—*Brief Paper* —

A NEW APPROACH TO TERMINAL SLIDING MODE CONTROL DESIGN

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ABSTRACT

In this paper, terminal sliding mode control design is considered. A control method, different from many existing terminal sliding model control design methods, is proposed based on a new switching law and continuous finite-time control ideas. Then terminal sliding mode control laws are constructed for some classes of nonlinear systems.

KeyWords: Terminal sliding mode control, finite-time convergence, non-smooth feedback.

I. INTRODUCTION

Sliding mode control has been studied for many decades and is now one of the most active areas of research on nonlinear control theory. Many reports on sliding mode control and its extensions can be found in the literature, including [1-4]. In recent years, an increasing interest has focused on terminal sliding mode control, with the goal of rendering finite-time convergence to equilibrium of the closed-loop system in question [5-7]. However, these control methods may cause singularity problems and influence the terminal sliding surfaces. In fact, [8] proposed a two-phase control scheme to avoid the singularity in their original control law.

The main contribution of this paper is as follows. Different from many existing recursive design procedures for terminal sliding modes, we propose a design method that combines a new switching scheme with some ideas related to continuous finite-time control given in [9,10]. The constructed terminal sliding mode controller is not in a recur-

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sive form and removes singularities outside of sliding surfaces.

The rest of the paper is organized as follows. In section 2, the problem formulation is presented, while some theoretic analysis is given in section 3. Then, in section 4, terminal sliding mode control laws are built for two classes of systems. Finally, concluding remarks are given in section 5.

II. PROBLEM FORMULATION

In this paper, we focus on a single-input control system of the form

$$
\begin{cases}
\dot{x}_1 = f_1(x) \\
\cdots \\
\dot{x}_{n-1} = f_{n-1}(x) \\
\dot{x}_n = f_n(x) + u
\end{cases}
\tag{1}
$$

with $x = (x_1, \ldots, x_n)^T \in \mathcal{R}^n$, $f_i(x)$ smooth, and $f_i(0) = 0$ for $i = 1, \ldots, n$.

Let $s: \mathcal{R}^n \to \mathcal{R}$ be a continuous function with $s(0) = 0$. The sliding surface is defined as $S = \{x \in \mathcal{R}^n : s(x) = 0\}$. The sliding mode design procedure can be described roughly as consisting of two main steps. First, a sliding surface satisfying $\frac{\partial s}{\partial x_n} \neq 0$ is found such that $s(x(t)) \equiv 0$ implies $x(t) \to$

0 as $t \to \infty$; then, a stabilizing feedback law is constructed in the form

$$
u(x) = \begin{cases} u^+(x), & \text{if} \quad s > 0 \\ u^-(x), & \text{if} \quad s < 0 \end{cases}.
$$

Most sliding mode control laws make the controlled system (1) convergent to the sliding surfaces in finite time; then, along the sliding surfaces, the systems converge to the equilibrium $x = 0$ of system (1) as time goes to infinity. However, terminal sliding mode control laws can achieve more by steering the states to equilibrium in finite time.

Definition 1. For system (1), $u = u(x)$ is called a finite-time convergent controller if the equilibrium $x = 0$ of the closed-loop system (1) under this feedback law is finite-time convergent; that is, for any initial condition $x(0) = x^0 \in \mathcal{R}^n$, there is a settling time $T \geq 0$, which satisfies

$$
\lim_{t \to T} x(t; 0, x^0) = 0,
$$

and

$$
x(t; 0, x^0) = 0, \quad \text{if} \quad t > T
$$

for every solution $x(t; 0, x^0)$ to the closed-loop system (1). Moreover, if the controller is also a sliding mode controller, then it is called a terminal sliding mode controller.

In this paper, we assume that $s(x)$ takes the following form:

$$
s(x) = x_n - h(x_1, ..., x_{n-1}), \quad h(0, ..., 0) = 0, \tag{2}
$$

where $h(x_1, ..., x_{n-1})$ is a continuous function satisfying

$$
|h(x_1, ..., x_{n-1})| < K_0 + L_0 \sum_{i=1}^{n-1} |x_i| \tag{3}
$$

for some positive numbers K_0 and L_0.

Many choices of sliding modes satisfy condition (3). For example, $s(x) = \sum_{i=1}^{n-1} a_i x_i$ is a widely used form, and taking $K_0 = 1$ and $L_0 = \max\{a_i, i = 1, ..., n-1\}$ will make (3) hold.

In the following, two lemmas will be introduced for the following analysis. Their proofs are quite obvious and, therefore, omitted here.

Lemma 1. For any $0 < \alpha < 1$ and $M_0 > 0$, $M_1 > 0$ exists such that $|z|^\alpha \leq M_0 + M_1 |z|$ holds for all $z \in \mathcal{R}$.

Lemma 2. Suppose that a, b, and $m > 1$ are all positive numbers. Then, $(a^m + b^m)^{1/m} \leq a + b$.

III. TERMINAL SLIDING MODE

First, we will present results for finite-time convergence to given sliding surfaces.

Theorem 1. If the sliding surface S is taken with $h(x)$ satisfying (3), then the system (1) reaches S in finite time by the control law

$$
u = -f_n(x) + v, \tag{4}
$$

where

$$
v = -K\left(1 + \sum_{i=1}^{n-1} |f_i(x)|\right)\text{sgn}(s), \quad K > \max\{L_0, 1\}, \quad s \neq 0, \tag{5}
$$

and $\text{sgn}(\cdot)$ is denoted as the sign function.

Proof. The task, in fact, is to prove that, for any initial condition $x(0) \neq 0$ with $s(x(0)) \neq 0$, the feedback (4) will lead to $s(x(t)) = 0$ in finite time.

Let $x(0) = x^0 = (x_1^0, ..., x_{n-1}^0, x_n^0)$ be the initial condition. The trajectory with the initial condition is denoted by $x(t; 0, x(0))$, or $x(t)$ for simplicity.

We will first study the case where $s(x(0)) > 0$; that is, $x_n^0 > h(x_1^0, ..., x_{n-1}^0)$. We will prove that $T > 0$ exists such that $s(x(T)) = 0$ by contradiction.

Suppose that $T > 0$ does not exist such that $s(x(T)) = 0$. Because $s(x(0)) > 0$ and $s(x(t))$ is continuous, $s(x(t)) > 0$ for any $t > 0$; that is, $x_n(t) > h(x_1(t), ..., x_{n-1}(t))$ for any $t > 0$. Therefore,

$$
\dot{x}_n = -K\left(1 + \sum_{i=1}^{n-1} |f_i(x)|\right)\text{sgn}(s) = -K\left(1 + \sum_{i=1}^{n-1} |f_i(x)|\right). \tag{6}
$$

Integrating both sides of (6) gives

$$
x_n(t) - x_n^0 = -Kt - K\sum_{i=1}^{n-1} \int_0^t |f_i(x)| \, dt
$$

$$
\leq -Kt - K\sum_{i=1}^{n-1} |\int_0^t \dot{x}_i \, dt| \leq -Kt - K\sum_{i=1}^{n-1} |x_i(t)| + K\sum_{i=1}^{n-1} |x_i^0|,
$$

which implies

$$
x_n(t) \leq -\left(Kt - x_n^0 - K\sum_{i=1}^{n-1} |x_i^0|\right) - K\sum_{i=1}^{n-1} |x_i(t)|.
$$

Recall (3), *i.e.*, $|h(x_1, ..., x_{n-1})| < K_0 + L_0 \sum_{i=1}^{n-1} |x_i|$. If we make t_1 large enough, say

$$
t_1 > \frac{K_0 + |x_n^0|}{K} + \sum_{i=1}^{n-1} |x_i^0|, \tag{7}
$$

then we have

$$
x_n(t_1) < -K_0 - L_0 \sum_{i=1}^{n-1} |x_i(t_1)| \leq h(x_1(t_1), ..., x_{n-1}(t_1));
$$

that is, $s(x(t_1)) < 0$. This contradiction shows that there exists a reaching time $T_1 < t_1$ such that $s(x(T_1)) = 0$.

Similarly, if $s(x(0)) < 0$, that is, $x_n^0 < h(x_1^0, ..., x_{n-1}^0)$,

then there is another reaching time $T_2 > 0$ such that $s(x(T_2)) = 0$.

Thus, the assertion follows. ∎

An important merit of sliding mode control is that it provides robustness to uncertainties. Consider system (1) with f_i, $i = 1, \ldots, n$ uncertain. Suppose

$$|f_i(x)| \leq \phi_i(x), \quad i = 1, .., n,$$

where the functions $\phi_i(x)$, $i = 1, \ldots, n$, are known and $\phi_i(0) = 0$. Then, using an idea that is similar to the one employed in the proof of Theorem 1, we have the following theorem.

Theorem 2. If $h(x_1, \ldots, x_{n-1})$ satisfies (3), then we can put the control law in the form

$$u(x) = -K\left(1 + \sum_{i=1}^{n} \phi_i(x)\right) \operatorname{sgn}(s), \quad s \neq 0 \tag{8}$$

with $K > \max\{K_0, L_0, 1\}$, such that the uncertain system (1) reaches S in finite time.

In fact, the goal of terminal sliding mode control is to find a suitable sliding surface such that the state outside of the selected sliding surface is finite-time convergent to it while the system state in the sliding surface tends to equilibrium in finite time. We will sketch our design idea in the following. We first select $s(x) = x_n - h(x_1, \ldots, x_{n-1})$ such that condition (3) holds and

$$\begin{cases} \dot{x}_1 = f_1(x_1, \ldots, x_{n-1}, h) \\ \ldots \\ \dot{x}_{n-1} = f_{n-1}(x_1, \ldots, x_{n-1}, h) \end{cases} \tag{9}$$

is finite time convergent to the origin on $S = \{x \in \mathcal{R}^n : s(x) = 0\}$. Then, we adopt the control laws given in the form of (4) or (8) to guarantee finite-time convergence to S and force the state to move along S, reaching the equilibrium in finite time.

Remark 1. In many conventional cases, we can construct $u = u_{eq}$ on $S = \{x \in \mathcal{R}^n : s(x) \equiv 0\}$ based on knowledge of f_i, $i = 1, \ldots, n$ in order to make S an invariant set or even manifold; that is, $s(x(t)) \equiv 0$ for any $t \geq 0$ with the initial condition $s(x(0)) = 0$. However, terminal sliding mode control may result in the singularity problem for u_{eq} and prevent the existence of the corresponding sliding surface S that is kept invariant by u_{eq} in some sense. This has been widely noted [5,6,8].

IV. TWO CLASSES OF SYSTEMS

In this section, based on the discussion in section 3, we will consider how to construct terminal sliding mode controllers for two classes of systems in form of (1).

4.1 Class 1

Consider the system

$$\begin{cases} \dot{x}_1 = x_2 \\ \ldots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = f_n(x) + u \end{cases} \tag{10}$$

As shown in [5,8], existing terminal sliding modes for system (10) have usually been constructed in a recursive structure as follows:

$$s_0 = x_1, \quad \ldots, \quad s_{n-1} = \dot{s}_{n-2} + b_{n-2}\, s_{n-2}^{q_{n-1}/p_{n-1}},$$

where $(n-i-1)/(n-i) < q_{i+1}/p_{i+1} < 1$ with p_{i+1}, q_{i+1}, $i = 0, \ldots, n-2$ as positive odd integers and $b_i > 0$, $i = 0, \ldots, n-2$. Then, the control law is taken as

$$u = u_{eq} - K \operatorname{sgn}(s_{n-1}). \tag{11}$$

$u_{eq}(x)$ is given to keep $s(x(t)) = 0$ if $s(x(0)) = 0$, but this may not be satisfactory because a singularity or uncertainty may occur in the feedback law (11). In [6], a terminal sliding mode controller was given as

$$u = -f_n(x) - \beta_0 \operatorname{sgn}(\phi_{n-1,n}), \quad \beta_0 > 0 \tag{12}$$

with properly chosen positive parameters β_i, positive definite functions $N_{i,n}(x_1, \ldots, x_i)$, $i = 1, \ldots, n-1$, and a (recursive) sliding mode:

$$\phi_{0,n} = x_1, \quad \ldots, \quad \phi_{i,n} = x_{i+1} + \beta_i N_{i,n}(x_1, \ldots, x_i) \operatorname{sgn}(\phi_{i-1,n})$$

for $i = 1, \ldots, n-1$.

Our approach is as follows. We take $s = x_n - h(x_1, \ldots, x_{n-1})$ such that

$$\begin{cases} \dot{x}_1 = x_2 \\ \ldots \\ \dot{x}_{n-2} = x_{n-1} \\ \dot{x}_{n-1} = h(x_1, \ldots, x_{n-1}) \end{cases} \tag{13}$$

is finite-time convergent to the origin. In fact, system (13) admits a continuous finite-time convergent feedback law $h = v_{n-1}(x_1, \ldots, x_{n-1})$ (e.g., [9,10]). Thus, in our design, we select $h = v_{n-1}$ in one of the forms provided in [9,10]. In this way, we make system (13) finite-time convergent to its origin on sliding surface S. Moreover, the condition (3) for this h can also be verified using Lemmas 1 and 2 (see the following examples for some details). Then, with the control law in the form of (4) or (8), the state outside of the sliding surface is finite-time convergent to the surface S. Hence, system (10) with $s(x) = x_n - v_{n-1}$ and its corresponding control law are finite-time convergent to the origin.

For the purpose of illustration, we will give two examples.

Example 1. Consider

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f_2(x) + u \end{cases}. \tag{14}$$

Take a continuous function $h(x_1) = -x_1^{3/5}$ and then $s = x_2 + x_1^{3/5}$. Thus, for the given h, (3) can be satisfied with $K_0 = 3$ and $L_0 = 2$, due to Lemma 1. Then, we can take

$$u(x) = \begin{cases} -f_2(x) - 3(1 + |x_2|)\,\mathrm{sgn}(s), & \text{if } s \neq 0 \\ u_{eq}(x) = -f_2(x) + \frac{3}{5} x_1^{1/5}, & \text{if } s = 0 \end{cases}. \tag{15}$$

As a result, the system converges to the sliding surface in finite time. Once it converges to the sliding surface ($s = 0$), we have $\dot{x}_1 = -x_1^{3/5}$, which is finite-time convergent to $x_1 = 0$, that is, the origin $x = 0$, because of $s = 0$. This directly leads to finite-time convergence of the closed-loop system.

Example 2. Consider

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = f_3(x) + u \end{cases}. \tag{16}$$

First, we employ the method in [5], and we have

$$s_0 = x_1, \quad s_1 = \dot{s}_0 + s_0^{7/9}, \quad s_2 = \dot{s}_1 + s_1^{5/7}.$$

Then the control law is given as

$$u(x) = u_{eq}(x) - K\,\mathrm{sgn}(s_2), \quad K > 0 \tag{17}$$

with

$$u_{eq}(x) = -f_3(x) - (\ddot{s}_0^{7/9}) - (\dot{s}_1^{5/7})$$

$$= -f_3(x) - \frac{7x_3}{9x_1^{2/9}} - \frac{5(x_3 + \frac{7x_2}{9x_1^{2/9}})}{7(x_2 + x_1^{7/9})},$$

which may contain singularities.

Next, consider our proposed method for terminal sliding mode control. Take a continuous function

$$s(x) = x_3 + x_1^{\alpha_1} + x_2^{\alpha_2}, \quad 0 < \alpha_1 < 1, \quad \alpha_2 = \frac{2\alpha_1}{1 + \alpha_1}. \tag{18}$$

with $\alpha_i = q_i / p_i$ and $p_i > 0$, $q_i > 0$ as suitable odd integers for $i = 1, 2$. For example, we take $\alpha_1 = 1/5$ or $\alpha_1 = 3/7$, and then $\alpha_2 = 1/3$ or $\alpha_2 = 3/5$. When $x \in S$ (*i.e.*, $s(x) = 0$), we have

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1^{\alpha_1} - x_2^{\alpha_2} \end{cases},$$

which is finite-time convergent [1]. Moreover, for $s(x)$ given in (18), (3) can be satisfied with $K_0 = 3$ and $L_0 = 2$, due to Lemma 1. Then, by Theorem 1, we can construct a feedback law which is of the form in (4) and different from that in (17):

$$u(x) = \begin{cases} -f_3(x) - 3(1 + |x_2| + |x_3|)\,\mathrm{sgn}(s), & \text{if } s \neq 0 \\ u_{eq} = -f_3(x) - \alpha_1 x_1^{\alpha_1 - 1} x_2 + \alpha_2 x_2^{\alpha_2 - 1} x_3, & \text{if } s = 0 \end{cases}. \tag{19}$$

Thus, the system converges to the sliding surface in finite time, and in the sliding surface, the state reaches equilibrium in finite time, which implies finite-time convergence of the closed-loop system. To avoid the singularity (referring to Remark 2), we change (19) to a simple form:

$$u(x) = \begin{cases} -f_3(x) - 3(1 + |x_2| + |x_3|)\,\mathrm{sgn}(s), & \text{if } s \neq 0 \\ -f_3(x), & \text{if } s = 0 \end{cases}. \tag{20}$$

Note that $u = -f_3(x)$ cannot keep $s \equiv 0$ when its initial state is already in the sliding surface S. Hence, S does not exist in the conventional sense.

4.2 Class 2

There are many (smooth) stabilizable nonlinear systems that cannot be stabilized by means of time-invariant smooth feedback. Usually, these nonlinear systems are hard to control effectively, and their variety makes it even harder to obtain a unified method of stabilization. Here, we will only consider a well-known system, but the proposed design idea may also be applicable to generalized systems with similar structures (such as those studied in [12]).

The following well-known smooth system has been widely studied (referring to [12]):

$$\begin{cases} \dot{x}_1 = x_1 + x_2^3 \\ \dot{x}_2 = f_2(x_1, x_2) + u \end{cases}. \tag{21}$$

This system cannot be stabilized by means of smooth feedback. In this case, conventional terminal sliding mode control design methods may also encounter some difficulties.

The guideline for selecting the switching surface $s = x_2 - h(x_1)$ for this system is based on the idea given in section 3: the given h makes $\dot{x}_1 = x_1 + h^3$ finite-time convergent. It is easy to see that we can take $h(x_1) = -(x_1^{7/9} + x_1)^{1/3}$, or equivalently, $s(x) = x_2 + (x_1^{7/9} + x_1)^{1/3}$.

Taking $K_0 = 1$ and $L_0 = 2.5$ yields condition (3) by Lemma 1. Therefore, when $s \neq 0$, the control law can be taken as $u = -3 (1+ | x_1 + x_2^3 |) \, \text{sgn}(s)$, according to Theorem 1. On the sliding surface, we can derive

$$u_{eq}(x) = -f_2(x_1, x_2) + \frac{x_1^{7/27} + \frac{7}{9} x_1^{1/27}}{3(1+x_1^{2/9})^{2/3}}, \qquad (22)$$

which forces the state to stay in the sliding surface, since it is governed by the dynamics $\dot{x}_1 = x_1 + h^3 = -x_1^{7/9}$, and it arrives at the origin in finite time. Therefore, a terminal sliding mode controller for system (21) is

$$u(x) = \begin{cases} -3 (1+ | x_1 + x_2^3 |) \, \text{sgn}(s), & \text{if} \quad s \neq 0 \\ u_{eq}(x) \text{ in } (22), & \text{if} \quad s \neq 0 \end{cases}.$$

V. CONCLUSION

In the paper, a new design method for terminal sliding mode control has been proposed. Terminal sliding mode controllers have been constructed. They are not in a recursive form, and they can remove singularities outside of sliding surfaces. Future work will focus on developing a complete theory for the construction of these controllers.

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