction, suppose that the two rectangles can be decomposed into each other by translations of rectangles:

$$R = [s, S] \times [t, T] = \cup_{i=1}^{n}[x_i, X_i] \times [y_i, Y_i],$$
$$R' = [s', S'] \times [t', T'] = \cup_{i=1}^{n}[x_i', X_i'] \times [y_i', Y_i'],$$

where the rectangles $[x_i, X_i] \times [y_i, Y_i]$ and $[x_i', X_i'] \times [y_i', Y_i']$ correspond to each other. We may assume that R and R' are in the upper half-plane, and that $[s, S] \cap [s', S'] = \emptyset$. Obviously R and R' must have the same area: $ab = cd$.

On the x-axis the points $x_1, X_1, \dots, x_n, X_n$ break $[s, S]$ up into subintervals $L_1, \dots, L_u$, while the points $x_1', X_1', \dots, x_n', X_n'$ partition $[s', S']$ into subintervals $L_{u+1}, \dots, L_N$. (By subintervals we mean closed and non-degenerate subintervals.) By Lemma 1 we can find positive integers $q, p_1, \dots, p_N$ such that

$$\left| l(L_i) - \frac{p_i}{q} \right| < \frac{1}{2Nq} \quad (i = 1, \dots, N).$$

As earlier we write $m(L_i) = p_i$ and call it the integer measure of the interval L_i. For a subset I of $\{1, \dots, N\}$ we extend the definition of m by defining

$$m(\cup_{i \in I} L_i) = \sum_{i \in I} m(L_i). \tag{17}$$

The important property of m is that it is "equality preserving," in the sense that for any subsets I_1 and I_2 of $\{1, \dots, N\}$, $m(\cup_{i \in I_1} L_i) = m(\cup_{j \in I_2} L_j)$ holds whenever $l(\cup_{i \in I_1} L_i) = l(\cup_{j \in I_2} L_j)$. This property follows from the idea presented at (8).

Since $l([x_i, X_i]) = l([x_i', X_i'])$ for each i, we have $m([x_i, X_i]) = m([x_i', X_i'])$, so the following two sums are equal:

$$\sum_{i=1}^{n} m([x_i, X_i]) l([y_i, Y_i]) = \sum_{i=1}^{n} m([x_i', X_i']) l([y_i', Y_i']). \tag{18}$$

If L_j $(j \in I^{(i)})$ denote the subintervals forming $[x_i, X_i]$, then

$$m([x_i, X_i]) = m(\cup_{j \in I^{(i)}} L_j) = \sum_{j \in I^{(i)}} m(L_j), \tag{19}$$

and

$$\sum_{i=1}^{n} m([x_i, X_i]) l([y_i, Y_i]) = \sum_{i=1}^{n} \sum_{j \in I^{(i)}} m(L_j) l([y_i, Y_i]) \tag{20}$$

We now interchange the order of summation. Fix an integer k in $\{1, \dots, u\}$. What is the coefficient of $m(L_k)$ in the double sum at (20)? Since k is in $I^{(i)}$ if and only if $L_k \subset [x_i, X_i]$, to obtain the coefficient of $m(L_k)$ we must add the heights $l([y_i, Y_i])$ of those rectangles $[x_i, X_i] \times [y_i, Y_i]$ that lie above L_k. Since this sum equals b we gain the following “additive property”:

$$\sum_{i=1}^{n} m([x_i, X_i])l([y_i, Y_i]) = \left(\sum_{k=1}^{u} m(L_k)\right)b. \tag{21}$$

Similarly,

$$\sum_{i=1}^{n} m([x_i', X_i'])l([y_i', Y_i']) = \left(\sum_{k=u+1}^{N} m(L_k)\right)d. \tag{22}$$

By (18), (21), and (22) d/b is rational, as is a/c $(= d/b)$. ■

Remark. The extension of m at (17) is not really necessary, for we can set up the double sum at (20) without it. But the extension of m lets us formulate (18), which makes the argument more elegant.

In 1903 Dehn proved the following statement [**7**]:

Corollary. *A rectangle with dimensions $a \times b$ can be tiled using finitely many squares if and only if a/b is rational.*

Proof. If a/b is rational then $a/u = b/v$ for positive integers u and v. Therefore the rectangle can be tiled with uv squares.

To prove the opposite direction assume that the rectangle is tiled with squares. Rotate the rectangle (together with its tiling) by 90°. The new rectangle has dimensions $b \times a$. Notice that the original rectangle and the new rectangle can be decomposed into each other by translations of rectangles (in fact, by using the squares of our tilings). Hence, by Theorem 6, a/b is rational. ■

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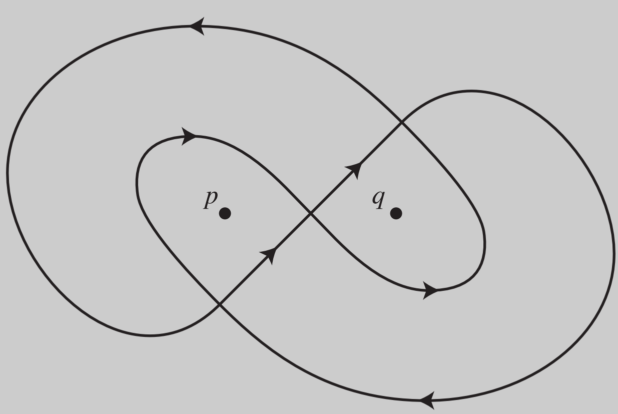
An Oral Exam Question



Let C be the contour shown above. At oral examinations of Ph.D. candidates, I often asked the following question: if f is analytic in $\mathbb{C} \setminus \{p, q\}$, is $\int_C f(z)\,dz = 0$? (Solution on p. 743)

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