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## A NEW APPROACH TO REPRESENTATION OF OBSERVABLES ON FUZZY QUANTUM POSETS

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*Summary.* We give a representation of an observable on a fuzzy quantum poset of type II by a pointwise defined real-valued function. This method is inspired by that of Kolesárová [6] and Mesiar [7], and our results extend representations given by the author and Dvurečenskij [4]. Moreover, we show that in this model, the converse representation fails, in general.

*Keywords:* Fuzzy quantum poset, fuzzy quantum space,  $q$ - $\sigma$ -algebra, observable

*AMS classification:* 81P15

### 1. INTRODUCTION

By a fuzzy set we understand a real-valued function  $a$  from a given non-void set  $\Omega$  into the interval  $[0, 1]$ , and we say that

$$\begin{aligned}\bigcap_i a_i &:= \inf_i a_i, \\ \bigcup_i a_i &:= \sup_i a_i, \\ a^\perp &:= 1 - a\end{aligned}$$

are the fuzzy intersection, the fuzzy union of the fuzzy sets  $a_i$ 's, and the fuzzy complement of the fuzzy set  $a$ , respectively.

Two models of fuzzy quantum posets were considered by A. Dvurečenskij, F. Chovanec, F. Kôpka and L. B. Long in [1, 2, 3, 4, 8], where two fuzzy sets  $a$  and  $b$  are said to be orthogonal, notation  $a \perp b$  iff  $a + b \leq 1$ , and fuzzy orthogonal, notation  $a \perp_F b$  iff  $a \cap b \leq \frac{1}{2}$ . (See also Mesiar [9].)

By a model I and model II of a fuzzy quantum poset we understand a couple  $(\Omega, M)$ , where  $\Omega$  is a non-void set and  $M \subset [0, 1]^\Omega$  is a system of fuzzy sets such that

- (i) If  $\mathbf{1}(\omega) = 1$  for any  $\omega \in \Omega$ , then  $\mathbf{1} \in M$ ;
- (ii) if  $a \in M$ , then  $a^\perp := 1 - a \in M$ ;
- (iii) if  $\frac{1}{2}(\omega) = \frac{1}{2}$  for any  $\omega \in \Omega$ , then  $\frac{1}{2} \notin M$ ;
- (iv) (model I) if  $\{a_n\}_{n=1}^\infty \subseteq M$ ,  $a_n \perp_F a_m$  for any  $n \neq m$  then  $\bigcup_{n=1}^\infty a_n \in M$ ;

(model II) if  $\{a_n\}_{n=1}^\infty \subseteq M$ ,  $a_n \perp a_m$  for any  $n \neq m$  then  $\bigcup_{n=1}^\infty a_n \in M$ .

If (iv) is replaced by a stronger form (iv)\*  $\bigcup_{n=1}^\infty a_n \in M$  for any sequence  $\{a_n\}_{n=1}^\infty \subseteq M$ , then  $(\Sigma, M)$  is said to be a fuzzy quantum space; this model was originally introduced by Riečan [12] as a new axiomatic model of quantum mechanics. Similarly, the model II was suggested by J. Pykacz [11].

It is obvious that a fuzzy quantum space is a fuzzy quantum poset, and a model I of a fuzzy quantum poset is a model II, but the converse is not true, in general, as we can see below.

Example 1. Put  $\Omega = [0, 1]$ . Consider

$$a(\omega) = \begin{cases} 0.7 & \text{if } 0 \leq \omega < 0.6 \\ 0.3 & \text{if } 0.6 \leq \omega \leq 1, \end{cases}$$

$$b(\omega) = \begin{cases} 0.4 & \text{if } 0 \leq \omega < 0.8 \\ 0.6 & \text{if } 0.8 \leq \omega \leq 1, \end{cases}$$

$$c = a \cup a^\perp; \quad d = b \cup b^\perp.$$

Put  $M = \{0, 1, a, b, c, d, a^\perp, b^\perp, c^\perp, d^\perp\}$ , then  $(\Omega, M)$  is a model II of a fuzzy quantum poset.

On the other hand, we see that  $a \perp_F b$  but  $a \cup b \notin M$ , hence  $(\Omega, M)$  is not a model I.

## 2. REPRESENTATIONS OF OBSERVABLES

We recall that an observable  $X$  on  $(\Omega, M)$  is a function from  $B(R)$ , the  $\sigma$ -algebra of Borel sets of the real numbers  $R$ , into  $M$  such that

- (i)  $X(E^c) = X(E)^\perp$  for any  $E \in B(R)$ .
- (ii)  $X(\bigcup_{i=1}^\infty E_i) = \bigcup_{i=1}^\infty X(E_i)$  for any  $E_i \in B(R)$ .

A simple example of an observable is a mapping  $X_a$ , where  $a$  is a fixed fuzzy element of  $M$ , defined by

$$(1) \quad X_a(E) = \begin{cases} a \cup a^\perp & \text{if } 0, 1 \in E \\ a & \text{if } 0 \notin E, 1 \in E \\ a^\perp & \text{if } 0 \in E, 1 \notin E \\ a \cap a^\perp & \text{if } 0, 1 \notin E \end{cases}$$

for any  $E \in B(R)$ .

$X_a$  plays the role of the indicator of the fuzzy set  $a \in M$ . If  $X$  is an observable on  $M$  and  $E, F \in B(R)$ ,  $E \cap F = \emptyset$ , then  $X(E) \perp X(F)$  as well as  $X(E) \perp_F X(F)$ .

Indeed, we have  $X(F^c) = X(E \cup (F^c \cap E^c)) = X(E) \cup X(F^c \cap E^c)$ , which entails the orthogonality of  $X(E)$  and  $X(F)$ .

Let us define

$$K(M) := \left\{ A \subseteq \Omega; \exists a \in M; \{a > \frac{1}{2}\} \subseteq A \subseteq \{a \geq \frac{1}{2}\} \right\},$$

where  $\{a > \frac{1}{2}\} := \{\omega \in \Omega; a(\omega) > \frac{1}{2}\}$ , analogously for  $\{a \geq \frac{1}{2}\}$ .

Let  $(\Omega, M)$  be a model I or II of fuzzy quantum posets. Let  $a$  be a given fuzzy set of  $M$ . Put

$$\begin{aligned} M_a &= \{b \in M; b \cup b^\perp = a \cup a^\perp\}, \\ \Omega_a &= \{\omega \in \Omega; a(\omega) \neq \frac{1}{2}\}, \\ \Omega_a(b) &= \{\omega \in \Omega_a; b(\omega) = (a \cup a^\perp)(\omega)\} \\ &= \{\omega \in \Omega_a; b(\omega) > \frac{1}{2}\}, \text{ for any } b \in M_a, \\ Q_a &= \{\Omega_a(b); b \in M_a\}. \end{aligned}$$

We see that if  $b \cup b^\perp = a \cup a^\perp$ , then  $a \perp b$  iff  $a \perp_F b$ . Therefore, Theorems and Lemmas 2.1, 2.2, 2.3, 2.4, 3.1, 3.2, 3.3 for model I proved in [4] (see also A. Dvurečenskij, F. Chovanec, F. Kôpka [2]) are still valid for model II.

We recall that  $\mathcal{C}$  is called a  $q$ - $\sigma$ -algebra of subsets of a non-void set  $\Omega$  if

- (i)  $\Omega \in \mathcal{C}$ ;
- (ii) if  $A \in \mathcal{C}$ , then  $\Omega - A \in \mathcal{C}$ ;
- (iii) if  $\{A_i\}_{i=1}^\infty \subseteq \mathcal{C}$ ,  $A_i \cap A_j = \emptyset$  for any  $i \neq j$ , then  $\bigcup_{i=1}^\infty A_i \in \mathcal{C}$ .

The following theorems for model II of fuzzy quantum posets can be proved by methods analogous to those in [4] and, therefore, their proofs are omitted.

**Theorem 2.** (i)  $Q_a$  is a  $q$ - $\sigma$ -algebra, for any  $a \in M$ .

(ii) The mapping  $\Omega_a(\cdot): M_a \rightarrow Q_a$  defined by  $b \mapsto \Omega_a(b)$  is a  $\sigma$ -orthoisomorphism, i.e., it is bijective and preserves the maximal elements, complements and joins of any sequences of mutually orthogonal elements.

**Theorem 3.** Let  $X$  be an observable on  $(\Omega, M)$ , then there is a unique function  $\varphi: \Omega_{X(R)} \rightarrow R$  such that  $\varphi$  is  $Q_{X(R)}$ -measurable and

$$(2) \quad \Omega_{X(R)}(X(E)) = \varphi^{-1}(E), \quad E \in B(R).$$

Conversely, for any  $Q_a$ -measurable mapping  $\varphi: \Omega_a \rightarrow R$ , where  $a \in M$ , there is a unique observable  $X$  of  $(\Omega, M)$  with  $X(R) = a \cup a^\perp$  such that (2) holds.

**Theorem 4.** Let  $X$  be an observable of a fuzzy quantum poset  $(\Omega, M)$ , and let  $Q$  be the set of all rational numbers. For any  $r \in Q$  denote  $B_X(r) = X((-\infty, r))$ . The system  $\{B_X(r); r \in Q\}$  fulfills the following conditions:

(i)  $B_X(s) < B_X(t)$  if  $s < t$ ;  $s, t \in Q$ ;

(ii)  $\bigcup_{r \in Q} B_X(r) = a$ ;  $\bigcap_{r \in Q} B_X(r) = a^\perp$ ;

(iii)  $\bigcup_{s < r} B_X(s) = B_X(r)$ ,  $r \in Q$ ;

(iv)  $B_X(r) \cup B_X(r)^\perp = a$ ,  $r \in Q$ ,

where  $a = X(R)$ ,  $a^\perp = X(\emptyset)$ .

Conversely, let  $\{B(r); r \in Q\}$  be a system of fuzzy sets from  $M$  fulfilling the conditions (i)–(iv) for some  $a \in M$ . Then there is a unique observable  $X$  on  $(\Omega, M)$  such that  $B_X(r) = B(r)$  for any  $r \in Q$  and  $X(R) = a$ .

A representation of fuzzy observables in model I of fuzzy quantum posets was given by the author and A. Dvurečenskij [4], and for fuzzy quantum spaces by A. Dvurečenskij [5]. In both cases the proofs have used an embeddings of  $M$  onto orthocomplemented,  $\sigma$ -orthocomplete, orthomodular poset and a Boolean  $\sigma$ -algebra  $M/J_0$  (see [4, 5]), respectively, and the representation of  $M/J_0$  by  $K(M)$ 's. An interesting direct method of representation of observables via pointwise,  $K(M)$ -measurable real-valued functions, has been presented by Kolesárová [6] and Kolesárová and Mesiar [7]. Applying this method we will give a representation of fuzzy observables on model II.

**Theorem 5.** Let  $(\Omega, M)$  be a model II of a fuzzy quantum poset, let  $X$  be an observable on  $M$ . Then there is a  $K(M)$ -measurable function  $f: \Omega \rightarrow R$  such that

$$(3) \quad \{X(E) > \frac{1}{2}\} \subseteq f^{-1}(E) \subseteq \{X(E) \geq \frac{1}{2}\}$$

for any Borel set  $E$ . Moreover, if  $g$  is any  $K(M)$ -measurable, real-valued function on  $\Omega$ , then  $g$  fulfills (3) iff

$$\{\omega \in \Omega; f(\omega) \neq g(\omega)\} \subseteq \{X(\emptyset) = \frac{1}{2}\}.$$

**Proof.** According to Kolesárová [6], for any given  $\omega$  from  $\Omega$  we consider  $X(\omega, \cdot): R \rightarrow [0, 1]$  defined by

$$T \mapsto X(\omega, t) = X((-\infty, t))(\omega).$$

From the properties of fuzzy observables (Theorem 4) it follows that  $X(\omega, \cdot)$  is a non-decreasing function with two values,  $X(R)(\omega)$  or  $1 - X(R)(\omega)$ . Therefore, there exists  $a_\omega \in R$  such that

$$X(\omega, t) = \begin{cases} X(R)(\omega) & \text{if } t > a_\omega, \\ 1 - X(R)(\omega) & \text{if } t \leq a_\omega, \end{cases}$$

$a_\omega = \sup\{t \in R; X(\omega, t) = 1 - X(R)(\omega)\}$  if  $X(R)(\omega) \neq \frac{1}{2}$ . In the case  $X(R)(\omega) = \frac{1}{2}$ ,  $a_\omega$  can be chosen arbitrarily.

It is clear that  $X((-\infty, a_\omega])(\omega) = X((-\infty, a_\omega))(\omega) \cup X(\{a_\omega\})(\omega)$ . Thus

$$X(\{a_\omega\})(\omega) = X(R)(\omega).$$

Further,  $X(E)(\omega)$  assumes only two values,  $X(R)(\omega)$  or  $1 - X(R)(\omega)$ . Hence  $X$  can be written in the form

$$(4) \quad X(E)(\omega) = \begin{cases} X(R)(\omega) & \text{if } a_\omega \in E \\ 1 - X(R)(\omega) & \text{if } a_\omega \notin E; \end{cases}$$

Now we consider a function  $f: \Omega \rightarrow R$  defined by

$$\omega \mapsto f(\omega) = a_\omega.$$

We claim to prove that  $f$  fulfills the conditions of the theorem. To prove that, it suffices to verify that

$$\{X(E) > \frac{1}{2}\} \subseteq f^{-1}(E) \subseteq \{X(E) \geq \frac{1}{2}\}.$$

This is straightforward from the definition of  $f$  and (4).

Now, let  $g$  be a  $K(M)$ -measurable function with the condition (3). Then

$$\begin{aligned} \{\omega \in \Omega; f(\omega) < g(\omega)\} &= \bigcup_{r \in Q} \{\omega \in \Omega; f(\omega) < r < g(\omega)\} \\ &= \bigcup_{r \in Q} \{\omega \in \Omega; f(\omega) < r\} \cap \{\omega \in \Omega; g(\omega) > r\} \\ &\subseteq \bigcup_{r \in Q} \{\omega \in \Omega; f(\omega) < r\} \cap \{\omega \in \Omega; g(\omega) \geq r\} \\ &\subseteq \bigcup_{r \in Q} \{\omega \in \Omega; X((-\infty, r) \cap [r, \infty))(\omega) \geq \frac{1}{2}\} \\ &= \{\omega \in \Omega; X(\emptyset)(\omega) = \frac{1}{2}\}, \end{aligned}$$

where  $Q$  is the set of all rational numbers. Similarly,  $\{\omega \in \Omega; f(\omega) > g(\omega)\} \subseteq \{\omega \in \Omega; X(\emptyset)(\omega) = \frac{1}{2}\}$ . Thus  $\{\omega \in \Omega; f(\omega) \neq g(\omega)\} \subseteq \{\omega \in \Omega; X(\emptyset)(\omega) = \frac{1}{2}\}$ . Conversely, if  $g$  is a  $K(M)$ -measurable function from  $\Omega$  into  $R$  such that  $A := \{\omega \in \Omega; f(\omega) \neq g(\omega)\} \subseteq \{\omega \in \Omega; X(\emptyset)(\omega) = \frac{1}{2}\}$  we claim to verify the condition (3). It is clear that if  $X(E)(\omega) > \frac{1}{2}$  then  $\omega \notin A$ . So  $\omega \in f^{-1}(E) \cap A^c = g^{-1}(E) \cap A^c$ , this means that  $\{X(E) > \frac{1}{2}\} \subseteq g^{-1}(E)$ . On the other hand, if  $\omega \in g^{-1}(E)$ , there are two cases:

- (i)  $\omega \in A$ , then  $X(\emptyset)(\omega) = \frac{1}{2} = X(E)(\omega)$ ;
- (ii)  $\omega \notin A$ , Then  $\omega \in f^{-1}(E)$ , therefore  $X(E)(\omega) \geq \frac{1}{2}$  which entails  $g^{-1}(E) \subseteq \{X(E) \geq \frac{1}{2}\}$ .  $\square$

The converse of Theorem 5 for model I of fuzzy quantum posets was proved in [4], but it is not true for model II, in general, as we show below.

**Counterexample 6.** Let  $\Omega = [0, 1]$ ,  $a, b, c, d$  be as in Example 1.

$$\text{Put } e(\omega) = \begin{cases} 0.1 & \text{if } 0 \leq \omega < 0.6 \text{ or } 0.8 \leq \omega \leq 1, \\ 0.9 & \text{if } 0.6 \leq \omega < 0.8; \end{cases}$$

$$f(\omega) = 0.9 \text{ for } 0 \leq \omega \leq 1;$$

and  $M = \{0, 1, a, b, c, d, e, f, a^\perp, b^\perp, c^\perp, d^\perp, e^\perp, f^\perp\}$ . Then  $(\Omega, M)$  is a type II of fuzzy quantum posets.

We see that  $K(M) = \{\emptyset, \Omega, A, B, C, A^c, B^c, C^c\}$ , where  $A = [0, 0.6]$ ;  $B = [0.8, 1]$ ;  $C = [0.6, 0.8]$ . Hence  $K(M)$  is a  $\sigma$ -algebra. Therefore, the function  $h = I_{A^c} + 2I_{B^c}$ , where  $I_{A^c}, I_{B^c}$  are indicators of the sets  $A^c$  and  $B^c$ , respectively, is  $K(M)$ -measurable and such that

$$h^{-1}(\{1\}) = A^c - C = B, h^{-1}(\{2\}) = B^c - C = A, h^{-1}(\{3\}) = C.$$

On the other hand, there exist unique  $a, b$  and  $e$  such that

$$\begin{aligned} \{a > \frac{1}{2}\} &\subseteq A \subseteq \{a \geq \frac{1}{2}\}, \\ \{b > \frac{1}{2}\} &\subseteq B \subseteq \{b \geq \frac{1}{2}\}, \\ \{e > \frac{1}{2}\} &\subseteq C \subseteq \{e \geq \frac{1}{2}\}, \end{aligned}$$

but  $a \cup a^\perp \neq b \cup b^\perp \neq e \cup e^\perp \neq a \cup a^\perp$ . Therefore, there exists no observable  $X$  on  $(\Omega, M)$  such that (3) holds.  $\square$

For any sequence  $\{a_n\}_{n=1}^\infty$  of fuzzy sets of a model II fuzzy quantum poset  $(\Omega, M)$  there exist  $1_K = \bigcap_{n=1}^\infty (a_n \cup a_n^\perp) \in M$  and  $0_K = 1_K^\perp \in M$ . However, in general  $a_n \cap 1_K \cup 0_K$  does not belong to  $M$  if  $(\Omega, M)$  is not a type I, which entails that the converse of Theorem 5 fails for type II, in general. Nevertheless, we have a converse of Theorem 5 in the following case.

**Theorem 7.** *Let  $(\Omega, M)$  be a model II fuzzy quantum poset such that*

$$(5) \quad a_n \cap \left( \bigcup_{m=1}^\infty (a_m \cup a_m^\perp) \right) \in M$$

for any  $n \geq 1$  and any sequence  $\{a_n\}_{n=1}^\infty$  of  $M$ . If  $f: \Omega \rightarrow R$  is any  $K(M)$ -measurable function, then there exists an observable  $X$  of  $(\Omega, M)$  with (3). If  $Y$  is any observable of  $(\Omega, M)$  with (3), then  $X(E) \perp_F Y(E^C)$  for any  $E \in B(R)$ .

The theorem can be proved by a method analogous to the proof of Theorem 5.2 in [4]. Therefore, the proof is omitted.

It should be noted that a model II of a fuzzy quantum poset with the condition (5) need not be a model I, see the following example.

**Example 8.** Let  $\Omega, a, b, c, d$ , be as in Example 1.

$$\text{Put } g(\omega) = \begin{cases} 0.6 & \text{if } 0 \leq \omega < 0.6, \\ 0.4 & \text{if } 0.6 \leq \omega \leq 1, \end{cases}$$

put  $M = \{0, 1, a, b, c, d, g, a^\perp, b^\perp, c^\perp, d^\perp, g^\perp, g \cup b, g^\perp \cap b^\perp\}$ . Then  $(\Omega, M)$  is a model II of a fuzzy quantum poset with (5) but it is not a model I.

**Theorem 9.** *Let  $X$  be an observable of a model II of fuzzy quantum poset  $(\Omega, M)$  and let  $\varphi: \Omega_{X(R)} \rightarrow R$  be a unique  $Q_{X(R)}$ -measurable function on  $\Omega$ . Then  $f$  fulfills*



condition (3) of Theorem 5 iff

$$f(\omega) = \begin{cases} \varphi(\omega) & \text{if } \omega \in \Omega_{X(R)}, \\ \varphi_0(\omega) & \text{if } \omega \in \Omega - \Omega_{X(R)}, \end{cases}$$

where  $\varphi_0$  is any mapping from  $\Omega - \Omega_{X(R)}$  into  $R$ .

Proof. Theorem is proved by a method similar to the proof of Theorem 5.4 [4].

□

Following the ideas from [7], we arrive at the following result.

**Theorem 10.** Let  $X$  be an observable on a model  $\Pi$  of a fuzzy quantum poset  $(\Omega, M)$ . Then there is a  $K(M)$ -measurable, real-valued function  $f$  and a fuzzy set  $c \in W_1(M)$  such that

$$(6) \quad X(E)(\omega) = \begin{cases} c(\omega) & \text{if } \omega \in f^{-1}(E) \\ 1 - c(\omega) & \text{if } \omega \notin f^{-1}(E) \end{cases}$$

for any  $E \in B(R)$ .

Conversely, if  $f$  is a  $K(M)$ -measurable real-valued function and  $c \in W_1(M)$  is such that for any  $E \in C$  the right-hand side of (6) determines fuzzy sets  $X(E)$  from  $M$ , where  $C$  is a  $\sigma$ -countable generator of  $B(R)$  which is closed with respect to finite intersection, for example  $C = \{(-\infty, r); r \in Q\}$ , then (6) defines a unique observable  $X$  of  $(\Omega, M)$ .

Proof. If  $X$  is an observable, then from Theorem 5 we have an  $f: \Omega \rightarrow R$  such that (3) holds. If we put  $c = X(R)$ , then (6) is true.

Conversely, let  $B$  be the set of all  $E \in B(R)$  such that (6) defines  $X(E) \in M$ .

Then

- (i)  $\emptyset, R \in B, C \subseteq B$ ;
- (ii) if  $E \in B$ , then  $E^c \in B$  and  $X(E^c) = X(E)^\perp$ ;
- (iii) if  $E, F \in C$  then  $E \cap F \in C \subseteq B$ ;
- (iv) if  $E, F \in B, E \cap F = \emptyset$  then  $X(E) \perp X(F)$ ;
- (v) if  $\{E_i\}_{i=1}^\infty \in B, E_i \cap E_j = \emptyset, i \neq j$ , then

$$X\left(\bigcup_{i=1}^\infty E_i\right) = \bigcup_{i=1}^\infty X(E_i) \in M.$$

Hence,

$$\bigcup_{i=1}^\infty E_i \in B.$$

Due to Proposition 4.13 and Theorem 4.20 by Neubrunn and Riečan [10], we see that  $B = B(R)$ . Therefore,  $X$  is an observable on  $(\Omega, M)$ .  $\square$

**Remark.** If  $(\Omega, M)$  is a type I of fuzzy quantum posets, then for any  $K(M)$ -measurable function  $f: \Omega \rightarrow R$  there exists  $c \in W_1(M)$  such that (6) always defines an observable  $X$  of  $(\Omega, M)$ .

### 3. SUMMABILITY OF OBSERVABLES

Let  $X, Y$  be two observables on a model I, II of fuzzy a quantum poset  $(\Omega, M)$ . Let  $Q$  be the set of all rational numbers.

$$\text{Put } B_z(t) = \bigcup_{r \in Q} B_x(r) \cap B_y(t-r), \quad t \in Q.$$

We see that if there exists  $B_z(t) \in M$  for any  $t \in Q$ , then  $\{B_z(t); t \in Q\}$  fulfills the conditions of Theorem 3. Therefore, there exists a unique observable called the sum of  $X$  and  $Y$  and we write  $Z = X + Y$ ;  $X$  and  $Y$  are said to be summable.

Let  $X$  be an observable on  $M$ . We write  $X \sim f$  and  $X \approx \varphi_X$  if  $f$  is defined by Theorem 4 and  $\varphi_X$  by Theorem 3.

**Proposition 11.** *Let  $X, Y$  be two observables on a model I, II of a fuzzy quantum poset  $(\Omega, M)$  and let  $X \sim f, Y \sim g, X \approx \varphi_X, Y \approx \varphi_Y$ . If  $X$  and  $Y$  are summable then  $f+g$  is also  $K(M)$ -measurable and  $X+Y \sim f+g, (X+Y)(R) = X(R) \cap Y(R)$ . Therefore  $\varphi_X + \varphi_Y$  is  $Q_{(X+Y)(R)}$ -measurable and  $X+Y \approx \varphi_X + \varphi_Y$ .*

**Proof.** Let  $X+Y \sim h$ . We see that:

$$\{B_{X+Y}(t) > \frac{1}{2}\} \subseteq \{h < t\} \subseteq \{B_{X+Y}(t) \geq \frac{1}{2}\}.$$

On the other hand, for any  $t, r \in Q$ .

$$\{B_X(r) \cap B_Y(t-r) > \frac{1}{2}\} \subseteq \{f < t\} \cap \{g < t-r\} \subseteq \{B_X(r) \cap B_Y(t-r) \geq \frac{1}{2}\}.$$

Hence

$$\begin{aligned} \left\{ \bigcup_{r \in Q} B_X(r) \cap B_Y(t-r) > \frac{1}{2} \right\} &\subseteq \bigcup_{r \in Q} \{f < t\} \cap \{g < t-r\} \\ &\subseteq \left\{ \bigcup_{r \in Q} B_X(r) \cap B_Y(t-r) \geq \frac{1}{2} \right\}, \end{aligned}$$

i.e.  $\{B_{X+Y}(t) > \frac{1}{2}\} \subseteq \{f + g < t\} \subseteq \{B_{X+Y}(t) \geq \frac{1}{2}\}$ . Therefore

$$\begin{aligned} \{f + g < h\} &\subseteq \bigcup_{r \in Q} \{f + g < r\} \cap \{h > r\} \\ &\subseteq \bigcup_{r \in Q} \{B_{X+Y}(t) \geq \frac{1}{2}\} \cap \{B_{X+Y}(t) \geq \frac{1}{2}\}^c \\ &\subseteq \bigcup_{r \in Q} \{B_{X+Y}(t) = \frac{1}{2}\} = \{(X + Y)(R) = \frac{1}{2}\} \end{aligned}$$

In the same way, we see also that  $\{f + g > h\} \subseteq \{(X + Y)(R) = \frac{1}{2}\}$ . Hence  $A := \{h = f + g\} \subseteq \{(X + Y)(R) = \frac{1}{2}\}$ . Thus  $A^c \supseteq \{(X + Y)(E) > \frac{1}{2}\}$  for any Borel set  $E$ . We see that  $(f + g)^{-1}(E) \cap A \in K(M)$  for any Borel set  $E$ . So  $(f + g)^{-1}(E) \cap A^c = h^{-1}(E) \cap A^c$

$$\{(X + Y)(E) > \frac{1}{2}\} \subseteq h^{-1}(E) \cap A^c \subseteq \{(X + Y)(E) \geq \frac{1}{2}\}.$$

On the other hand,

$$(f + g)^{-1}(E) = ((f + g)^{-1}(E) \cap A) \cup ((f + g)^{-1}(E) \cap A^c).$$

Hence  $\{(X + Y)(E) > \frac{1}{2}\} \subseteq (f + g)^{-1}(E) \subseteq \{(X + Y)(E) \geq \frac{1}{2}\}$  for any Borel set  $E$ , which entails  $K(M)$ -measurability of  $f + g$  and  $X + Y \sim f + g$ .

It can be proved that if  $X, Y$  are summable then  $(X + Y)(R) = X(R) \cap Y(R)$ . Therefore,  $\varphi_X + \varphi_Y$  is  $Q_{(X+Y)(R)}$ -measurable and  $X + Y \approx \varphi_X + \varphi_Y$ , by Theorem 9.  $\square$

As we can see from the following, the sum of two observables on a fuzzy quantum poset need not always exist.

**Example 12.** Let  $\Omega = [0, 1]$ . Put  $M = \{0, 1, a_0, b_0, c, d, a_0^\perp, b_0^\perp, c^\perp, d^\perp\}$ , where

$$\begin{aligned} a_0(\omega) &= \begin{cases} 0.7 & \text{if } 0 \leq \omega < 0.6, \\ 0.3 & \text{if } 0.6 \leq \omega \leq 1, \end{cases} \\ b_0(\omega) &= \begin{cases} 0.4 & \text{if } 0 \leq \omega < 0.8, \\ 0.6 & \text{if } 0.8 \leq \omega \leq 1, \end{cases} \end{aligned}$$

$$\begin{aligned} c(\omega) &= 0.7 \quad 0 \leq \omega \leq 1, \\ d(\omega) &= 0.6 \quad 0 \leq \omega \leq 1. \end{aligned}$$

Then  $(\Omega, M)$  is a model II of a fuzzy quantum poset, and

$$K(M) = (\emptyset, \Omega, A, B, A^c, B^c),$$

where  $A = [0, 0.6]$ ;  $B = [0.8, 1]$ .  $K(M)$  consists only from two proper sub  $\sigma$ -algebras  $\{\emptyset, \Omega, A, A^c\}$ ,  $\{\emptyset, \Omega, B, B^c\}$ . Therefore  $f: \Omega \rightarrow R$  is  $K(M)$ -measurable iff  $f = \alpha I_A + \beta I_{A^c}$  or  $f = \gamma I_B + \delta I_{B^c}$ , where  $\alpha, \beta, \gamma, \delta \in R$ .

Let  $X_{a_0}, X_{b_0}$  be two observables on  $(\Omega, M)$  defined via (1). Then it is clear that  $X_{a_0} \sim I_A, X_{b_0} \sim I_B$ . But  $I_A + I_B$  is not  $K(M)$ -measurable. Therefore, it follows from Proposition 10 that  $X_{a_0}$  and  $X_{b_0}$  are not summable on  $(\Omega, M)$ .  $\square$

**Definition 12.** Let  $(\Omega, M)$  be a model I or II of a fuzzy quantum poset. Let  $X_i \sim f_i$  for  $i = 1, 2, \dots, N$ , where  $N$  can be either integer or  $\infty$ , and let  $F: R^N \rightarrow R$  be a Borel measurable function. We define  $F(X_1, \dots, X_N)$  as any observable  $X$  of  $(\Omega, M)$  such that

- (i)  $F(f_1, \dots, f_N)$  is  $K(M)$ -measurable;
- (ii)  $X \sim F(f_1, \dots, f_N)$ ;
- (iii)  $X(R) = \bigcap_{i=1}^N X_i(R)$ .

It is easy to verify that such an  $X$  is unique. We recall that if  $N = \infty$ , then by  $F$  we mean some limit expressions, or convergence, respectively.

This definition enables us to define a calculus of observables. For example, let  $F(u, v) = u + v$ ;  $u, v \in R$ . If  $X$  and  $Y$  are summable and  $X \sim f, Y \sim g$ , then  $X + Y \sim F(f, g)$ , i.e.  $X + Y = F(X, Y)$ . If we consider  $G(u, v) = u.v$ ;  $u, v \in R$  and there exists  $X.Y$  then  $X.Y \sim G(f, g), X.Y = G(X, Y)$ .

If  $(\Omega, M)$  is a quantum space, the conditions of Definition 12 are fulfilled for any  $F: R^N \rightarrow R$ . Consequently, in this case we can always define  $F(X_1, \dots, X_N)$ . In a model I, II of fuzzy quantum posets, this is not true, in general. (It suffices take into account the summation.)

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