A new approach to Sobolev spaces and connections to Γ-convergence

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Abstract

We study how the existence of the limit

$$\int_\Omega\int_\Omega \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_\varepsilon(x-y)\,dx\,dy \quad \text{as } \varepsilon \downarrow 0 \tag{$*$}$$

for $\omega : [0,\infty) \to [0,\infty)$ continuous and $(\rho_\varepsilon) \subset L^1(\mathbb{R}^N)$ converging to δ_0 is related to the weak regularity of $f \in L^1_{\text{loc}}(\Omega)$. This approach gives an alternative way of defining the Sobolev spaces $W^{1,p}$. We also briefly discuss the Γ-convergence of $(*)$ with respect to the $L^1(\Omega)$-topology.

1 Introduction and main results.

Let $\Omega \subset \mathbb{R}^N$ be an open set such that $\partial\Omega$ is compact and Lipschitz. Given a function $f \in L^1_{\text{loc}}(\Omega)$ we consider the functional

$$\int_\Omega\int_\Omega \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_\varepsilon(x-y)\,dx\,dy, \tag{1.1}$$

where $\omega : [0,\infty) \to [0,\infty)$ is continuous and $(\rho_\varepsilon) \subset L^1(\mathbb{R}^N)$ is a family of functions satisfying the following properties

$$\begin{cases} \rho_\varepsilon \geq 0 \quad \text{a.e. in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} \rho_\varepsilon = 1 \quad \forall \varepsilon > 0, \\ \displaystyle\lim_{\varepsilon\downarrow 0} \int_{|h|>\delta} \rho_\varepsilon(h)\,dh = 0 \quad \forall \delta > 0. \end{cases} \tag{1.2}$$

We show that in general there exists a sequence $\varepsilon_j \downarrow 0$ for which the pointwise limit in (1.1) exists. By imposing an extra condition on (ρ_ε) we obtain new characterizations for the Sobolev spaces $W^{1,p}$, $1 \leq p < \infty$, and BV. At the end we prove the Γ-convergence of (1.1). As we will see our results can also be used to get some information about noncoercive functionals.

We have been inspired by the simplified proofs presented in [3], following a suggestion of E. Stein (see Lemma 5.4). Our approach not only unifies the proofs of some well-known results, including in the BV-case, but it also deals with more general families $(\rho_\varepsilon) \subset L^1(\mathbb{R}^N)$ (see [1, 2, 3, 6, 10, 11, 14]).

1.1 Construction of the subsequence $\varepsilon_j \downarrow 0$.

We start with a family of functions (ρ_ε) in $L^1(\mathbb{R}^N)$ satisfying (1.2).

To each $\varepsilon > 0$ we associate the positive Radon measure μ_ε on S^{N-1} defined by

$$\mu_\varepsilon(E) := \int_{\mathbb{R}_+ E} \rho_\varepsilon \quad \text{for each Borel set } E \subset S^{N-1}, \tag{1.3}$$

where $\mathbb{R}_+ E := \{rx : r \geq 0 \text{ and } x \in E\}$ is the cone generated by E with respect to the origin.

The family (μ_ε) is bounded in $M(S^{N-1})$ (the space of Radon measures on S^{N-1}), so there exist a sequence $\varepsilon_j \downarrow 0$ and $\mu \in M(S^{N-1})$ such that

$$\mu_{\varepsilon_j} \rightharpoonup \mu \quad \text{in } M(S^{N-1}). \tag{1.4}$$

In particular, $\mu \geq 0$ on S^{N-1} and $\mu(S^{N-1}) = 1$.

In Section 3 we present some examples of admissible families (ρ_ε) for which the measure μ can be written down explicitly from the construction above.

Given $\omega : [0, \infty) \to [0, \infty)$ continuous, we define

$$\omega_\mu(v) := \int_{S^{N-1}} \omega\big(|v \cdot \sigma|\big)\, d\mu(\sigma) \quad \forall v \in \mathbb{R}^N. \tag{1.5}$$

1.2 The pointwise limit of (1.1) as $\varepsilon_j \downarrow 0$.

With the particular choice $\omega(t) = t^p$, for some $p \geq 1$, we have

Theorem 1.1 *If $f \in W^{1,p}(\Omega)$, $p \geq 1$, then there exists $C > 0$ such that*

$$\int_\Omega\int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_\varepsilon(x - y)\, dx\, dy \leq C \quad \forall \varepsilon > 0. \tag{1.6}$$

Moreover,

$$\lim_{j\to\infty} \int_\Omega\int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_{\varepsilon_j}(x - y)\, dx\, dy = \int_\Omega \left(\int_{S^{N-1}} |Df \cdot \sigma|^p\, d\mu(\sigma) \right).$$

The case $p = 1$ can be further extended to include the case of BV-functions:

Theorem 1.2 *If $f \in BV(\Omega)$, then there exists $C > 0$ such that*

$$\int_\Omega\int_\Omega \frac{|f(x) - f(y)|}{|x - y|} \rho_\varepsilon(x - y)\, dx\, dy \leq C \quad \forall \varepsilon > 0. \tag{1.7}$$

In addition, we have

$$\lim_{j\to\infty} \int_\Omega\int_\Omega \frac{|f(x) - f(y)|}{|x - y|} \rho_{\varepsilon_j}(x - y)\, dx\, dy = \int_{S^{N-1}} \left(\int_\Omega |Df \cdot \sigma| \right) d\mu(\sigma).$$

(We point out that the right-hand side of the identity above is well-defined since the function $\sigma \in S^{N-1} \mapsto \int_\Omega |Df \cdot \sigma|$ is continuous).

There are special choices of (ρ_ε) which give some very interesting expressions (see Section 3, and also [3]). Taking for instance ρ_ε to be a radial function for each $\varepsilon > 0$, we get the following (see [2, 6])

Corollary 1.3 *Suppose that ρ_ε is radial for each $\varepsilon > 0$.*
If $f \in W^{1,p}(\Omega)$, $p > 1$, or if $f \in BV(\Omega)$ and $p = 1$, then

$$\lim_{\varepsilon \downarrow 0} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_\varepsilon\big(|x-y|\big)\, dx\, dy = K_{p,N} \int_\Omega |Df|^p, \tag{1.8}$$

where $K_{p,N} = \fint_{S^{N-1}} |e_1 \cdot \sigma|^p \, d\mathcal{H}^{N-1}$.

Choosing the family (ρ_ε) of the form $\rho_\varepsilon(x) := \frac{1}{\varepsilon^N} \rho\left(\frac{x}{\varepsilon}\right)$ for some fixed nonnegative function $\rho \in L^1(\mathbb{R}^N)$, we obtain the following limit (see [11]):

Corollary 1.4 *Let $\rho \in L^1(\mathbb{R}^N)$, $\rho \geq 0$ a.e. in $\mathbb{R}^N$.*
If $f \in W^{1,p}(\Omega)$, $p > 1$, or if $f \in BV(\Omega)$ and $p = 1$, then

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^N} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho\Big(\frac{x-y}{\varepsilon}\Big)\, dx\, dy = \int_{\mathbb{R}^N} \bigg(\int_\Omega \Big| Df \cdot \frac{z}{|z|} \Big|^p \bigg) \rho(z)\, dz.$$

In the special case where $\Omega = \mathbb{R}^N$, by a simple change of variables we may rewrite the above identity as

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x + \varepsilon h) - f(x)|^p}{|\varepsilon h|^p} \rho(h)\, dx\, dh = \int_{\mathbb{R}^N} \bigg(\int_{\mathbb{R}^N} \Big| Df \cdot \frac{z}{|z|} \Big|^p \bigg) \rho(z)\, dz.$$

We can also take families (ρ_ε) which privilege certain directions. Let for instance

$$\rho_\varepsilon := \frac{1}{2^N \varepsilon^{2N-1}} \chi_{(-\varepsilon,\varepsilon)\times(-\varepsilon^2,\varepsilon^2)^{(N-1)}};$$

we have (see Example 3.3)

Corollary 1.5 *If $f \in W^{1,p}(\Omega)$, $p > 1$, or if $f \in BV(\Omega)$ and $p = 1$, then*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{2N-1}} \int_\Omega \int_{\substack{\Omega \\ |x_1 - y_1| < \varepsilon \\ |x_i - y_i| < \varepsilon^2 \\ i = 2, \ldots, N}} \frac{|f(x) - f(y)|^p}{|x-y|^p}\, dx\, dy = 2^N \int_\Omega \left| \frac{\partial f}{\partial x_1} \right|^p. \tag{1.9}$$

Assuming that ω is asymptotic linear at infinity we obtain the following result which extends Theorem 1.2:

Theorem 1.6 *Let $\omega : [0,\infty) \to [0,\infty)$ be a continuous function satisfying*

$$\omega^\infty := \lim_{t\to\infty} \frac{\omega(t)}{t} \in [0,\infty). \tag{1.10}$$

If $\Omega \subset \mathbb{R}^N$ is unbounded, suppose in addition that there exists $C > 0$ such that

$$|\omega(t)| \le Ct \quad \forall t \ge 0. \tag{1.11}$$

If $f \in BV(\Omega)$, then

$$\begin{aligned}\lim_{j\to\infty} \int_\Omega\int_\Omega \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_{\varepsilon_j}(x-y)\,dx\,dy = \\ = \int_\Omega \omega_\mu(D^{\mathrm{a}}f) + \omega^\infty \int_{S^{N-1}} \left(\int_\Omega |D^{\mathrm{s}}f\cdot\sigma|\right) d\mu(\sigma),\end{aligned} \tag{1.12}$$

where $Df = D^{\mathrm{a}}f\,\mathcal{L}^N + D^{\mathrm{s}}f$ is the Radon-Nikodym decomposition of Df with respect to the Lebesgue measure.

Remark 1.1 The results in this section rely heavily on the Lipschitz regularity of $\partial\Omega$. In fact, take for instance $N = 2$ and $\Omega := B_1(0)\backslash\big\{(x_1,0) : 0 \le x_1 < 1\big\}$. On Ω one can easily construct a smooth function $f \in W^{1,p}(\Omega)$ such that

$$\lim_{\substack{x_2\downarrow 0\\ \frac12<x_1<1}} f(x_1,x_2) = 1 \quad \text{and} \quad \lim_{\substack{x_2\uparrow 0\\ \frac12<x_1<1}} f(x_1,x_2) = 0.$$

However, taking (ρ_ε) radial we have

$$\lim_{\varepsilon\downarrow 0} \int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_\varepsilon\big(|x-y|\big)\,dx\,dy = +\infty. \tag{1.13}$$

See Theorem 1.8; note that $f \notin W^{1,p}(B_1)$ while the integral above is actually being computed on $B_1 \times B_1$ (since $\big\{(x_1,0) : 0 \le x_1 < 1\big\}$ is a null set in $\mathbb{R}^2$). See also Remark 1.4.

We conclude this section with the following generalization of Theorem 1.1:

Theorem 1.7 *Assume $\tilde\omega : [0,\infty) \to [0,\infty)$ is convex and increasing, and let $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$ be such that $\tilde\omega\big(|Df|\big) \in L^1(\mathbb{R}^N)$.*

For any continuous function ω,

$$0 \le \omega(t) \le \tilde\omega(t) \quad \forall t \ge 0, \tag{1.14}$$

we have

$$\lim_{j\to\infty} \int_\Omega\int_\Omega \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_{\varepsilon_j}(x-y)\,dx\,dy = \int_\Omega \omega_\mu(Df). \tag{1.15}$$

Remark 1.2 The assumption $\tilde{\omega}(|Df|) \in L^1(\mathbb{R}^N)$ was made just for simplicity. In fact, if $\tilde{\omega}(|Df|) \in L^1(\Omega)$ we can always extend u to $\mathbb{R}^N$ so that $\tilde{\omega}(\alpha|Df|) \in L^1(\mathbb{R}^N)$ for some $\alpha > 0$ (which may be much smaller than 1).

Remark 1.3 There are several important functionals which cannot be pointwise approximated by using (1.1). An example is the Mumford-Shah functional

$$MS(f) := \lambda_1 \int_{\mathbb{R}^N} |D^{\mathrm{a}} f|^2 + \lambda_2 \mathcal{H}^{N-1}(S_f) \quad \forall f \in SBV(\mathbb{R}^N) \subset BV(\mathbb{R}^N),$$

where $\mathcal{H}^{N-1}(S_f)$ denotes the $(N-1)$-dimensional Hausdorff measure of the set of essential discontinuity of f and $\lambda_1, \lambda_2 > 0$ are constants.

Nevertheless, it can be approximated by (see [11, Example 7.4])

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \omega_{|x-y|}\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_\varepsilon(x-y)\,dx\,dy, \tag{1.16}$$

where

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^N}\rho\Big(\frac{x}{\varepsilon}\Big), \quad \rho \in L^1(\mathbb{R}^N) \text{ radial, } \rho \geq 0 \text{ a.e.} \tag{1.17}$$

and

$$\omega_\tau(s) = \frac{1}{\tau}\arctan(\tau s^2) \quad \forall s \geq 0 \quad \forall \tau > 0. \tag{1.18}$$

It would be interesting to study the pointwise (and also the Γ-) convergence of (1.16) for more general families of continuous functions (ω_τ) but **especially** for any (ρ_ε) which satisfy (1.2).

If (ρ_ε) is an arbitrary family of radial functions, the convergence of (1.16) seems to be unknown even in the special case where (ω_τ) is given by (1.18) (see however [11]).

1.3 Some new characterizations of $W^{1,p}$, $p \geq 1$, and BV.

By our previous results, we know that if $u \in W^{1,p}(\Omega)$ and $1 < p < \infty$ then

$$\limsup_{\varepsilon \downarrow 0} \int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_\varepsilon(x-y)\,dx\,dy < +\infty \tag{1.19}$$

(if $p = 1$ then $W^{1,1}(\Omega)$ may be replaced by $BV(\Omega)$). In order to prove the converse, we shall impose the following condition on the family (ρ_ε):

$$\begin{cases} \text{there exist linearly independent vectors } v_1, \dots, v_N \in \mathbb{R}^N \\ \text{and } \delta > 0 \text{ such that} \\ C_\delta(v_i) \cap C_\delta(v_j) = \phi \quad \text{if } i \neq j, \\ \displaystyle\limsup_{\varepsilon \downarrow 0} \int_{C_\delta(v_i)} \rho_\varepsilon > 0 \quad \forall i = 1, \dots, N. \end{cases} \tag{1.20}$$

Here, for any $v \in \mathbb{R}^N \backslash \{0\}$ and $\delta > 0$, $C_\delta(v)$ denotes the cone

$$C_\delta(v) := \left\{ w \in \mathbb{R}^N \backslash \{0\} : \frac{v}{|v|} \cdot \frac{w}{|w|} > (1-\delta) \right\}. \tag{1.21}$$

We have

Theorem 1.8 *Let $f \in L^p(\Omega)$, $p \geq 1$. Suppose*

$$\limsup_{\varepsilon \downarrow 0} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_\varepsilon(x-y)\, dx\, dy < \infty, \tag{1.22}$$

where (ρ_ε) satisfies (1.2) *and* (1.20)*.*
Then $f \in W^{1,p}(\Omega)$ if $p > 1$, and $f \in BV(\Omega)$ if $p = 1$. Moreover, there exists $\alpha > 0$ (depending only on (ρ_ε)) such that

$$\alpha \int_\Omega |Df|^p \leq \limsup_{\varepsilon \downarrow 0} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_\varepsilon(x-y)\, dx\, dy. \tag{1.23}$$

In particular, we obtain the following characterization (see [2]):

Corollary 1.9 *Suppose that ρ_ε is radial for each $\varepsilon > 0$.*
Let $f \in L^p(\Omega)$, $p \geq 1$. If

$$\liminf_{\varepsilon \downarrow 0} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_\varepsilon\big(|x-y|\big)\, dx\, dy < \infty, \tag{1.24}$$

then $f \in W^{1,p}(\Omega)$ if $p > 1$, and $f \in BV(\Omega)$ if $p = 1$. In particular, (1.8) *holds.*

Theorem 1.8 also implies the (see [11])

Corollary 1.10 *Let $\rho \in L^1(\mathbb{R}^N)$, $\rho \geq 0$ a.e. in $\mathbb{R}^N$, be such that $\int \rho > 0$.*
Let $f \in L^p(\Omega)$, $p \geq 1$. If

$$\liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^N} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho\Big(\frac{x-y}{\varepsilon}\Big)\, dx\, dy < \infty, \tag{1.25}$$

then $f \in W^{1,p}(\Omega)$ if $p > 1$, and $f \in BV(\Omega)$ if $p = 1$. In particular, Corollary 1.4 can be applied to f.

Another application of Theorem 1.8 is the following criterion to decide whether a measurable function f, defined on an open connected set $A \subset \mathbb{R}^N$, is constant or not; this extends some of the results in [3] (see also [13]):

Corollary 1.11 *Assume $A \subset \mathbb{R}^N$ is an open connected set. Let $f : A \to \mathbb{R}$ be a measurable function such that*

$$\lim_{\varepsilon \downarrow 0} \int_A \int_A \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_\varepsilon(x - y)\, dx\, dy = 0, \tag{1.26}$$

where $p \geq 1$ and (ρ_ε) satisfies (1.2) *and* (1.20).

Then $f = const$ a.e. in A.

We first note that it suffices to prove the corollary for $f \in L^\infty(A)$. In fact, one can always replace f by its truncation

$$T_M f(x) := \begin{cases} f(x) & \text{if } |f(x)| \leq M \\ M & \text{if } f(x) > M \\ -M & \text{if } f(x) < -M \end{cases}$$

for every $M > 0$. Applying Theorem 1.8 on the open balls $B_r \subset A$, we conclude that $f \in W^{1,p}_{\text{loc}}(\Omega)$ and $Df = 0$ a.e.. Since A is connected, we must have $f = \text{const}$ a.e. in A.

Remark 1.4 A careful inspection in the proof of Theorem 1.8 shows that it still holds without any assumption on the regularity of Ω. As we have seen the converse statement relies heavily on the smoothness of $\partial\Omega$. It would be interesting to find an expression similar to (1.19) which characterizes $W^{1,p}(\Omega)$ without any additional assumptions on Ω. The example in Remark 1.1 suggests the following

Open problem 1 *Suppose (ρ_ε) is a family of radial functions satisfying* (1.2). *Let $f \in L^p(\Omega)$ be such that*

$$\limsup_{\varepsilon \downarrow 0} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_\varepsilon\big(d_\Omega(x, y)\big)\, dx\, dy < +\infty, \tag{1.27}$$

where d_Ω denotes the geodesic distance in Ω. Can one conclude that $u \in W^{1,p}(\Omega)$ without assuming any regularity of Ω?

The answer to this problem does not seem to be known even in the case of a disk without a line segment.

Let $\omega : [0, \infty) \to [0, \infty)$ be a nondecreasing convex function such that $\omega(0) = 0$ and satisfying the coercivity condition

$$\lim_{t \to \infty} \frac{\omega(t)}{t} = \infty. \tag{1.28}$$

The Orlicz spaces are defined as

$$L^\omega(\Omega) := \left\{ f \in L^1_{\text{loc}}(\Omega) : \int_\Omega \omega\big(\alpha |f|\big) < \infty \quad \text{for some } \alpha > 0 \right\}. \tag{1.29}$$

Analogously, we have the Orlicz-Sobolev spaces (see e.g. [15])

$$W^{1,\omega}(\Omega) := \Big\{ f \in L^\omega(\Omega) : |Df| \in L^\omega(\Omega) \Big\}. \tag{1.30}$$

We have the following characterization for these spaces:

Theorem 1.12 *Suppose that* (1.2) *and* (1.20) *hold.*
Let $f \in L^\omega(\Omega)$. *Then* $f \in W^{1,\omega}(\Omega)$ *if, and only if, there exists* $\beta > 0$ *such that*

$$\limsup_{\varepsilon \downarrow 0} \int_\Omega \int_\Omega \omega \left(\beta \frac{|f(x) - f(y)|}{|x - y|} \right) \rho_\varepsilon(x - y)\, dx\, dy < \infty. \tag{1.31}$$

The description of the Sobolev space $W^{1,1}(\Omega)$ is more delicate since it is not reflexive, and so bounded sequences do not necessarily converge weakly to an element in $W^{1,1}(\Omega)$ (but they do converge weakly in $BV(\Omega)$).

We first recall that given $g \in L^1(\Omega)$ there exists a nondecreasing convex function $\omega_g : [0, \infty) \to [0, \infty)$ such that $g \in L^{\omega_g}(\Omega)$ (see e.g. [8]). In particular, $W^{1,1}(\Omega)$ can be written as the union of all Orlicz-Sobolev spaces. More precisely,

$$W^{1,1}(\Omega) = \bigcup_{\substack{\omega \text{ convex} \\ \text{and coercive}}} W^{1,\omega}(\Omega). \tag{1.32}$$

This gives an indirect characterization of $W^{1,1}(\Omega)$, by means of the Orlicz-Sobolev spaces, in terms of (1.31).

1.4 Properties of f under no additional assumptions on (ρ_ε).

Let us now assume that only (1.2) holds. We can still derive some information about f if (1.19) is satisfied.

In order to simplify our notation, we state our results in the special case $\Omega = \mathbb{R}^N$:

Theorem 1.13 *Let* $f \in L^p(\mathbb{R}^N)$, $p \geq 1$, *be such that*

$$\liminf_{\varepsilon \downarrow 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_\varepsilon(x - y)\, dx\, dy < \infty. \tag{1.33}$$

Then there exists a vector subspace $E \subset \mathbb{R}^N$, $\dim E \geq 1$, *such that*

$$f|_{E+w} \in W^{1,p}(E + w) \quad \text{for a.e. } w \in E^\perp \text{ if } p > 1,$$
$$f|_{E+w} \in BV(E + w) \quad \text{for a.e. } w \in E^\perp \text{ if } p = 1.$$

In addition, there exists $\alpha > 0$ *such that*

$$\alpha \int_{\mathbb{R}^N} |D_E f|^p \leq \liminf_{\varepsilon \downarrow 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_\varepsilon(x - y)\, dx\, dy. \tag{1.34}$$

In particular we have

Corollary 1.14 *Assume $f \in L^p_{\text{loc}}(\mathbb{R}^N)$, $p \geq 1$, is such that*

$$\liminf_{\varepsilon \downarrow 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_\varepsilon(x-y)\, dx\, dy = 0. \tag{1.35}$$

Then there exist a vector space $E \subset \mathbb{R}^N$, $\dim E = k \geq 1$, and a function $\tilde{f} \in L^p_{\text{loc}}(\mathbb{R}^{N-k})$ such that

$$f(v+w) = \tilde{f}(w) \quad \textit{for a.e. } v \in E \textit{ and a.e. } w \in E^\perp. \tag{1.36}$$

In other words, f is a function of $(N-k)$-variables.

Note that by Corollary 1.5 this is the best we can expect from f under our assumptions on (ρ_ε).

1.5 Some remarks about the Γ-convergence of (1.1).

Let us first recall the definition of Γ-lower and upper limits (with respect to the $L^1(\Omega)$-topology; see for instance [7]).

Given a bounded open set $A \subset \mathbb{R}^N$, let (F_j) be any sequence of functionals $F_j : L^1(A) \to [0, +\infty]$. For each $f \in L^1(A)$ we set

$$\Gamma^-_{L^1(A)}\text{-}\liminf_{j\to\infty} F_j(f) := \min\left\{ \liminf_{j\to\infty} F_j(f_j) :\ f_j \to f \ \text{ in } L^1(A) \right\}, \tag{1.37}$$

$$\Gamma^-_{L^1(A)}\text{-}\limsup_{j\to\infty} F_j(f) := \min\left\{ \limsup_{j\to\infty} F_j(f_j) :\ f_j \to f \ \text{ in } L^1(A) \right\}. \tag{1.38}$$

(A standard diagonalization argument shows that both minima are really attained).

If both limits are equal at some point $f \in L^1(A)$, we say that the sequence (F_j) Γ-converges at f and we denote this common number by $\Gamma^-_{L^1(A)}\text{-}\lim_{j\to\infty} F_j(f)$.

Given $F : L^1(A) \to [0, +\infty]$, the lower semicontinuous envelope of F, $\text{sc}^-_{L^1(A)} F$, is the greatest $L^1(A)$-lower semicontinuous functional less than or equal to F. In terms of the Γ-convergence we have

$$\text{sc}^-_{L^1(A)} F(f) = \min\left\{ \liminf_{j\to\infty} F(f_j) :\ f_j \to f \ \text{ in } L^1(A) \right\}. \tag{1.39}$$

We recall that ω^{**} denotes the convex lower semicontinuous envelope of $\omega : [0,\infty) \to [0,\infty)$ (which in our case coincides with the greatest convex function less than or equal to ω).

Theorem 1.15 *Assume $\Omega \subset \mathbb{R}^N$ is a bounded open set with Lipschitz boundary and $\omega : [0,\infty) \to [0,\infty)$ is continuous. If*

$$\omega_\mu^{**} = (\omega^{**})_\mu \quad \text{in } \mathbb{R}^N, \tag{1.40}$$

then

$$\Gamma^-_{L^1(\Omega)}\text{-}\lim_{j\to\infty} \int_\Omega\int_\Omega \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_{\varepsilon_j}(x-y)\,dx\,dy = \mathrm{sc}^-_{L^1(\Omega)} F(f) \tag{1.41}$$

for every $f \in L^1(\Omega)$, where $F : L^1(\Omega) \to [0,+\infty]$ is given by

$$F(f) = \begin{cases} \displaystyle\int_\Omega \omega_\mu(Df) & \text{if } f \in C^1(\overline{\Omega}), \\ +\infty & \text{otherwise.} \end{cases} \tag{1.42}$$

The theorem above reduces the problem of studying the Γ-convergence of our functionals to a relaxation problem, namely to determine the lower semicontinuous envelope of F.

By relaxation, we know that (see [4, Theorems 4.2.8 and 4.4.1])

$$\mathrm{sc}^-_{L^1(\Omega)} F(f) = \int_\Omega \omega_\mu^{**}(Df) \quad \forall f \in C^1(\overline{\Omega}). \tag{1.43}$$

More generally, let $\left(\omega_\mu^{**}\right)^\infty : \mathbb{R}^N \to [0,+\infty]$ be defined as

$$\left(\omega_\mu^{**}\right)^\infty(v) := \lim_{t\to\infty} \frac{\omega_\mu^{**}(tv)}{t} \quad \forall v \in \mathbb{R}^N \tag{1.44}$$

(the limit above always exists in $[0,+\infty]$ since ω_μ^{**} is convex). Applying Theorem 4.7 in [5] to (1.43) we get

$$\mathrm{sc}^-_{L^1(\Omega)} F(f) = \int_\Omega \omega_\mu^{**}(D^{\mathrm{a}} f)\,dx + \int_\Omega \left(\omega_\mu^{**}\right)^\infty\left(\frac{dD^{\mathrm{s}} f}{d|D^{\mathrm{s}} f|}\right) d|D^{\mathrm{s}} f| \tag{1.45}$$

for every $f \in BV(\Omega)$. Here, $Df = D^{\mathrm{a}} f\,\mathcal{L}^N + D^{\mathrm{s}} f$ is the Radon-Nikodym decomposition of Df and $\frac{dD^{\mathrm{s}} f}{d|D^{\mathrm{s}} f|}$ denotes the Radon-Nikodym derivative of $D^{\mathrm{s}} f$ with respect to $|D^{\mathrm{s}} f|$.

In view of Theorem 1.15 and (1.45), we have the following

Corollary 1.16 *Under the assumptions of Theorem 1.15, we have*

$$\begin{aligned} \Gamma^-_{L^1(\Omega)}\text{-}\lim_{j\to\infty} \int_\Omega\int_\Omega \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_{\varepsilon_j}(x-y)\,dx\,dy &= \\ = \int_\Omega \omega_\mu^{**}(D^{\mathrm{a}} f)\,dx + \int_\Omega \left(\omega_\mu^{**}\right)^\infty\left(\frac{dD^{\mathrm{s}} f}{d|D^{\mathrm{s}} f|}\right) d|D^{\mathrm{s}} f| & \end{aligned} \tag{1.46}$$

for every $f \in BV(\Omega)$.

Remark 1.5 As in [12], given a vector-valued Radon measure ν in $\mathbb{R}^N$ with values in $\mathbb{R}^N$, for each Borel set $A \subset \mathbb{R}^N$ we define

$$\omega_\mu^{**}\nu(A) := \sup \left\{ \sum_i |A_i| \, \omega_\mu^{**}\Big(\frac{\nu(A_i)}{|A_i|}\Big) \right\}, \tag{1.47}$$

where the supremum is taken over all finite disjoint partitions $A = \cup A_i$ in terms of Borel sets A_i, and $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^N$ (if $|A_i| = 0$, the term $|A_i| \, \omega_\mu^{**}\big(\frac{\nu(A_i)}{|A_i|}\big)$ is to be understood as the limit $\big(\omega_\mu^{**}\big)^\infty\big(\nu(A_i)\big)$). With such definition, $\omega_\mu^{**}\nu$ is a positive measure in $\mathbb{R}^N$ and (see [12, Theorem 2′])

$$\omega_\mu^{**}\nu(A) = \int_A \omega_\mu^{**}(\nu^{\mathrm{a}}) \, dx + \int_A \big(\omega_\mu^{**}\big)^\infty \Big(\frac{d\nu^{\mathrm{s}}}{d|\nu^{\mathrm{s}}|}\Big) \, d|\nu^{\mathrm{s}}| \tag{1.48}$$

for any Borel set $A \subset \mathbb{R}^N$.

Applying (1.48) with $\nu = Df$, $f \in BV(\Omega)$, and $A = \Omega$, we can rewrite (1.46) in the more elegant form

$$\Gamma^-_{L^1(\Omega)}\text{-}\lim_{j\to\infty} \int_\Omega \int_\Omega \omega\left(\frac{|f(x) - f(y)|}{|x-y|}\right) \rho_{\varepsilon_j}(x-y) \, dx \, dy = \int_\Omega \omega_\mu^{**} Df.$$

We observe that (1.40) holds for any nonnegative $\mu \in M(S^{N-1})$ in the following cases:

1) ω is convex;

2) ω is concave;

3) $\displaystyle\lim_{t\to\infty} \frac{\omega(t)}{t} = 0$ and $\omega(0) = 0$.

Identity (1.40) also holds for any continuous function $\omega : [0,\infty) \to [0,\infty)$ if we take

4) $\displaystyle\mu = \sum_{i=1}^N \alpha_i \delta_{e_i}$, where $\alpha_i \geq 0$ for each $i = 1, \ldots, N$.

In particular, we see that for any $\omega : [0,\infty) \to [0,\infty)$ continuous and any $f \in W^{1,1}(\Omega)$ we have

$$\Gamma^-_{L^1(\Omega)}\text{-}\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^{2N-1}} \int_\Omega \int_\Omega\limits_{\substack{|x_1-y_1|<\varepsilon \\ |x_i-y_i|<\varepsilon^2 \\ i=2,\ldots,N}} \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right) dx \, dy = 2^N \int_\Omega \omega^{**}\left(\Big|\frac{\partial f}{\partial x_1}\Big|\right).$$

Identity (1.40) does not hold in general. In fact, take $\mu = \frac{1}{|S^{N-1}|}\mathcal{H}^{N-1}\lfloor_{S^{N-1}}$ (which corresponds to a family (ρ_ε) of radial functions); then one can construct a continuous function ω which is not convex, while ω_μ is.

It would be interesting to know if condition (1.40) is really necessary to prove the Γ-convergence of (1.1) for the subsequence $\varepsilon_j \downarrow 0$ we have constructed. We can state the following

Open problem 2 Under the hypotheses of Theorem 1.15, but **without** assuming condition (1.40), does

$$\Gamma^-_{L^1(\Omega)}\text{-}\lim_{j\to\infty} \int_\Omega\int_\Omega \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_{\varepsilon_j}(x-y)\,dx\,dy = \mathrm{sc}^-_{L^1(\Omega)}\,F(f)$$

still hold for any $f \in L^1(\Omega)$ and any continuous function $\omega : [0,\infty) \to [0,\infty)$?

2 Notation.

For any set $A \subset \mathbb{R}^N$, we denote by χ_A the characteristic function of A.

For an open set $\Omega \subset \mathbb{R}^N$ and $r > 0$ we write

$$\Omega_r := \{x \in \Omega : \mathrm{dist}\,(x, \partial\Omega) > r\},$$
$$N_r(\Omega) := \{x \in \mathbb{R}^N : \mathrm{dist}\,(x, \Omega) < r\}.$$

Let us fix a radial nonnegative function $\eta \in C_0^\infty(\mathbb{R}^N)$ such that $\int \eta = 1$ and $\mathrm{supp}\,\eta \subset B_1$. Given $\delta > 0$, for any $f \in L^1(\Omega)$ we define

$$f_\delta(x) := \frac{1}{\varepsilon^N}\int_\Omega \eta\Big(\frac{x-y}{\varepsilon}\Big) f(y)\,dy \quad \forall x \in \Omega_\delta.$$

Given a locally compact topological space X (in our case we shall take X to be $\mathbb{R}^N$ or S^{N-1}), we denote by $M(X)$ the vector space of finite Radon measures on X. We shall endow $M(X)$ with the norm

$$\|\nu\| := \int_X d|\nu| = \sup\left\{\int_\Omega \varphi\,d\nu :\ \varphi \in C_0(X),\ |\varphi| \le 1 \text{ in } \Omega\right\}.$$

We shall also use the standard notation for averages:

$$⨍_X v\,d\mu := \frac{\int_X v\,d\mu}{\int_X d\mu}.$$

By abuse of notation, $|S^{N-1}|$ denotes the $(N-1)$-Hausdorff measure of S^{N-1}.

Given $\omega : [0,\infty) \to [0,\infty)$ continuous and $\mu \in M(S^{N-1})$, let $\omega_\mu : \mathbb{R}^N \to \mathbb{R}$ be the continuous function given by

$$\omega_\mu(v) := \int_{S^{N-1}} \omega\big(|v\cdot\sigma|\big)\,d\mu(\sigma) \quad \forall v \in \mathbb{R}^N.$$

3 Determining the measure $\mu \in M(S^{N-1})$. Some examples.

Before proceeding, we point out that the family (μ_ε) we defined in Section 1 is absolutely continuous with respect to the Hausdorff measure $\mathcal{H}^{N-1}\lfloor_{S^{N-1}}$ in S^{N-1}, that is, $\mu_\varepsilon \in L^1(S^{N-1})$ and it is given by

$$\mu_\varepsilon(\sigma) = \int_0^\infty \rho_\varepsilon(t\sigma)t^{N-1}\,dt \quad \text{for a.e. } \sigma \in S^{N-1}. \tag{3.1}$$

In particular, $\mu_\varepsilon \geq 0$ a.e. in S^{N-1} and $\int_{S^{N-1}} \mu_\varepsilon = 1$ for every $\varepsilon > 0$. Since $\mu_{\varepsilon_j} \rightharpoonup \mu$ in $M(S^{N-1})$, these properties imply that the Radon measure μ itself is nonnegative and $\int_{S^{N-1}} d\mu = 1$.

Example 3.1 Suppose that ρ_ε is radial for every $\varepsilon > 0$. Then $\mu_\varepsilon = \frac{1}{|S^{N-1}|}$ $\forall \varepsilon > 0$, and so $\mu = \frac{1}{|S^{N-1}|}\mathcal{H}^{N-1}\lfloor_{S^{N-1}}$. Therefore,

$$\omega_\mu(v) = \fint_{S^{N-1}} \omega\big(|v \cdot \sigma|\big)\,d\mathcal{H}^{N-1}(\sigma) \quad \forall v \in \mathbb{R}^N. \tag{3.2}$$

Taking in particular $\omega(t) = t^p$, $p > 0$, and using the symmetry of S^{N-1} we have

$$\omega_\mu(v) = K_{p,N}|v|^p \quad \forall v \in \mathbb{R}^N, \tag{3.3}$$

where

$$K_{p,N} = \fint_{S^{N-1}} |e_1 \cdot \sigma|^p\,d\mathcal{H}^{N-1} = \frac{1}{\pi^{1/2}} \frac{\Gamma\big(\frac{N}{2}\big)\Gamma\big(\frac{p+1}{2}\big)}{\Gamma\big(\frac{N+p}{2}\big)}. \tag{3.4}$$

Example 3.2 Let $\rho \in L^1(\mathbb{R}^N)$, $\rho \geq 0$ a.e. in $\mathbb{R}^N$, be such that $\int \rho = 1$. For each $\varepsilon > 0$, define

$$\rho_\varepsilon(x) := \frac{1}{\varepsilon^N}\rho\left(\frac{x}{\varepsilon}\right) \quad \text{for a.e. } x \in \mathbb{R}^N.$$

Therefore,

$$\mu(\sigma) = \mu_\varepsilon(\sigma) = \int_0^\infty \rho(t\sigma)t^{N-1}\,dt \quad \text{for a.e. } \sigma \in S^{N-1}.$$

The function ω_μ may be written in this case as

$$\omega_\mu(v) = \int_{\mathbb{R}^N} \omega\left(\left|v \cdot \frac{z}{|z|}\right|\right)\rho(z)\,dz \quad \forall v \in \mathbb{R}^N. \tag{3.5}$$

Example 3.3 Let

$$\rho_\varepsilon := \frac{1}{2^N \varepsilon^{2N-1}} \chi_{(-\varepsilon,\varepsilon)\times(-\varepsilon^2,\varepsilon^2)^{(N-1)}}.$$

It is easy to see that $\mu = \dfrac{\delta_{e_1} + \delta_{-e_1}}{2}$, whence

$$\omega_\mu(v) = \omega\big(|v_1|\big) \quad \forall v \in \mathbb{R}^N. \tag{3.6}$$

More generally, let $1 \le k \le N$ be a fixed integer and write $\mathbb{R}^N = \mathbb{R}^k \oplus \mathbb{R}^{N-k}$. We now define

$$\rho_\varepsilon := \frac{1}{|B^k_\varepsilon| \times |B^{N-k}_{\varepsilon^2}|} \chi_{B^k_\varepsilon \times B^{N-k}_{\varepsilon^2}}.$$

We observe that $\operatorname{supp} \mu \subset S^{k-1}$, μ is uniform on S^{k-1} and $\mu(S^{k-1}) = 1$. We then conclude that $\mu = \dfrac{1}{|S^{k-1}|} \mathcal{H}^{k-1} \lfloor_{S^{k-1}}$.

Taking in particular $\omega(t) = t^p$, $p > 0$, we get

$$\omega_\mu(v) = K_{p,k} |v'|^p \quad \forall v = (v', v'') \in \mathbb{R}^N, \tag{3.7}$$

where $K_{p,k}$ is defined in Example 3.1.

In the next example we show that given any nonnegative measure $\mu \in M(S^{N-1})$, $\mu(S^{N-1}) = 1$, one can find a family (ρ_ε) satisfying (1.2) for which

$$\mu_\varepsilon \rightharpoonup \mu \quad \text{in } M(S^{N-1}). \tag{3.8}$$

Example 3.4 Let $\mu \in M(S^{N-1})$, $\mu \ge 0$, be such that $\mu(S^{N-1}) = 1$. We define

$$\rho_\varepsilon(x) := \frac{1}{\varepsilon^N} \int_{S^{N-1}} \eta_\varepsilon \left(\frac{x}{\varepsilon} - y \right) d\mu(y) \quad \forall x \in \mathbb{R}^N, \tag{3.9}$$

where $\eta \in C_0^\infty(\mathbb{R}^N)$ is a nonnegative function such that $\int \eta = 1$ and $\operatorname{supp} \eta \subset B_1$.

Notice that $\rho_\varepsilon \in C_0^\infty(\mathbb{R}^N)$, $\rho_\varepsilon \ge 0$ in $\mathbb{R}^N$, $\int \rho_\varepsilon = 1$ and $\operatorname{supp} \rho_\varepsilon \subset B_{2\varepsilon}$. Thus (ρ_ε) satisfies (1.2). In addition, one can easily check that (3.8) holds for such family.

We conclude this section with the following remark which will be useful in some of the proofs:

Remark 3.1 Assume $\theta \in C(S^{N-1})$. For each $\varepsilon > 0$ we have

$$\int_{\mathbb{R}^N} \theta \left(\frac{h}{|h|} \right) \rho_\varepsilon(h)\, dh = \int_{S^{N-1}} \theta(\sigma)\, d\mu_\varepsilon(\sigma). \tag{3.10}$$

In particular,

$$\lim_{j\to\infty} \int_{\mathbb{R}^N} \theta \left(\frac{h}{|h|} \right) \rho_{\varepsilon_j}(h)\, dh = \int_{S^{N-1}} \theta(\sigma)\, d\mu(\sigma). \tag{3.11}$$

4 The regular case.

The next proposition implies that (1.1) always converges (up to the fixed subsequence $\varepsilon_j \downarrow 0$ we have constructed) if Ω is bounded and f is smooth. More precisely,

Proposition 4.1 *Assume $\Omega \subset \mathbb{R}^N$ is bounded and let $\omega : [0,\infty) \to [0,\infty)$ be a continuous function.*
If $f \in C^2(\overline{\Omega})$, then

$$\lim_{j\to\infty} \int_\Omega\int_\Omega \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right) \rho_{\varepsilon_j}(x-y)\,dx\,dy = \int_\Omega \omega_\mu(Df). \tag{4.1}$$

Proof. For each $f \in C^2(\overline{\Omega})$, we set $M_f := \|Df\|_{L^\infty}$. Since ω is uniformly continuous in $[0, M_f]$, given any $\delta > 0$ there exists $C_\delta > 0$ such that

$$|\omega(s) - \omega(t)| \le C_\delta |s-t| + \delta \quad \forall s,t \in [0, M_f].$$

In particular, we have

$$\begin{aligned}\left|\omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right) - \omega\left(\left|Df(x)\cdot\frac{x-y}{|x-y|}\right|\right)\right| &\le \\ &\le C_\delta \frac{\big|f(x)-f(y)-Df(x)\cdot(x-y)\big|}{|x-y|} + \delta \\ &\le C_\delta |x-y| + \delta \qquad \forall x,y \in \mathbb{R}^N,\ x\neq y.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_\Omega\int_\Omega \left|\omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right) - \omega\left(\left|Df(x)\cdot\frac{x-y}{|x-y|}\right|\right)\right| \rho_\varepsilon(x-y)\,dx\,dy \le \\ \le |\Omega| \left\{ C_\delta \int_{|h|\le 1} |h|\rho_\varepsilon(h)\,dh + \delta + \max_{[0,M_f]} \omega \cdot \int_{|h|>1} \rho_\varepsilon(h)\,dh \right\}.\end{aligned}$$

As $\varepsilon \downarrow 0$, by (1.2) the first and the last terms in the right-hand side tend to zero for every fixed $\delta > 0$. By taking $\delta \downarrow 0$ in the resulting expression, we conclude that

$$\lim_{\varepsilon\downarrow 0} \int_\Omega\int_\Omega \left|\omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right) - \omega\left(\left|Df(x)\cdot\frac{x-y}{|x-y|}\right|\right)\right| \rho_\varepsilon(x-y)\,dx\,dy = 0.$$

In other words, to prove (4.1), it suffices to show that

$$\lim_{j\to\infty} \int_\Omega\int_\Omega \omega\left(\left|Df(x)\cdot\frac{x-y}{|x-y|}\right|\right) \rho_{\varepsilon_j}(x-y)\,dx\,dy = \int_\Omega \omega_\mu(Df). \tag{4.2}$$

We first write

$$\int_\Omega\int_{\mathbb{R}^N} \omega\left(\left|Df(x)\cdot\frac{h}{|h|}\right|\right)\rho_{\varepsilon_j}(h)\,dx\,dh$$
$$= \int_\Omega\int_\Omega \omega\left(\left|Df(x)\cdot\frac{x-y}{|x-y|}\right|\right)\rho_{\varepsilon_j}(x-y)\,dx\,dy + \int_\Omega\int_{\mathbb{R}^N\setminus\Omega} \omega\left(\left|Df(x)\cdot\frac{x-y}{|x-y|}\right|\right)\rho_{\varepsilon_j}(x-y)\,dx\,dy. \tag{4.3}$$

To estimate the last term in (4.3), fix $\lambda > 0$. We have

$$\int_\Omega\int_{\mathbb{R}^N\setminus\Omega} \omega\left(\left|Df(x)\cdot\frac{x-y}{|x-y|}\right|\right)\rho_\varepsilon(x-y)\,dx\,dy \le$$
$$\le \max_{[0,M_f]}\omega\cdot\left\{|\Omega|\int_{|h|>\lambda}\rho_\varepsilon(h)\,dh + |\Omega\setminus\Omega_\lambda|\int_{|h|\le\lambda}\rho_\varepsilon(h)\,dh\right\}.$$

We first take $\varepsilon \downarrow 0$ and then $\lambda \downarrow 0$ to get

$$\lim_{\varepsilon\downarrow 0}\int_\Omega\int_{\mathbb{R}^N\setminus\Omega} \omega\left(\left|Df(x)\cdot\frac{x-y}{|x-y|}\right|\right)\rho_\varepsilon(x-y)\,dx\,dy = 0. \tag{4.4}$$

By Remark 3.1, (4.3) and (4.4), we conclude that (4.2) holds.

The next two remarks will be used in Section 12 to study the Γ-convergence of (1.1):

Remark 4.1 It follows from the proof of Proposition 4.1 that the convergence in (4.1) is uniform on the bounded subsets of $C^2(\overline{\Omega})$.

Remark 4.2 A slight modification in the argument above shows that (4.1) still holds for any $f \in C^1(\overline{\Omega})$.

5 Some useful estimates.

The following lemmas will be used throughout this paper. Since they have been extensively applied (see [2, 3, 6]), we shall only sketch their proofs.

Lemma 5.1 *Assume* $\omega : [0,\infty) \to [0,\infty)$ *is convex.*
If $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$*, then*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_\varepsilon(x-y)\,dx\,dy \le \le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \omega\left(\left|Df(x)\cdot\frac{h}{|h|}\right|\right)\rho_\varepsilon(h)\,dx\,dh \quad \forall\varepsilon > 0. \tag{5.1}$$

Proof. Let $\delta > 0$. For any $R > 0$, it follows from a standard application of the Fundamental Theorem of Calculus and Jensen's inequality that

$$\begin{aligned}\int_{B_R}\int_{B_R} \omega\left(\frac{|f_\delta(x) - f_\delta(y)|}{|x-y|}\right) \rho_\varepsilon(x-y)\, dx\, dy \leq \\ \leq \int_{B_R}\int_{\mathbb{R}^N} \omega\left(\left|Df_\delta(x)\cdot\frac{h}{|h|}\right|\right) \rho_\varepsilon(h)\, dx\, dh \qquad (5.2)\\ \leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \omega\left(\left|Df(x)\cdot\frac{h}{|h|}\right|\right) \rho_\varepsilon(h)\, dx\, dh.\end{aligned}$$

Taking $\delta \downarrow 0$ and then $R \to \infty$ we obtain (5.1).

Lemma 5.2 *Assume $f \in W^{1,p}(\Omega)$, $p \geq 1$. Let $\bar{f} \in W^{1,p}(\mathbb{R}^N)$ be an extension of f in $\mathbb{R}^N$. For every $r, \varepsilon > 0$ we have*

$$\begin{aligned}\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_\varepsilon(x-y)\, dx\, dy \leq \\ \leq \int_{N_r(\Omega)}\int_{|h|<r} \left|D\bar{f}(x)\cdot\frac{h}{|h|}\right|^p \rho_\varepsilon(h)\, dx\, dh + \frac{2^p\|f\|_{L^p}^p}{r^p}\int_{|h|\geq r}\rho_\varepsilon.\end{aligned} \qquad (5.3)$$

Proof. For any $\delta \in (0, r)$ we have

$$\begin{aligned}\int_\Omega\int_\Omega \frac{|\bar{f}_\delta(x)-\bar{f}_\delta(y)|^p}{|x-y|^p}\rho_\varepsilon(x-y)\, dx\, dy \leq \\ \leq \int_\Omega\int_{\substack{\Omega \\ |x-y|<r}} \frac{|\bar{f}_\delta(x)-\bar{f}_\delta(y)|^p}{|x-y|^p}\rho_\varepsilon(x-y)\, dx\, dy + \frac{2^p\|f\|_{L^p}^p}{r^p}\int_{|h|\geq r}\rho_\varepsilon.\end{aligned} \qquad (5.4)$$

Proceeding as before to estimate the first term in the right-hand side of the inequality (note that if $x, y \in \Omega$ and $|x-y| < r$ then $tx + (1-t)y \in N_r(\Omega)$ for every $t \in [0,1]$), we obtain (5.3).

The next lemma can be proved exactly as above. Actually, applying Jensen's inequality as in the last estimate in (5.2), we can avoid the weak convergence $D\bar{f}_\delta \rightharpoonup D\bar{f}$ in $M(\mathbb{R}^N)$.

Lemma 5.3 *Assume $f \in BV(\Omega)$. Let $\bar{f} \in BV(\mathbb{R}^N)$ be an extension of f in $\mathbb{R}^N$. For every $r, \varepsilon > 0$ we have*

$$\begin{aligned}\int_\Omega\int_\Omega \frac{|f(x)-f(y)|}{|x-y|}\rho_\varepsilon(x-y)\, dx\, dy \leq \\ \leq \int_{|h|<r}\left(\int_{N_r(\Omega)} \left|D\bar{f}\cdot\frac{h}{|h|}\right|\right)\rho_\varepsilon(h)\, dh + \frac{2\|f\|_{L^1}}{r}\int_{|h|\geq r}\rho_\varepsilon.\end{aligned} \qquad (5.5)$$

The following lemma was pointed out by E. Stein. It comes from a simple application of Jensen's inequality and a change of variables.

Lemma 5.4 *Assume $\omega : [0,\infty) \to [0,\infty)$ is convex, and let $f \in L^1_{\rm loc}(\Omega)$. For each $r > 0$ we have*

$$\begin{aligned}\int_{\Omega_r}\int_{\Omega_r} \omega\left(\frac{|f_\delta(x)-f_\delta(y)|}{|x-y|}\right)\rho_\varepsilon(x-y)\,dx\,dy \le \\ \le \int_\Omega\int_\Omega \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_\varepsilon(x-y)\,dx\,dy \quad \forall\delta\in(0,r).\end{aligned} \tag{5.6}$$

6 Proof of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Given $f \in W^{1,p}(\Omega)$, we take an extension $\bar f \in W^{1,p}(\mathbb{R}^N)$ of f. For any $g \in C_0^\infty(\mathbb{R}^N)$, using the triangle inequality and Lemma 5.1 we have

$$\begin{aligned}\Bigg|\Big(\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_{\varepsilon_j}(x-y)\,dx\,dy\Big)^{1/p} - \\ -\Big(\int_\Omega\int_\Omega \frac{|g(x)-g(y)|^p}{|x-y|^p}\rho_{\varepsilon_j}(x-y)\,dx\,dy\Big)^{1/p}\Bigg| \le \\ \le \Big(\int_\Omega\int_\Omega \frac{|(f-g)(x)-(f-g)(y)|^p}{|x-y|^p}\rho_{\varepsilon_j}(x-y)\,dx\,dy\Big)^{1/p} \\ \le \Big(\int_{\mathbb{R}^N}|D\bar f - Dg|^p\Big)^{1/p}.\end{aligned}$$

Let $j \to \infty$. We conclude the proof by using a variant of Proposition 4.1 for C_0^∞-functions and the density of $C_0^\infty(\mathbb{R}^N)$ in $W^{1,p}(\mathbb{R}^N)$.

Proof of Theorem 1.2. Given $f \in BV(\Omega)$, there exists an extension $\bar f \in BV(\mathbb{R}^N)$ such that $\displaystyle\int_{\partial\Omega}|D\bar f| = 0$ (see e.g. [9]; this last property can be obtained by a local reflexion across the boundary). Applying Lemma 5.3 we see that (1.7) holds.

By Lemmas 5.3 and 5.4 we have for any $0 < \delta < r$ that

$$\begin{aligned}\int_{\Omega_r\cap B_{1/r}}\int_{\Omega_r\cap B_{1/r}} \frac{|f_\delta(x)-f_\delta(y)|}{|x-y|}\rho_{\varepsilon_j}(x-y)\,dx\,dy \le \\ \le \int_\Omega\int_\Omega \frac{|f(x)-f(y)|}{|x-y|}\rho_{\varepsilon_j}(x-y)\,dx\,dy \le \\ \le \int_{\mathbb{R}^N}\left(\int_{N_r(\Omega)}\left|D\bar f\cdot\frac{h}{|h|}\right|\right)\rho_{\varepsilon_j}(h)\,dh + \frac{2\|f\|_{L^1}}{r}\int_{|h|\ge r}\rho_{\varepsilon_j}.\end{aligned} \tag{6.1}$$

We make the following remarks:

$$\sigma \in S^{N-1} \longmapsto \int |D\bar f\cdot\sigma| \in \mathbb{R} \quad \text{is continuous;} \tag{6.2}$$

$$\int |Df_\delta\cdot\sigma| \xrightarrow{\delta\downarrow 0} \int |Df\cdot\sigma| \quad \text{uniformly with respect to } \sigma\in S^{N-1}. \tag{6.3}$$

By (6.2), Remark 3.1 and the outer regularity of Radon measures, we get

$$\lim_{r\downarrow 0}\lim_{j\to\infty}\int_{\mathbb{R}^N}\left(\int_{N_r(\Omega)}\left|D\bar f\cdot\frac{h}{|h|}\right|\right)\rho_{\varepsilon_j}(h)\,dh = \\ = \int_{S^{N-1}}\left(\int_{\overline{\Omega}}|D\bar f\cdot\sigma|\right)d\mu(\sigma). \tag{6.4}$$

Proposition 4.1, (6.3) and the inner regularity of Radon measures give us

$$\lim_{r\downarrow 0}\lim_{\delta\downarrow 0}\lim_{j\to\infty}\int_{\Omega_r\cap B_{1/r}}\int_{\Omega_r\cap B_{1/r}}\frac{|f_\delta(x)-f_\delta(y)|}{|x-y|}\rho_{\varepsilon_j}(x-y)\,dx\,dy = \\ = \int_{S^{N-1}}\left(\int_{\Omega}|Df\cdot\sigma|\right)d\mu(\sigma). \tag{6.5}$$

We now take $j\to\infty$, $\delta\downarrow 0$ and then $r\downarrow 0$ in (6.1). Using (6.4), (6.5) and $\int_{\partial\Omega}|D\bar f| = 0$ we obtain the result.

7 Proof of Theorem 1.6.

Theorem 1.6 is an immediate consequence of Theorem 1.2 and the following lemma applied to the function $\beta(t) := \omega(t) - \omega^\infty t$, $t\in[0,\infty)$.

Lemma 7.1 *Let $\beta : [0,\infty)\to\mathbb{R}$ be a continuous function such that*

$$\lim_{t\to\infty}\frac{\beta(t)}{t} = 0. \tag{7.1}$$

If $\Omega\subset\mathbb{R}^N$ is unbounded, suppose in addition that there exists $C>0$ such that

$$|\beta(t)|\le Ct \quad \forall t\ge 0. \tag{7.2}$$

If $f\in BV(\Omega)$, then

$$\lim_{j\to\infty}\int_\Omega\int_\Omega\beta\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_{\varepsilon_j}(x-y)\,dx\,dy = \int_\Omega\beta_\mu(D^{\mathrm{a}}f), \tag{7.3}$$

where $D^{\mathrm{a}}f$ is the absolutely continuous part of Df with respect to the Lebesgue measure in $\mathbb{R}^N$, and $\beta_\mu(v) := \int_{S^{N-1}}\beta\big(|v\cdot\sigma|\big)\,d\mu(\sigma) \quad \forall v\in\mathbb{R}^N$.

In order to prove Lemma 7.1 we shall need the next two simple remarks:

Remark 7.1 Let $\nu_1,\nu_2\in M(\mathbb{R}^N)$ be such that $\nu_1\le\nu_2$ in $\mathbb{R}^N$, then

$$\nu_1^{\mathrm{a}}\le\nu_2^{\mathrm{a}} \quad\text{and}\quad \nu_1^{\mathrm{s}}\le\nu_2^{\mathrm{s}} \quad\text{in } \mathbb{R}^N, \tag{7.4}$$

where $\nu_i = \nu_i^{\mathrm{a}}\,\mathcal{L}^N + \nu_i^{\mathrm{s}}$ is the Radon-Nikodym decomposition of ν_i, $i=1,2$.

Remark 7.2 If (ν_j) is a sequence of nonnegative measures in $M(S^{N-1})$ such that $\nu_j \rightharpoonup \nu$ in $M(S^{N-1})$, then

$$\int_A d\nu \le \liminf_{j\to\infty} \int_A d\nu_j \le \limsup_{j\to\infty} \int_A d\nu_j \le \int_{\bar{A}} d\nu \quad \forall A \subset S^{N-1} \text{ open}. \tag{7.5}$$

This is a simple consequence of the inner and outer regularity of Radon measures (see e.g. [9]).

Proof of Lemma 7.1. Let $f \in BV(\Omega)$. After extending f to the whole space $\mathbb{R}^N$ (take for instance $f = 0$ in $\mathbb{R}^N \backslash \Omega$), we may suppose that $f \in BV(\mathbb{R}^N)$. We define (see [6])

$$\nu_j(x) := \int_{\mathbb{R}^N} \beta\left(\frac{|f(x+h) - f(x)|}{|h|}\right) \rho_{\varepsilon_j}(h)\, dh \quad \text{for a.e. } x \in \mathbb{R}^N. \tag{7.6}$$

In particular, (ν_j) is bounded in $L^1_{\text{loc}}(\mathbb{R}^N)$ (by Lemma 5.3) so that, up to a subsequence, there exists $\nu \in M_{\text{loc}}(\mathbb{R}^N)$ such that

$$\nu_j \rightharpoonup \nu \quad \text{in } M_{\text{loc}}(\mathbb{R}^N).$$

We shall prove that ν is absolutely continuous with respect to the Lebesgue measure, and $\nu = \beta_\mu(D^{\mathrm{a}} f)$ a.e. in $\mathbb{R}^N$.

Step 1. $\nu^{\mathrm{s}} = 0$ in $\mathbb{R}^N$.

By (7.1), for each $\delta > 0$ there exists $C_\delta > 0$ such that

$$|\beta(s)| \le \delta s + C_\delta \quad \forall s \ge 0. \tag{7.7}$$

We now take $x_0 \in \mathbb{R}^N$ and $R > 0$. For $r \in (0, R)$, it follows from (7.7) and Lemma 5.3 that

$$\begin{aligned} \int_{B_{R-r}(x_0)} \nu_j &\le \delta \int_{B_{R-r}(x_0)} \int_{\mathbb{R}^N} \frac{|f(x+h) - f(x)|}{|h|} \rho_{\varepsilon_j}(h)\, dx\, dh + C_\delta |B_R| \\ &\le \delta \int_{B_R(x_0)} |Df| + \frac{2\delta}{r} \|f\|_{L^1} \int_{|h|>r} \rho_{\varepsilon_j} + C_\delta |B_R|. \end{aligned}$$

Take $j \to \infty$ and then $r \downarrow 0$; Remark 7.2 implies that

$$\int_{B_R(x_0)} \nu \le \delta \int_{B_R(x_0)} |Df| + C_\delta |B_R| \quad \forall x_0 \in \mathbb{R}^N \quad \forall R > 0.$$

In particular, by Remark 7.1,

$$0 \le \nu^{\mathrm{s}} \le \delta |D^{\mathrm{s}} f| \quad \text{in } \mathbb{R}^N \quad \forall \delta > 0.$$

We now let $\delta \downarrow 0$ to conclude that $\nu^{\mathrm{s}} = 0$ in $\mathbb{R}^N$.

Step 2. $\nu^{\mathrm{a}} = \beta_\mu(D^{\mathrm{a}} f)$ a.e. in $\mathbb{R}^N$.

Let $\delta > 0$. By (7.1) and the continuity of β we have

$$\big|\beta(s) - \beta(t)\big| \le C_\delta |s - t| + \delta(1 + s + t) \quad \forall s, t \ge 0, \tag{7.8}$$

for some $C_\delta > 0$.

Let $x_0 \in \mathbb{R}^N$ and $R > 0$. For $r \in (0, R)$ fixed, using (7.8) we can estimate

$$\int_{B_{R-r}(x_0)} \int_{\mathbb{R}^N} \left| \beta\left(\frac{|f(x+h) - f(x)|}{|h|} \right) - \beta\left(\left| D^{\mathrm{a}} f(x) \cdot \frac{h}{|h|} \right| \right) \right| \rho_{\varepsilon_j}(h)\, dx\, dh$$

by an expression of the form

$$C_\delta A_1 + \delta A_2, \tag{7.9}$$

where

$$A_1 := \int_{B_{R-r}(x_0)} \int_{\mathbb{R}^N} \frac{\big|f(x+h) - f(x) - D^{\mathrm{a}} f(x) \cdot h\big|}{|h|} \rho_{\varepsilon_j}(h)\, dx\, dh,$$

$$A_2 := \int_{B_{R-r}(x_0)} \int_{\mathbb{R}^N} \left\{ 1 + \frac{\big|f(x+h) - f(x)\big|}{|h|} + \big|D^{\mathrm{a}} f(x)\big| \right\} \rho_{\varepsilon_j}(h)\, dx\, dh.$$

In order to estimate A_1 we write

$$\begin{aligned}
A_1 \le & \int_{B_{R-r}(x_0)} \int_{|h|<r} \left\{ \int_0^1 \big|D^{\mathrm{a}} f(x+th) - D^{\mathrm{a}} f(x)\big|\, dt \right\} \rho_{\varepsilon_j}(h)\, dx\, dh + \\
& \qquad + \int_{B_R(x_0)} |D^{\mathrm{s}} f| + \left(\frac{2\|f\|_{L^1}}{r} + \int_{B_R} |D^{\mathrm{a}} f| \right) \int_{|h|\ge r} \rho_{\varepsilon_j} \le \\
\le & \sup_{v \in B_r} \left\{ \int_{B_R(x_0)} \big|D^{\mathrm{a}} f(x+v) - D^{\mathrm{a}} f(x)\big|\, dx \right\} + \\
& \qquad + \int_{B_R(x_0)} |D^{\mathrm{s}} f| + \left(\frac{2\|f\|_{L^1}}{r} + \int_{B_R} |D^{\mathrm{a}} f| \right) \int_{|h|\ge r} \rho_{\varepsilon_j}.
\end{aligned}$$

(One may verify these inequalities first for the smooth functions $f_\lambda := \eta_\lambda * f$, observing that $Df_\lambda = (D^{\mathrm{a}} f)_\lambda + (D^{\mathrm{s}} f)_\lambda$, and then using Jensen's inequality before letting $\lambda \downarrow 0$).

On the other hand, using Lemma 5.3 we have

$$\begin{aligned}
A_2 \le & |B_R| + \int_{B_{R-r}(x_0)} \int_{|h|<r} \frac{\big|f(x+h) - f(x)\big|}{|h|} \rho_{\varepsilon_j}(h)\, dx\, dh + \\
& \qquad + \frac{2\|f\|_{L^1}}{r} \int_{|h|\ge r} \rho_{\varepsilon_j} + \int_{B_R(x_0)} |D^{\mathrm{a}} f| \le \\
\le & |B_R| + 2 \int_{B_R(x_0)} |Df| + \frac{2\|f\|_{L^1}}{r} \int_{|h|\ge r} \rho_{\varepsilon_j}.
\end{aligned}$$

Taking $j \to \infty$ and then $r \downarrow 0$, it follows from Remarks 3.1 and 7.2, and the estimates above that

$$\left| \int_{B_R(x_0)} \big[\nu - \beta_\mu(D^{\mathrm{a}} f)\big] \right| \le C_\delta \int_{B_R(x_0)} |D^{\mathrm{s}} f| + \delta \left(|B_R| + 2 \int_{B_R(x_0)} |Df| \right)$$

for all $x_0 \in \mathbb{R}^N$ and $R > 0$.

In particular, by Remark 7.1,

$$\big|\nu^{\mathrm{a}} - \beta_\mu(D^{\mathrm{a}} f)\big| \le \delta\big(1 + 2|D^{\mathrm{a}} f|\big) \quad \text{a.e. in } \mathbb{R}^N \quad \forall \delta > 0.$$

We let $\delta \downarrow 0$ to conclude that $\nu^{\mathrm{a}} = \beta_\mu(D^{\mathrm{a}} f)$ a.e. in $\mathbb{R}^N$.

Step 3. Proof of Lemma 7.1 completed.

If follows from Steps 1 and 2 that $\nu = \beta_\mu(D^{\mathrm{a}} f)$ a.e. in $\mathbb{R}^N$. In order to prove (7.3), for a fixed $r > 0$ we write

$$\begin{aligned} \int_{\Omega_r} \nu_j \le & \int_\Omega \int_\Omega \beta \left(\frac{|f(x) - f(y)|}{|x-y|} \right) \rho_{\varepsilon_j}(x-y)\, dx\, dy + \\ & + \int_{\Omega_r} \int_{\mathbb{R}^N \setminus \Omega} \beta \left(\frac{|f(x) - f(y)|}{|x-y|} \right) \rho_{\varepsilon_j}(x-y)\, dx\, dy \le \\ \le & \int_\Omega \nu_j . \end{aligned} \tag{7.10}$$

We now observe that dist $\big(\Omega_r, \mathbb{R}^N \backslash \Omega\big) = r > 0$. Applying (7.7) (take for instance $\delta = 1$) if Ω is bounded, or (7.2) if not, it is easy to check that the term of the form $\int_{\Omega_r} \int_{\mathbb{R}^N \setminus \Omega}$ in the expression above tends to 0 as $j \to \infty$. We obtain (7.3) by letting $j \to \infty$ and then $r \downarrow 0$ in (7.10).

8 Proof of Theorem 1.7.

Step 1. (1.15) holds if ω is convex.

For any $\delta \in (0, r)$, $r > 0$ fixed, it follows from Lemmas 5.1 and 5.4 that

$$\begin{aligned} \int_{\Omega_r \cap B_{1/r}} \int_{\Omega_r \cap B_{1/r}} & \omega \left(\frac{|f_\delta(x) - f_\delta(y)|}{|x-y|} \right) \rho_{\varepsilon_j}(x-y)\, dx\, dy \le \\ & \le \int_\Omega \int_\Omega \omega \left(\frac{|f(x) - f(y)|}{|x-y|} \right) \rho_{\varepsilon_j}(x-y)\, dx\, dy \le \\ & \le \int_\Omega \int_{\mathbb{R}^N} \omega \left(\left| Df(x) \cdot \frac{h}{|h|} \right| \right) \rho_{\varepsilon_j}(h)\, dx\, dh. \end{aligned} \tag{8.1}$$

Note that

$$\omega\big(|Df \cdot \sigma|\big) \le \tilde{\omega}\big(|Df|\big) \in L^1(\mathbb{R}^N) \quad \forall \sigma \in S^{N-1}.$$

Thus

$$\sigma \in S^{N-1} \longmapsto \int \omega\big(|Df \cdot \sigma|\big) \in \mathbb{R} \quad \text{is continuous.}$$

We now let $j \to \infty$, $\delta \downarrow 0$ and $r \downarrow 0$ in (8.1). Applying Remark 3.1 and Proposition 4.1, we see that (1.15) holds in this case.

Step 2. (1.15) holds if ω is convex on $[R, \infty)$ for some $R > 0$.

It suffices to write ω as $\omega = \omega_1 + \omega_2$ in $[0, \infty)$, where

$$\omega_1(t) = \begin{cases} 0 & \text{if } 0 \le t \le R \\ \omega(t) - \omega(R) & \text{if } t > R \end{cases}.$$

In particular, ω_1 is convex and ω_2 is bounded (moreover, if Ω is unbounded we have $\omega_2(t) \le \tilde{\omega}(t) \le Ct$ for $t \ge 0$ small). We now apply the previous step to ω_1 and Lemma 7.1 to ω_2. This gives (1.15).

Step 3. Proof of Theorem 1.7 completed.

Let $R > 0$ fixed. For an arbitrary continuous function ω satisfying (1.14) we take two continuous functions $0 \le \underline{\omega} \le \omega \le \overline{\omega}$ such that $\underline{\omega} = \omega = \overline{\omega}$ on $[0, R]$, and $\underline{\omega} = 0$, $\overline{\omega} = \tilde{\omega}$ on $[R+1, \infty)$.

Applying Step 2 and Lemma 7.1 we conclude that

$$\begin{aligned}
\int_\Omega \underline{\omega}_\mu(Df) &\le \liminf_{j\to\infty} \int_\Omega\int_\Omega \omega\left(\frac{|f(x) - f(y)|}{|x-y|}\right) \rho_{\varepsilon_j}(x-y)\,dx\,dy \\
&\le \limsup_{j\to\infty} \int_\Omega\int_\Omega \omega\left(\frac{|f(x) - f(y)|}{|x-y|}\right) \rho_{\varepsilon_j}(x-y)\,dx\,dy \\
&\le \int_\Omega \overline{\omega}_\mu(Df).
\end{aligned}$$

Taking $R \to \infty$ we obtain (1.15) from the Dominated Convergence Theorem.

9 A characterization of $W^{1,p}$, $p > 1$, and BV. Proof of Theorem 1.8.

Suppose (ρ_ε) is a family of functions in $L^1(\mathbb{R}^N)$ satisfying (1.2).

Let $p \ge 1$ and $f \in L^p(\Omega)$ be such that

$$\int_\Omega\int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_\varepsilon(x-y)\,dx\,dy \le C \quad \forall \varepsilon > 0 \text{ small,} \tag{9.1}$$

for some $C > 0$.

It follows from Lemma 5.4 that for any $0 < \delta < r$ we have

$$\int_{\Omega_r \cap B_{1/r}} \int_{\Omega_r \cap B_{1/r}} \frac{|f_\delta(x) - f_\delta(y)|^p}{|x-y|^p} \rho_\varepsilon(x-y)\,dx\,dy \le C \quad \forall \varepsilon > 0 \text{ small.} \tag{9.2}$$

By Proposition 4.1 and Jensen's inequality (recall that $\mu(S^{N-1}) = 1$) we get

$$\begin{aligned}\int_{\Omega_r \cap B_{1/r}} \left\{ \int_{S^{N-1}} \left| Df_\delta(x) \cdot \sigma \right| d\mu(\sigma) \right\}^p dx \le \\ \le \int_{\Omega_r \cap B_{1/r}} \int_{S^{N-1}} \left| Df_\delta(x) \cdot \sigma \right|^p dx\, d\mu(\sigma) \le C,\end{aligned} \tag{9.3}$$

for every $\delta \in (0, r)$.

In the special case of Examples 3.1 and 3.2, it is easy to see that the measure μ satisfies the coercivity condition

$$\alpha |v| \le \int_{S^{N-1}} |v \cdot \sigma|\, d\mu(\sigma) \quad \forall v \in \mathbb{R}^N \tag{9.4}$$

for some $\alpha > 0$.

By (9.3) and (9.4) we conclude that

$$\int_{\Omega_r \cap B_{1/r}} |Df_\delta|^p \le \frac{C}{\alpha^p} \quad \forall \delta \in (0, r). \tag{9.5}$$

Therefore, $f \in W^{1,p}(\Omega)$ if $p > 1$, and $f \in BV(\Omega)$ if $p = 1$. In addition, the following estimate holds

$$\int_\Omega |Df|^p \le \frac{C}{\alpha^p}. \tag{9.6}$$

This proves Corollaries 1.9 and 1.10.

More generally, the above argument shows that in order to characterize the elements in $W^{1,p}(\Omega)$ for $p > 1$, or $BV(\Omega)$ for $p = 1$, by using (9.1) it suffices to show that (9.4) holds.

Let $I_\mu : S^{N-1} \to \mathbb{R}_+$ be the function given by

$$I_\mu(v) := \int_{S^{N-1}} |v \cdot \sigma|\, d\mu(\sigma) \quad \forall v \in S^{N-1}, \tag{9.7}$$

so that I_μ is continuous and (9.4) holds if, and only if, $I_\mu > 0$ in S^{N-1}. Conversely, $I_\mu(v_0) = 0$ for some $v_0 \in S^{N-1}$ if, and only if, $v_0 \perp \operatorname{supp} \mu$, i.e. $\operatorname{supp} \mu$ is contained in an $(N-1)$-dimensional vector space.

This simple remark implies the following

Lemma 9.1 (9.4) *holds if, and only if,* $\operatorname{supp} \mu$ *contains a basis of* $\mathbb{R}^N$*.*

Using the same reasoning as above, Theorem 1.8 follows easily from Lemma 9.1

10 Proof of Theorem 1.12.

Step 1. If $f \in W^{1,\omega}(\Omega)$, then there exists $\beta > 0$ such that (1.31) holds.

Using the Lipschitz regularity of $\partial\Omega$, we can extend f to the whole space $\mathbb{R}^N$ so that $f \in W^{1,\omega}(\mathbb{R}^N)$. By the definition of the Orlicz-Sobolev spaces, there exists $\beta > 0$ such that $\omega\big(\beta|Df|\big) \in L^1(\mathbb{R}^N)$. Estimate (1.31) now follows immediately from Lemma 5.1 applied to the function βf.

Step 2. If (1.31) is satisfied, then $f \in W^{1,\omega}(\Omega)$.

Let $\varepsilon_j \downarrow 0$ and $\mu \in M(S^{N-1})$ be as in (1.4). Without loss of generality, we may assume that there exists $\alpha > 0$ such that

$$\alpha|v| \leq \int_{S^{N-1}} |v \cdot \sigma| \, d\mu(\sigma) \quad \forall v \in \mathbb{R}^N. \tag{10.1}$$

Take $C > 0$ such that

$$\int_\Omega\int_\Omega \omega\left(\beta\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_{\varepsilon_j}(x-y)\,dx\,dy \leq C \quad \forall j \geq 1. \tag{10.2}$$

Proceeding as in Section 9 and using (10.1) we have

$$\int_{\Omega_r \cap B_{1/r}} \omega\big(\alpha\beta|Df_\delta|\big)\,dx \leq C \quad \forall \delta \in (0,r) \quad \forall r > 0. \tag{10.3}$$

In particular, we conclude that $f \in BV(\Omega)$. On the other hand, (1.28) implies that the family Df_δ is equi-integrable on the compact subsets of Ω. Therefore, $Df \in L^1_{\mathrm{loc}}(\Omega)$, and so $Df_\delta \to Df$ a.e. in Ω. Letting $\delta \downarrow 0$ we conclude that $\omega\big(\alpha\beta|Df|\big) \in L^1(\Omega)$.

11 Proof of Theorem 1.13.

Let $\varepsilon_j \downarrow 0$ and $\mu \in M(S^{N-1})$ be as in (1.4) and such that there exists $C > 0$ satisfying

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_{\varepsilon_j}(x-y)\,dx\,dy \leq C \quad \forall j \in \mathbb{N} \text{ large enough.} \tag{11.1}$$

Arguing as in Section 9 we conclude that

$$\int_{\mathbb{R}^N}\left\{\int_{S^{N-1}} \big|Df_\delta(x)\cdot\sigma\big|\,d\mu(\sigma)\right\}^p dx \leq C \quad \forall \delta > 0. \tag{11.2}$$

Define

$$F := \left\{w \in \mathbb{R}^N : \int_{S^{N-1}} |w \cdot \sigma|\,d\mu(\sigma) = 0\right\}, \tag{11.3}$$

so that F is a vector subspace and $F \subsetneqq \mathbb{R}^N$ since $\mu \geq 0$ and $\mu(S^{N-1}) = 1$.
Let $k := \dim F^\perp \geq 1$. Given $v = v' + v'' \in F \oplus F^\perp = \mathbb{R}^N$, we have

$$\begin{aligned}\int_{S^{N-1}} |v \cdot \sigma| \, d\mu(\sigma) &= \int_{S^{N-1} \cap F^\perp} |v'' \cdot \sigma''| \, d\mu(\sigma) \\ &= \int_{S^{k-1}} |v'' \cdot \sigma''| \, d\mu(\sigma'') \\ &\geq \tilde{\alpha} |v''| \quad \text{for some } \tilde{\alpha} > 0.\end{aligned} \tag{11.4}$$

By (11.3) and (11.4), we conclude that

$$\tilde{\alpha}^p \int_{\mathbb{R}^N} |D_{F^\perp} f_\delta|^p \leq C \quad \forall \delta > 0, \tag{11.5}$$

from which the theorem follows by letting $\delta \downarrow 0$ and taking $E = F^\perp$.

12 Proof of Theorem 1.15.

Throughout this section we shall assume that $\Omega \subset \mathbb{R}^N$ is bounded.
For each $j = 1, 2, \ldots$ we take

$$F_j(f) := \int_\Omega \int_\Omega \omega \left(\frac{|f(x) - f(y)|}{|x - y|} \right) \rho_{\varepsilon_j}(x - y) \, dx \, dy \quad \forall f \in L^1(\Omega). \tag{12.1}$$

Theorem 1.15 will be a consequence of the following two lemmas:

Lemma 12.1

$$\Gamma^-_{L^1(\Omega)}\text{-}\limsup_{j \to \infty} F_j(f) \leq \mathrm{sc}^-_{L^1(\Omega)} F(f) \quad \forall f \in L^1(\Omega), \tag{12.2}$$

where F is the functional given by (1.42).

Proof. Let $f \in C^1(\overline{\Omega})$. Taking the constant sequence $f_j := f$ for each $j \geq 1$ in (1.38), it follows from Remark 4.2 that

$$\Gamma^-_{L^1(\Omega)}\text{-}\limsup_{j \to \infty} F_j(f) \leq \lim_{j \to \infty} F_j(f) = \int_\Omega \omega_\mu(Df), \tag{12.3}$$

whence

$$\Gamma^-_{L^1(\Omega)}\text{-}\limsup_{j \to \infty} F_j(f) \leq F(f) \quad \forall f \in L^1(\Omega). \tag{12.4}$$

Since $\Gamma^-_{L^1(\Omega)}\text{-}\limsup_{j \to \infty} F_j$ is lower semicontinuous in $L^1(\Omega)$ (see [7]), (12.2) follows.

Lemma 12.2

$$\Gamma^-_{L^1(\Omega)}\text{-}\liminf_{j\to\infty} F_j(f) \geq \mathrm{sc}^-_{L^1(\Omega)} G_\Omega(f) \quad \forall f \in L^1(\Omega), \tag{12.5}$$

where for each open set $A \subset \mathbb{R}^N$ the functional G_A is defined as

$$G_A(g) = \begin{cases} \displaystyle\int_A (\omega^{**})_\mu(Dg) & \text{if } g \in C^1(\overline{A}), \\ +\infty & \text{if } g \in L^1(A)\backslash C^1(\overline{A}). \end{cases} \tag{12.6}$$

Proof. Fix $0 < \delta < r$. Let $f \in L^1(\Omega)$ and $(f_j) \subset L^1(\Omega)$ be such that $f_j \to f$ in $L^1(\Omega)$. Applying Lemma 5.4, for each $j \geq 1$ we have

$$\begin{aligned} F_j(f_j) &\geq \int_\Omega\int_\Omega \omega^{**}\left(\frac{|f_j(x)-f_j(y)|}{|x-y|}\right)\rho_{\varepsilon_j}(x-y)\,dx\,dy \\ &\geq \int_{\Omega_r}\int_{\Omega_r} \omega^{**}\left(\frac{|f_{j,\delta}(x)-f_{j,\delta}(y)|}{|x-y|}\right)\rho_{\varepsilon_j}(x-y)\,dx\,dy. \end{aligned} \tag{12.7}$$

Note that for each $\delta > 0$ fixed we have $f_{j,\delta} \to f_\delta$ in $C^2(\overline{\Omega}_r)$. It follows from Remark 4.1 that

$$\int_{\Omega_r}\int_{\Omega_r} \omega^{**}\left(\frac{|f_{j,\delta}(x)-f_{j,\delta}(y)|}{|x-y|}\right)\rho_{\varepsilon_j}(x-y)\,dx\,dy \overset{j\to\infty}{\longrightarrow} \int_{\Omega_r}(\omega^{**})_\mu(Df_\delta).$$

Therefore,

$$\liminf_{j\to\infty} F_j(f_j) \geq \int_{\Omega_r} \omega^{**}_\mu(Df_\delta) \quad \forall \delta \in (0,r). \tag{12.8}$$

Given $A \subset\subset \Omega$, let $r > 0$ sufficiently small so that $A \subset \Omega_r$. We have

$$\liminf_{j\to\infty} F_j(f_j) \geq G_A(f_\delta) \geq \mathrm{sc}^-_{L^1(A)} G_A(f_\delta) \quad \forall \delta \in (0,r). \tag{12.9}$$

Letting $\delta \downarrow 0$ and using the lower semicontinuity of $\mathrm{sc}^-_{L^1(A)} G_A$ in $L^1(A)$, we conclude that

$$\liminf_{j\to\infty} F_j(f_j) \geq \sup\left\{ \mathrm{sc}^-_{L^1(A)} G_A(f) : A \subset\subset \Omega\right\}. \tag{12.10}$$

Since $\Omega \subset \mathbb{R}^N$ is a bounded open set with Lipschitz boundary and $(\omega^{**})_\mu$ is convex, we can apply Theorem 4.4 in [5] which implies that

$$\sup\left\{ \mathrm{sc}^-_{L^1(A)} G_A(f) :\ A \subset\subset \Omega\right\} = \mathrm{sc}^-_{L^1(\Omega)} G_\Omega(f). \tag{12.11}$$

Since the sequence $f_j \to f$ in $L^1(\Omega)$ was arbitrary, (12.5) follows from (12.10) and (12.11).

Proof of Theorem 1.15. Let

$$\tilde{G}(f) = \begin{cases} \int_\Omega \omega_\mu^{**}(Df) & \text{if } f \in C^1(\overline{\Omega}), \\ +\infty & \text{if } f \in L^1(\Omega)\backslash C^1(\overline{\Omega}), \end{cases} \tag{12.12}$$

then

$$\mathrm{sc}^-_{L^1(\Omega)} F(f) = \mathrm{sc}^-_{L^1(\Omega)} \tilde{G}(f) \quad \forall f \in L^1(\Omega). \tag{12.13}$$

(This follows from (1.39) and (1.43)).

By hypothesis, $\omega_\mu^{**} = (\omega^{**})_\mu$, so that $G_\Omega(f) = \tilde{G}(f)$ for every $f \in L^1(\Omega)$. Theorem 1.15 now follows from (12.2), (12.5) and (12.13).

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