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A new approach to stability and stabilization analysis for continuous-time Takagi-Sugeno fuzzy systems with time delay

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Abstract—Till now, there are lots of stability and stabilization results about T-S (Takagi-Sugeno) fuzzy systems with time delay, but most of them are independent of the analysis of membership functions. Since the membership functions are an essential component to make a fuzzy system different from others, the conditions without its information are conservative. In this brief paper, a new Lyapunov-Krasovskii functional is designed to investigate the stability and stabilization of continuous-time T-S fuzzy systems with time delay. Different from the existing results in the literature, the integrand of the Lyapunov-Krasovskii functional in this paper depends not only on the integral variable but also on the membership functions, and thus, the information of the time-derivative of membership functions can also be used to reduce the conservativeness of finding the maximum delay bounds. Utilizing the information of the time-derivative of membership, a bunch of controllers are designed according to their sign, and then a switching idea is applied to stabilize the fuzzy system. In the end, two examples are given to illustrate the feasibility and validity of the design and analysis.

Keywords: Takagi–Sugeno's fuzzy model, Time delay, Parallel distributed compensation law, Membership dependent Lyapunov-Krasovskii function.

I. INTRODUCTION

In the recent decades, the stability and stabilization analysis for fuzzy systems ([24]-[30]) especially for Takagi-Sugeno fuzzy model [1] with time delay has been a hot topic (see [2]-[13], [22], [23] and the references therein) and the focus is on how to maximize the tolerance of time delay. All kinds of Lyapunov-Krasovskii functional have been developed to reduce the conservativeness of finding the maximum delay bounds, for example, an augmented Lyapunov-Krasovskii functional is proposed in [5] and an improved Jensen's inequality which is a more general and tighter bounding technology is used to deal with the cross product terms. The results in [5] are improved by [8] where a fuzzy weighting-dependent Lyapunov-Krasovskii functional is designed for

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uncertain T-S fuzzy systems with time-varying delays. The results in [8] are less conservative than [5] because of an input-output approach and the fuzzy weighting-dependent Lyapunov-Krasovskii functional. A discretized Lyapunov-Krasovskii functional is proposed in [2] where the number of free variables increases as the discretization level increases. These variables help to reduce the conservativeness but increase the computing burden heavily. For the case of constant time delay, the results in [5], [8] and [2] are further improved by [11] where new simple and effective stabilization conditions are proposed by applying the Wirtinger inequality [14] and fuzzy line-integral Lyapunov functional [15]. The results in [11] are improved by [17] where a new augmented Lyapunov-Krasovskii functional is constructed and some new cross terms are included. For the first time, a distributed fuzzy optimal control law relied on actual physical meaning by adaptive dynamic programming algorithm is proposed in [20] which gives a new thought for the optimization of multi-agent system. However, all of the results mentioned above are independent of the analysis of membership functions which will lead to conservativeness because the membership functions are an essential component to make a fuzzy system different from other systems. More recently, a membership function dependent Lyapunov-Krasovskii functional is designed in [18] and some less conservative results are obtained by analyzing the time derivative of the membership function, however the drawback is that the local stabilization region becomes smaller as the delay increases.

Based on the above discussions, the contributions of this paper are as follows: 1), A new membership-function dependent Lyapunov-Krasovskii functional is designed in this paper which is different from the existing ones such as [11], [17] where the integrands are independent of the membership functions. 2), A new method is used to deal with the time derivative of the Lyapunov-Krasovskii functional. Since the integrand of the Lyapunov-Krasovskii functional depends not only on the integral variable but also on the membership functions, a switching idea is applied to ensure that the time derivative of the Lyapunov-Krasovskii functional is negative. Due to the above contributions, the obtained stability and stabilization criteria improve the existing results. In the end, two examples are provided for verification. One example is for stability and the other one is for stabilization. Both examples show that less conservative results can be obtained in this paper than the existing ones in the literature.

II. PRELIMINARIES AND BACKGROUNDS

Consider the following nonlinear model

$$\begin{aligned}
\dot{x}(t) &= f_1(z(t))x(t) + f_2(z(t))x(t-\tau) \\
&\quad + f_3(z(t))u(t), \\
x(t) &= \phi(t), t \in [-\tau \quad 0]
\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input, $z(t) \in \mathbb{R}^p$ is known premise variables. $f_1(\cdot)$, $f_2(\cdot)$ and $f_3(\cdot)$ are nonlinear functions or matrix functions with proper dimensions, $\phi(t)$ is the initial condition and the time delay τ is assumed to be constant. Applying the sector nonlinearity method or local approximation method in [16], one has the following well known time delay T-S fuzzy model:

$$\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{r} h_i(z(t))A_i x(t) + \sum_{i=1}^{r} h_i(z(t))A_{\tau i} x(t-\tau) \\
&\quad + \sum_{i=1}^{r} h_i(z(t))B_i u(t), \\
x(t) &= \phi(t), t \in [-\tau \quad 0],
\end{aligned} \tag{2}$$

where $A_i \in \mathbb{R}^{n \times n}$, $A_{\tau i} \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$ are known matrices and $h_i(z(t))$ are membership functions. To lighten the notation, we will drop the time t, for instance, we will use x instead of $x(t)$. For simplicity, single sums are written as $X_h = \sum_{i=1}^{r} h_i X_i$. For any matrix X, $\text{He}(X) = X + X^T$.

In this paper, the following controller (3) is used to stabilize the fuzzy system

$$u = K_h x(t) + K_{\tau h} x(t-\tau). \tag{3}$$

The following Lemma 1 is useful in this paper.

Lemma 1: (see [19]) For a function $F(t) = \int_{\alpha_1(t)}^{\alpha_2(t)} f(s,t)\,d_s$, if $\alpha_1(t)$, $\alpha_2(t)$ are differentiable for t and $f(s,t)$ is continuous for s and patrially derivative for t, we have

$$\begin{aligned}
\frac{dF(t)}{dt} &= \dot{\alpha}_2(t)f(\alpha_2(t),t) - \dot{\alpha}_1(t)f(\alpha_1(t),t) \\
&\quad + \int_{\alpha_1(t)}^{\alpha_2(t)} \frac{\partial f(s,t)}{\partial t}\,d_s.
\end{aligned} \tag{4}$$

In the following, we will discuss how to ensure $\dot{X}_h \leq 0$, $\dot{Y}_h \leq 0$ and $\dot{Z}_h \leq 0$ where $X_i > 0$, $Y_i > 0$ and $Z_i > 0$. Note

$$\dot{X}_h = \sum_{i=1}^{r} \dot{h}_i X_i = \sum_{k=1}^{r-1} \dot{h}_k(X_k - X_r), \tag{5}$$

$$\dot{Y}_h = \sum_{i=1}^{r} \dot{h}_i Y_i = \sum_{k=1}^{r-1} \dot{h}_k(Y_k - Y_r), \tag{6}$$

$$\dot{Z}_h = \sum_{i=1}^{r} \dot{h}_i Z_i = \sum_{k=1}^{r-1} \dot{h}_k(Z_k - Z_r), \tag{7}$$

where $\dot{h}_k$ are the time-derivative of membership functions and are negative or positive as time goes by. Since X_i, Y_i and Z_i are variables to be designed, we can use a switching idea to

ensure $\dot{X}_h \leq 0$, $\dot{Y}_h \leq 0$ and $\dot{Z}_h \leq 0$ as follows:

$$\begin{cases}
\text{if } \dot{h}_k \leq 0, \text{then } X_k - X_r \geq 0, Y_k - Y_r \geq 0, Z_k - Z_r \geq 0, \\
\text{if } \dot{h}_k > 0, \text{then } X_k - X_r < 0, Y_k - Y_r < 0, Z_k - Z_r < 0.
\end{cases} \tag{8}$$

There are 2^{r-1} possible cases in (8). Let H_l, $l = 1, 2, \cdots, 2^{r-1}$ be the set that contains the possible permutations of $\dot{h}_k$ and C_l be the set that contains the constraints of X_i, Y_i and Z_i, (8) can be presented as

$$\text{if } H_l, \text{then } C_l. \tag{9}$$

For example, if $r = 3$, we have

$$\begin{aligned}
\dot{X}_h &= \dot{h}_1(X_1 - X_3) + \dot{h}_2(X_2 - X_3), \\
\dot{Y}_h &= \dot{h}_1(Y_1 - Y_3) + \dot{h}_2(Y_2 - Y_3), \\
\dot{Z}_h &= \dot{h}_1(Z_1 - Z_3) + \dot{h}_2(Z_2 - Z_3).
\end{aligned}$$

There are 2^2 constraints C_l, $l = 1, 2, 3, 4$ to ensure $\dot{X}_h \leq 0$, $\dot{Y}_h \leq 0$, $\dot{Z}_h \leq 0$ and (9) is expressed as following

If H_1, then C_1; If H_2, then C_2;

If H_3, then C_3; If H_4, then C_4,

where

$$\begin{aligned}
H_1 &: \dot{h}_1 \leq 0, \dot{h}_2 \leq 0; \quad H_2 : \dot{h}_1 \leq 0, \dot{h}_2 > 0; \\
H_3 &: \dot{h}_1 > 0, \dot{h}_2 \leq 0; \quad H_4 : \dot{h}_1 > 0, \dot{h}_2 > 0,
\end{aligned}$$

$$C_1 : \begin{cases} X_1 \geq X_3, X_2 \geq X_3, Y_1 \geq Y_3, \\ Y_2 \geq Y_3, Z_1 \geq Z_3, Z_2 \geq Z_3. \end{cases},$$

$$C_2 : \begin{cases} X_1 \geq X_3, X_2 < X_3, Y_1 \geq Y_3, \\ Y_2 < Y_3, Z_1 \geq Z_3, Z_2 < Z_3. \end{cases},$$

$$C_3 : \begin{cases} X_1 < X_3, X_2 \geq X_3, Y_1 < Y_3, \\ Y_2 \geq Y_3, Z_1 < Z_3, Z_2 \geq Z_3. \end{cases},$$

$$C_4 : \begin{cases} X_1 < X_3, X_2 < X_3, Y_1 < Y_3, \\ Y_2 < Y_3, Z_1 < Z_3, Z_2 < Z_3. \end{cases}.$$

Based on the above discussion, we get the following lemma.

Lemma 2: For some membership function dependent matrices X_h, Y_h and Z_h where $X_i > 0$, $Y_i > 0$ and $Z_i > 0$ are free variables, we have $\dot{X}_h \leq 0$, $\dot{Y}_h \leq 0$ and $\dot{Z}_h \leq 0$, if the switching rules (9) are satisfied where $l = 1, 2, \cdots, 2^{r-1}$.

If Lemma 2 is used to get some stabilization conditions, for different C_l and H_l, $l = 1, \cdots, 2^{r-1}$, the corresponding controller is

$$u_l = K_{l,h}x(t) + K_{l,\tau h}x(t-\tau), \tag{10}$$

$$K_{l,h} = \sum_{i=1}^{r} h_i K_{l,i}, K_{l,\tau h} = \sum_{i=1}^{r} h_i K_{l,\tau i}.$$

The final controller (3) becomes a switching controller below

$$u : \begin{cases}
u_1 = K_{1,h}x(t) + K_{1,\tau h}x(t-\tau) \text{ for } H_1, \\
\quad\quad\quad\quad \vdots \\
u_l = K_{l,h}x(t) + K_{l,\tau h}x(t-\tau) \text{ for } H_l.
\end{cases} \tag{11}$$

III. Main Results

In this section, the stability and stabilization problem for T-S fuzzy system with time delay is considered by utilizing new Lyapunov-Krasovskii functional and a switching control method.

A. Stability

Theorem 1: For a given scalar $\tau > 0$, the closed-loop T-S fuzzy system (2) is globally asymptotically stable if there exist matrices $\begin{bmatrix} P_{11i} & P_{12i} \\ * & P_{22i} \end{bmatrix} > 0, Q_i > 0, Z_i > 0$, M_{1i}, M_{2i}, M_{3i}, M_{4i}, such that the following inequalities hold for $i, j = 1, \cdots, r$,

$$\dot{Q}_h \leq 0, \dot{Z}_h \leq 0, \dot{\mathbb{P}}_h \leq 0, \tag{12}$$

$$\mathbb{P}_h = \begin{bmatrix} P_{11h} & P_{12h} \\ * & P_{22h} \end{bmatrix},$$

$$\Theta_{ij} + \Theta_{ji} \leq 0, \tag{13}$$

$$\Theta_{ij} = \begin{bmatrix} \Theta_{11ij} & \Theta_{12ij} & \Theta_{13ij} & \Theta_{14ij} \\ * & \Theta_{22ij} & \Theta_{23ij} & \Theta_{24ij} \\ * & * & -12\tau^{-2}Z_i & \Theta_{34ij} \\ * & * & * & \Theta_{44ij} \end{bmatrix},$$

$$\begin{aligned}
\Theta_{11ij} &= \text{He}\,(P_{12i} + M_{1i}A_j) + Q_i - 4Z_i, \\
\Theta_{12ij} &= -P_{12i} - 2Z_i + M_{1i}A_{\tau j} + A_j^T M_{3i}^T, \\
\Theta_{13ij} &= P_{22i} + 6\tau^{-1}Z_i + A_j^T M_{4i}^T, \\
\Theta_{14ij} &= P_{11i} - M_{1i} + A_j^T M_{2i}^T, \\
\Theta_{22ij} &= -Q_i - 4Z_i + \text{He}\,(M_{3i}A_{\tau j}), \\
\Theta_{23ij} &= -P_{22i} + 6\tau^{-1}Z_i + A_{\tau j}^T M_{4i}^T, \\
\Theta_{24ij} &= A_{\tau j}^T M_{2i}^T - M_{3i}, \\
\Theta_{34ij} &= P_{12i}^T - M_{4i}, \Theta_{44ij} = \tau^2 Z_i - \text{He}\,(M_{2i}).
\end{aligned}$$

Proof: Choose the Lyapunov-Krasovskii functional as

$$V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t), \tag{14}$$

$$V_1(x_t) = \rho^T \mathbb{P}_h \rho, \rho^T = \begin{bmatrix} x(t)^T & \int_{t-\tau}^t x(s)^T ds \end{bmatrix},$$

$$V_2(x_t) = \int_{t-\tau}^t x(s)^T Q_h x(s) ds,$$

$$V_3(x_t) = \tau \int_{-\tau}^0 \int_{t+\theta}^t \dot{x}(s)^T Z_h \dot{x}(s) ds d\theta.$$

It follows that

$$\dot{V}_1(x_t) = 2\rho^T \mathbb{P}_h \dot{\rho} + \rho^T \dot{\mathbb{P}}_h \rho.$$

Applying Lemma 1, we have

$$\begin{aligned}
\dot{V}_2(x_t) &= x^T Q_h x - x(t-\tau)^T Q_h x(t-\tau) \\
&\quad + \int_{t-\tau}^t x(s)^T \dot{Q}_h x(s) ds, \tag{15}
\end{aligned}$$

$$\begin{aligned}
\dot{V}_3(x_t) &= \tau^2 \dot{x}^T Z_h \dot{x} - \tau \int_{t-\tau}^t \dot{x}(s)^T Z_h \dot{x}(s) ds \\
&\quad + \tau \int_{-\tau}^0 \int_{t+\theta}^t \dot{x}(s)^T \dot{Z}_h \dot{x}(s) ds d\theta. \tag{16}
\end{aligned}$$

Considering the constraints in (12), we have

$$\begin{aligned}
\dot{V}(x_t) &\leq 2\rho^T \mathbb{P}_h \begin{bmatrix} \dot{x}(t) \\ x(t) - x(t-\tau) \end{bmatrix} \\
&\quad + x(t)^T Q_h x(t) - x(t-\tau)^T Q_h x(t-\tau) \\
&\quad + \tau^2 \dot{x}(t)^T Z_h \dot{x}(t) - \tau \sum_{i=1}^r h_i \int_{t-\tau}^t \dot{x}(s)^T Z_i \dot{x}(s) ds.
\end{aligned}$$

Using the zero equation

$$2\mathbb{M} \times (A_h x + A_{\tau h} x(t-\tau) - \dot{x}) = 0, \tag{17}$$

$$\mathbb{M} = x^T M_{1h} + \dot{x}^T M_{2h} + x(t-\tau)^T M_{3h} + \int_{t-\tau}^t x(s)^T ds M_{4h},$$

and the results in [14] to deal with the term $-\tau \int_{t-\tau}^t \dot{x}(s)^T Z_i \dot{x}(s) ds$, we have $\dot{V}(x_t) < \eta^T \Omega \eta$ which can be ensured by (13), where

$$\eta^T = \begin{bmatrix} x(t)^T & x(t-\tau)^T & \int_{t-\tau}^t x(s)^T ds & \dot{x}(t)^T \end{bmatrix},$$

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} \\ * & \Omega_{22} & \Omega_{23} & \Omega_{24} \\ * & * & -12\tau^{-2}Z_h & \Omega_{34} \\ * & * & * & \Omega_{44} \end{bmatrix},$$

$$\begin{aligned}
\Omega_{11} &= \text{He}\,(P_{12h} + M_{1h}A_h) + Q_h - 4Z_h, \\
\Omega_{12} &= -P_{12h} - 2Z_h + M_{1h}A_{\tau h} + A_h^T M_{3h}^T, \\
\Omega_{13} &= P_{22h} + 6\tau^{-1}Z_h + A_h^T M_{4h}^T, \\
\Omega_{14} &= P_{11h} - M_{1h} + A_h^T M_{2h}^T, \\
\Omega_{22} &= -Q_h - 4Z_h + \text{He}\,(M_{3h}A_{\tau h}), \\
\Omega_{23} &= -P_{22h} + 6\tau^{-1}Z_h + A_{\tau h}^T M_{4h}^T, \\
\Omega_{24} &= A_{\tau h}^T M_{2h}^T - M_{3h}, \\
\Omega_{34} &= P_{12h}^T - M_{4h}, \Omega_{44} = \tau^2 Z_h - \text{He}\,(M_{2h}).
\end{aligned}$$

∎

Remark 1: Note, the Lyapunov-Krasovskii functional used in Theorem 1 is different from the one used in the literature such as [11], [17], [18]. In $V_2(x_t)$ and $V_3(x_t)$ of this paper, Q_h and Z_h are independent of the integral variable s but on the time t, so the information of the time-derivative of the membership function can be used to obtain new less conservative stability conditions, at the same time, there comes a problem that how to deal with the integral part $\int_{t-\tau}^t x(s)^T \dot{Q}_h x(s) ds$ and $\tau \int_{-\tau}^0 \int_{t+\theta}^t \dot{x}(s)^T \dot{Z}_h \dot{x}(s) ds d\theta$. A simple and direct method is letting $\dot{Q}_h \leq 0$, $\dot{Z}_h \leq 0$ and $\dot{\mathbb{P}}_h \leq$

0. Since $\dot{Q}_h = \sum_{k=1}^{r-1} \dot{h}_k (Q_k - Q_r)$, $\dot{Z}_h = \sum_{k=1}^{r-1} \dot{h}_k (Z_k - Z_r)$, $\dot{\mathbb{P}}_h = \sum_{k=1}^{r-1} \dot{h}_k (\mathbb{P}_k - \mathbb{P}_r)$, then $\dot{Q}_h \leq 0$, $\dot{Z}_h \leq 0$ and $\dot{\mathbb{P}}_h \leq 0$ can be ensured by the switching rules (9). Then, based on Lemma 2, applying (13) with each C_l, we obtain a maximum delay denoted as τ_l, and the final delay obtained by Theorem 1 is $\tau = \min_{1 \leq l < 2^{r-1}} (\tau_l)$. Since Theorem 1 does not depend on the initial conditions of the fuzzy systems, the switching method is feasible. In this paper, $\dot{X}_h$, $\dot{Y}_h$ and $\dot{Z}_h$ are presented as (5), (6) and (7). Since $\sum_{i=1}^{r} \dot{h}_i = 0$, there are some other expressions, for example, $\sum_{i=1}^{r-l-1} \dot{h}_i + \sum_{j=r-l+1}^{r} \dot{h}_j = -\dot{h}_{r-l}$, $l = 1, 2, \cdots r-1$. We can also use these expressions to derive the results.

B. Stabilization

Theorem 2: For a given scalar $\tau > 0$ and parameters λ_1, λ_2, the closed-loop T-S fuzzy system (2) is asymptotically stabilized by the controller (3), if there exist matrices $\begin{bmatrix} \bar{P}_{11j} & \bar{P}_{12j} \\ * & \bar{P}_{22j} \end{bmatrix} > 0$, $\bar{Q}_i > 0$, $\bar{Z}_i > 0$, $\bar{M}$, such that (12) and the following LMIs hold for $i, j = 1, \cdots, r$,

$$\bar{\Omega}_{ij} + \bar{\Omega}_{ji} \leq 0, \qquad (18)$$

$$\bar{\Omega}_{ij} = \begin{bmatrix} \bar{\Omega}_{11ij} & \bar{\Omega}_{12ij} & \bar{\Omega}_{13ij} & \bar{\Omega}_{14ij} \\ * & \bar{\Omega}_{22ij} & \bar{\Omega}_{23ij} & \bar{\Omega}_{24ij} \\ * & * & \bar{\Omega}_{33ij} & \bar{P}_{12i}^T \\ * & * & * & \bar{\Omega}_{44ij} \end{bmatrix},$$

$$\begin{aligned} \bar{\Omega}_{11ij} &= \text{He}\left(\bar{P}_{12i} + \bar{\mathcal{A}}\right) + \bar{Q}_i - 4\bar{Z}_i, \\ \bar{\Omega}_{12ij} &= -\bar{P}_{12i} - 2\bar{Z}_i + \bar{\mathcal{A}}_\tau + \lambda_2 \bar{\mathcal{A}}^T, \\ \bar{\Omega}_{13ij} &= \bar{P}_{22i} + 6\tau^{-1}\bar{Z}_i, \\ \bar{\Omega}_{14ij} &= \bar{P}_{11i} - \bar{M} + \lambda_1 \bar{\mathcal{A}}^T, \\ \bar{\Omega}_{22ij} &= -\bar{Q}_i - 4\bar{Z}_i + \text{He}\left(\lambda_2 \bar{\mathcal{A}}_\tau\right), \\ \bar{\Omega}_{23ij} &= -\bar{P}_{22i} + 6\tau^{-1}\bar{Z}_i, \bar{\Omega}_{24ij} = \lambda_1 \bar{\mathcal{A}}_\tau^T - \lambda_2 \bar{M}, \\ \bar{\Omega}_{33ij} &= -12\tau^{-2}\bar{Z}_i, \bar{\Omega}_{44ij} = \tau^2 \bar{Z}_i - \text{He}\left(\lambda_1 \bar{M}\right), \end{aligned}$$

$$\bar{\mathcal{A}} = A_j \bar{M} + B_j \bar{K}_i, \bar{\mathcal{A}}_\tau = A_{\tau j}\bar{M} + B_j \bar{K}_{\tau i},$$

and the controller gains are $K_i = \bar{K}_i \bar{M}^{-1}$, $K_{\tau i} = \bar{K}_{\tau i}\bar{M}^{-1}$.

Proof: The extension from stability to stabilization is straightforward. In order to avoid too many parameters, let $\bar{M} = M_1^{-T}$, $M_{2h} = \lambda_1 M_1$, $M_{3h} = \lambda_2 M_1$, $M_{4h} = 0$, using the controller (3) and replacing A_h and $A_{\tau h}$ with $\mathcal{A}_h$ and $\mathcal{A}_{\tau h}$ respectively where

$$\mathcal{A} = A_h + B_h K_h, \mathcal{A}_\tau = A_{\tau h} + B_h K_{\tau h}.$$

Pre-and post-multiplying both sides of $\Omega < 0$ with $\text{diag}\{M_1^{-1}, M_1^{-1}, M_1^{-1}, M_1^{-1}\}$ and its transpose respectively and defining $K_i \bar{M} = \bar{K}_i$, $K_{\tau i}\bar{M} = \bar{K}_{\tau i}$, $\bar{M}^T P_{11h}\bar{M} = \bar{P}_{11h}$, $\bar{M}^T P_{12h}\bar{M} = \bar{P}_{12h}$, $\bar{M}^T P_{22h}\bar{M} = \bar{P}_{22h}$, $\bar{M}^T Q_h \bar{M} = \bar{Q}_h$, $\bar{M}^T Z_h \bar{M} = \bar{Z}_h$, we get (18). ∎

Remark 2: Similar to the analysis in Remark 1, applying (18) with C_l, we obtain a maximum delay denoted as τ_l,

$l = 1, \cdots, 2^{r-1}$ and the corresponding controller denoted as (10) The final maximal delay is $\tau = \min_{l=1,\cdots,2^{r-1}} (\tau_l)$ and the final controller is (11). For any initial states $\phi(0)$, it can be driven to the origin by a sequence of controller u_l activated by H_l. Note, there are 2^{r-1} constraints in C_l and we need compute $2^r - 1$ times to get the maximum delay bound. For each time, the number of LMIs is $3 \times 2^{r-1} + 3 + r(r+1)/2$ and the computation burden may increase as the number of rules increase. The number of decision variables of Theorem 1 is $n(3n+2)r + 4n^2 r$ and Theorem 2 is $n(3n+2)r + n^2 r + 2mnr$.

Remark 3: Comparing with the existing results in the literature, the merit of this paper is that the Lyapunov-Krasovskii functional is dependent on the membership functions and the time derivatives of the membership functions are also considered. If we let Q_h, Z_h and $\mathbb{P}_h$ be membership function independent as Q, Z, $\mathbb{P}$ and the slack variables be $M_{3h} = 0$, $M_{4h} = 0$, Theorem 1 and Theorem 2 in this paper will become the results in [11], so this brief paper contains [11] as a special case. Another possible choice is letting $V_2(x_t) = \int_{t-\tau}^{t} x(s)^T Q_{h(s)} x(s) ds$ where $Q_{h(s)} = \sum_{i=1}^{r} h_i (z(s)) Q_i$ is dependent on the integral variable s, but it has been shown in [18] that this choice could not help reduce the conservativeness.

Remark 4: The differences between this paper and [18] are as follows: 1), The chosen Lyapunov-Krasovskii functional is different. In this paper, the integrand of $V_2(x_t)$ and $V_3(x_t)$ depends not only on the integral variable s but also on the membership functions, while in [18], $V_2(x_t)$ and $V_3(x_t)$ are independent of the membership function. 2), The method to deal with the time derivatives of the membership function is completely different. In this paper, a novel switching idea is applied to ensure the time-derivative of the Lyapunov-Krasovskii functional is negative, while in [18], the upper bounds on the time-derivative of the membership functions defined a priori by using the mod function and floor function $\lfloor \cdot \rfloor$. 3), The applicable scope is different. In this paper, Theorem 1 is global stability and Theorem 2 is global stabilization, while in [18], the results are only local and the obtained local stabilization region becomes smaller as the delay increases.

IV. NUMERICAL EXAMPLES

In this section, two examples are presented to demonstrate the effectiveness of the proposed method. Example 1 is for stability and Example 2 is for stabilization.

Example 1: Consider the following two-rule fuzzy system that has been studied in *[2], [5], [8], [11], [17]*,

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 0.5 \\ 0 & -1 \end{bmatrix},$$

$$A_{\tau 1} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, A_{\tau 2} = \begin{bmatrix} -1 & 0 \\ 0.1 & -1 \end{bmatrix}.$$

This open loop fuzzy system has been studied extensively in the literatures and the goal is to compute the maximum delay τ under which the fuzzy system is still stable. Table 1 shows the maximum delay τ obtained by different methods. From this table, we can see the best result is obtained by applying

the method proposed in this paper. In this paper, λ_1 and λ_2 are searched by using the most common brute-force algorithm and the searching scope is $[-1 \quad 3]$ with step 0.01. For example, the largest delay $\tau = 2.5932$ is obtained by using the latest method in *[17]*, while applying Theorem 1 with constraints $C_1 : \{Q_1 \geq Q_2, Z_1 \geq Z_2, \mathbb{P}_1 \geq \mathbb{P}_2\}$, we get $\tau_1 = 3.6610$ and with constraint $C_2 : \{Q_1 < Q_2, Z_1 < Z_2, \mathbb{P}_1 < \mathbb{P}_2\}$, we get $\tau_2 = 3.4859$, so the final maximal delay is $\tau = \min_{1 \leq l \leq 2} (\tau_l) = 3.4859$. Obviously, the method in this paper is much less conservative than the existing ones.

In order to show what leads to the less conservative results, we consider several cases: **(I)** means $M_{1h} = M_1$, $M_{2h} = M_2$, $M_{3h} = M_3$, $M_{4h} = M_4$; **(II)** means $M_{1h} = M_1$, $M_{2h} = M_2$, $M_{3h} = 0$, $M_{4h} = 0$; **(III)** means $M_{1h} = M_1$, $M_{2h} = M_2$, $M_{3h} = 0$, $M_{4h} = 0$, $\mathbb{P}_h = \mathbb{P}$, $Q_h = Q$ and $Z_h = Z$.

For **(I)**, applying Theorem 1 with constraint C_1 and C_2, we get $\tau_1 = 3.6384$ and $\tau_2 = 3.4859$ respectively, so the final maximal delay is $\tau = \min_{1 \leq l \leq 2} (\tau_l) = 3.4859$ which shows that whether the variables M_{1h}, M_{2h}, M_{3h}, M_{4h} are dependent on the membership function has no effect on the results; for **(II)**, we get $\tau = 2.9709$ which shows M_{3h}, M_{4h} can help reduce the conservativeness; for **(III)**, we only get $\tau = 2.0183$ which is more conservative than the existing results and shows that $\mathbb{P}_h$, Q_h, Z_h are very important in getting the less conservative results.

TABLE I
THE MAXIMUM DELAY τ OBTAINED BY DIFFERENT METHODS

Method	Max τ	Method	Max τ
[8]	2.1563	Theorem 1	3.4859
[2]	2.2943	Theorem 1 with (I)	3.4859
[11]	2.2944	Theorem 1 with (II)	2.9709
[17]	2.5932	Theorem 1 with (III)	2.0183

Example 2: Consider the two-rule fuzzy system that has been studied in [2], [10], [11], [17], [18] where the system and input matrices are as follows:

$$A_1 = \begin{bmatrix} 0 & 0.6 \\ 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, B_1 = B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$A_{\tau 1} = \begin{bmatrix} 0.5 & 0.9 \\ 0 & 2 \end{bmatrix}, A_{\tau 2} = \begin{bmatrix} 0.9 & 0 \\ 1 & 1.6 \end{bmatrix}.$$

This two-rule fuzzy system has been studied extensively in the literature in the past several years and the goal is to compute the maximum delay τ under which the fuzzy system can be stabilized by the designed controller. Using different methods to compute the maximum delay τ we get Table 2 which shows that better results can be obtained by using the method proposed in this paper than the ones in the literatures. For example, the delay $\tau = 1.4257$ ($\varepsilon = 1.26$) is obtained by using the method in *[11]*, while applying Theorem 2 with constraints C_1, and searching $\lambda_1 = 2.8$, $\lambda_2 = 1.1$ we get $\tau_1 = 1.6499$ and with constraint C_2, $\lambda_1 = 2.8$, $\lambda_2 = 1.1$, we also get $\tau_2 = 1.6499$, so the final maximum delay is $\tau = \min_{1 \leq l \leq 2} (\tau_l) = 1.6499$ which is much less conservative than the existing ones. The result in [18] ($\tau = 1.6421$) is comparable to the result

obtained in this paper, but it is only a local stabilization and the stabilization region is very small as $\tau = 1.6421$.

TABLE II
THE MAXIMUM DELAY τ OBTAINED BY DIFFERENT METHODS

Method	Max τ	Method	Max τ
[18]	1.6421	Theorem 2	1.6499
[17]	1.4214	Theorem 2 with(I)	1.5854
[11]	1.4257	Theorem 2 with(II)	1.2431

Similar to Example 1, we consider two cases: **(I)** means $\lambda_2 = 0$; **(II)** means $\lambda_2 = 0$, $\mathbb{P}_h = \mathbb{P}$, $Q_h = Q$ and $Z_h = Z$.

For **(I)**, applying Theorem 2 with constraint C_1, $\lambda_1 = 1.2$, we get $\tau_1 = 1.5854$ and applying Theorem 2 with C_2, $\lambda_1 = 1.98$, we get $\tau_2 = 1.6386$, so $\tau = \min_{1 \leq l \leq 2} (\tau_l) = 1.5854$ which shows λ_2 can improve the result but the improvement is not obvious. Similarly, for **(II)**, applying Theorem 2 with $\lambda_1 = 1.28$ we get $\tau_1 = 1.2431$ which shows that $\mathbb{P}_h$, Q_h and Z_h are very important for Theorem 2.

Suppose the membership functions are $h_1 = \dfrac{1}{1 + e^{-2x_1}}$ and $h_2 = \dfrac{e^{-2x_1}}{1 + e^{-2x_1}}$. Applying Theorem 2 with C_1 and C_2, we get the maximum delay $\tau = 1.6499$ and two controllers u_1 and u_2 which correspond to $H_1 : \dot{h}_1 \leq 0$ and $H_2 : \dot{h}_1 > 0$ respectively

$$u : \begin{cases} u_1 = K_{1,h}x(t) + K_{1,\tau h}x(t-\tau) \text{ for } H_1, \\ u_2 = K_{2,h}x(t) + K_{2,\tau h}x(t-\tau) \text{ for } H_2, \end{cases}$$

where

$$\begin{aligned} K_{1,1} &= \begin{bmatrix} 14.4427 & -44.1025 \end{bmatrix}, \\ K_{1,2} &= \begin{bmatrix} 4.0527 & -15.4395 \end{bmatrix}, \\ K_{1,\tau 1} &= \begin{bmatrix} -0.1537 & -1.5824 \end{bmatrix}, \\ K_{1,\tau 2} &= \begin{bmatrix} -1.3308 & -1.4258 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} K_{2,1} &= \begin{bmatrix} 5.2612 & -16.3162 \end{bmatrix}, \\ K_{2,2} &= \begin{bmatrix} 4.3167 & -16.5431 \end{bmatrix}, \\ K_{2,\tau 1} &= \begin{bmatrix} -0.3104 & -1.1512 \end{bmatrix}, \\ K_{2,\tau 2} &= \begin{bmatrix} -1.2354 & -1.5659 \end{bmatrix}. \end{aligned}$$
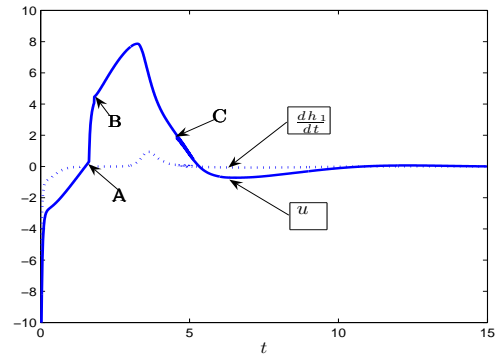

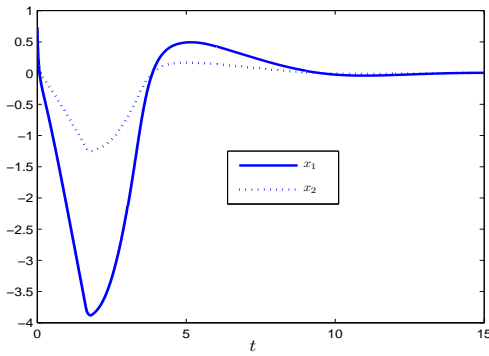Fig. 1. The evolution of $\dfrac{dh_1}{dt}$ and control input u.

Fig. 2.   The evolution of the system states.

Given $\phi(0) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and choosing $u = u_1$, we get $\dot{h}_1(0) = -5.7434$ which satisfies H$_1$. Figure 1 shows the evolution of $\dfrac{dh_1}{dt}$ and control input u. The time of point **A** is $t = 1.6499$ which is the maximal delay obtained by Theorem 2 and $\dot{h}_1(1.6499) = -0.0021613$, so at this point, the controller is still $u = u_1$. The time of point **B** is $t = 1.82$ and $\dot{h}_1(1.82) = 1.1647e - 005 > 0$, so at this point the controller is switched to u_2. The time of point **C** is $t = 4.66$ and at this point the controller is switched to u_1 again. Figure 2 shows the trajectories of the system states which are stabilized by the switching controller.

V. Conclusions

In this paper, we have studied the stability and stabilization for continuous-time T-S fuzzy systems with time-delay. A new membership dependent Lyapunov-Krasovskii functional has been designed to deal with the time delay and a switching control strategy is applied to stabilize the fuzzy systems. Comparing with the recent work, less conservative results can be obtained by using the new method proposed in this paper. In the end, the effectiveness of the proposed results are demonstrated by two examples. The method is simple but effective and can be used to deal with different problems such as filtering design, observer design and descriptor systems[21].

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