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A new approach toward geometrical concept of black hole thermodynamics

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Abstract Motivated by the energy representation of Riemannian metric, in this paper we study different approaches toward the geometrical concept of black hole thermodynamics. We investigate thermodynamical Ricci scalar of Weinhold, Ruppeiner and Quevedo metrics and show that their number and location of divergences do not coincide with phase transition points arisen from heat capacity. Next, we introduce a new metric to solve these problems. We show that the denominator of the Ricci scalar of the new metric contains terms which coincide with different types of phase transitions. We elaborate the effectiveness of the new metric and shortcomings of the previous metrics with some examples. Furthermore, we find a characteristic behavior of the new thermodynamical Ricci scalar which enables one to distinguish two types of phase transitions. In addition, we generalize the new metric for the cases of more than two extensive parameters and show that in these cases the divergencies of thermodynamical Ricci scalar coincide with phase transition points of the heat capacity.

1 Introduction

One of the succinct semi-classical approaches for investigating the quantum nature of gravity is via black hole thermodynamics in AdS spacetime which is dual to that of a field theory in one dimension fewer [1–3]. Besides, an interesting development in the black holes studies comes from the fact that black holes are akin to thermodynamical system which can be essentially described by the laws of thermodynamics re-expressed in terms of properties of black holes [4–10].

Scientific researches in the black hole physics have presented a very deep and fundamental relationship between quantum gravity, holography and thermodynamics. Among these researches, the special interest is attracted by AdS black holes because some of them lead to the very interesting phenomenon called Hawking–Page phase transition [11–14]. Phase transition plays an important role in order to explore thermodynamical properties of a system near the critical point. Usually, phase transitions are denoted by a discontinuity of a state space variable, specially heat capacity [15].

Motivated by large applications of the geometrical concept of thermodynamics in the black hole phase transition, in this paper, we investigate strengths and shortcomings of different approaches toward the matter. First attempt was done by Weinhold [16,17] which introduced a metric on the space of equilibrium states where its components are given as the Hessian of the internal energy. Then, Ruppeiner introduced a metric which is defined as the negative Hessian of entropy with respect to the internal energy and other extensive quantities of a thermodynamical system [18,19]. It was shown that these two metrics are conformally equivalent to each other where the temperature is the conformal factor [20]. Recently, Quevedo [21,22] proposed a Legendre invariant metric, in which solved some of problems in Weinhold/Ruppeiner methods.

The basic motivation for considering the geometrothermodynamics comes from the fact that this formalism describes in an invariant way the thermodynamic properties of a given thermodynamical system in terms of geometric structures. Although extracting phase transition in terms of curvature singularity is the main reason to consider the geometrical approach in thermodynamics, there are several examples in which the curvature singularities of the known metrics (Weinhold and Ruppeiner metrics and their Legendre transformations) are not located at the phase transition points and they have number of singularities in which are before/after the phase transition points [23–26].

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In this paper, in order to solve the mentioned problems, we introduce a metric where its curvature singularities are, exactly, located at the phase transition points. As we see later, this effective metric has different structure from the other known metrics. For this claim, we consider the black hole solutions with Born–Infeld (BI) and Maxwell sources in three and four dimensions and investigate phase transitions in various geometrothermodynamical methods.

2 Heat capacity

In order to investigate the local stability of a black hole, one can use various methods related to different ensembles. In principle thermal stability can be carried out in the grand canonical ensemble by finding the determinant of the Hessian matrix of $M(X_i)$ with respect to its extensive variables X_i [27,28]. For static charged black holes, we usually regard the mass M as a function of the entropy S and the charge Q. The number of thermodynamic variables depends on the ensemble that is used. The most well known approach for studying phase transition is in the context of canonical ensemble, hence the heat capacity of systems. Depending on the number of extensive parameters, the behavior of Hessian matrix is highly sensitive [29–32] and therefore most of physicists prefer to work in canonical ensemble. In this ensemble, the positivity of the heat capacity is sufficient to ensure thermal stability. In addition the system is considered to be in fixed charged and the heat capacity has the following form

$$C_Q = \frac{M_S}{M_{SS}}, \tag{1}$$

where $M_S = \left(\frac{\partial M}{\partial S}\right)_Q$ and $M_{SS} = \left(\frac{\partial^2 M}{\partial S^2}\right)_Q$ are regular functions. Now we regard two different types of phase transitions. In type one, the changes in signature of the heat capacity is representing a phase transition. In other words, the roots of the heat capacity in this case are representing phase transition points which means one should solve $M_S = 0$. Phase transition concerning to the divergency of the heat capacity is denoted by type two. It means the singular points of the heat capacity are representing the phase transitions. Thus, we should consider $M_{SS} = 0$ to obtain the phase transition of type two.

In order to have a fitting geometrical approach for studying phase transitions, the thermodynamical Ricci scalar (TRS) must diverge in the mentioned both types of phase transitions. In what follows, we present a brief review study of several geometrical approaches with their shortcomings, and then, we propose a new effective metric concerning this issue.

3 Weinhold metric

In Weinhold method, one is considering appropriate extensive parameters such as entropy, electric charge and angular momentum, and their related intensive quantities such as temperature, electric potential and angular velocity with the mass of the black holes as a potential. The Weinhold metric is given by [16,17]

$$\mathrm{d}s_W^2 = M g_{ab}^W \mathrm{d}X^a \mathrm{d}X^b, \tag{2}$$

where $g_{ab}^W = \partial^2 M\,(X^c)\,/\partial X^a \partial X^b$ and also $X^a \equiv X^a(S, N^i)$, where N^i denotes other extensive variables of the system. Considering a static charged black hole, one can find the following expression for the denominator of Weinhold Ricci scalar

$$\mathrm{denom}(\mathcal{R}_W) = \left(M_{SS}M_{QQ} - M_{SQ}^2\right)^2 M^2\,(S, Q)\,, \tag{3}$$

where $M_{QQ} = \left(\frac{\partial^2 M}{\partial Q^2}\right)_S$ and $M_{SQ} = \frac{\partial^2 M}{\partial S \partial Q}$ and for consistency (with respect to the heat capacity results), the roots of the Eq. (3) should coincide with two types of the mentioned phase transitions in the heat capacity. It is easy to find that due to the structure of the Eq. (3), only in special case $M_{SQ} = 0$ and nonzero M_{QQ}, the divergence points of the heat capacity coincide with divergencies of the Weinhold Ricci scalar. In order to obtain consistent results for the type one phase transition, the following fine tuning must be hold

$$M_{SS}M_{QQ} - M_{SQ}^2 = M_S. \tag{4}$$

Regarding various case studies and calculating M and its derivatives, we should note that Eq. (4) is not always satisfied. In addition, it is evident that for the case of $M_{SS} = \frac{M_{SQ}^2}{M_{QQ}}$, there will be extra divergencies for $\mathcal{R}_W$ which are not related to any phase transition of the heat capacity. Therefore, the structure of this part of denominator is in a way that may present extra divergencies that do not coincide with any type of phase transition points of the heat capacity.

4 Ruppeiner metric

The Ruppeiner metric is defined as [18,19]

$$\mathrm{d}s_R^2 = g_{ab}^R \mathrm{d}Y^a \mathrm{d}Y^b, \tag{5}$$

where $g_{ab}^R = -\partial^2 S(Y^c)/\partial Y^a \partial Y^b$ and $Y^a \equiv Y^a(M, N^i)$. It was proved that Weinhold and Ruppeiner metrics are related to each other by a Legendre transformation [21,22]

$$\mathrm{d}s_R^2 = -MT^{-1} g_{ab}^W \mathrm{d}X^a \mathrm{d}X^b. \tag{6}$$

Applying this transformation, one can find the following relation for the denominator of the Ricci scalar for this case

$$\mathrm{denom}(\mathcal{R}_R) = \left(M_{SS}M_{QQ} - M_{SQ}^2\right)^2 T(S, Q)M^2(S, Q). \tag{7}$$

This equation shows that the phase transitions of the heat capacity coincide with the singularities of Ruppeiner Ricci scalar only when $M_{SQ}^2 = 0$ and nonzero M_{QQ}. In addition, regarding type one phase transitions, one finds due to existence of the $T(S, Q)$ and the fact that $T(S, Q) = M_S$, the roots of the heat capacity and some of the divergence points of Ruppeiner Ricci scalar coincide. On the other hand, there may be some divergence points of $\mathcal{R}_R$ that do not coincide with phase transition points. These extra phase transition points are originated from the zeroes of $(M_{SS}M_{QQ} - M_{SQ}^2)$. As we mentioned before, in the case of $M_{SQ}^2 = 0$, type two phase transitions of the heat capacity ($M_{SS} = 0$) are covered by divergencies of the Ricci scalar of the Ruppeiner metric. But in this case, we encounter with extra divergencies related to the roots of $M_{QQ} = 0$ which are not related to any phase transition of the heat capacity. In addition, in the case of nonzero M_{SQ}^2, the possible real roots of $M_{QQ} = \frac{M_{SQ}^2}{M_{SS}}$ lead to the same extra divergencies, which were observed in Weinhold metric.

5 Quevedo metrics

In order to remove the failures of the Weinhold and Ruppeiner metrics, Quevedo proposed a new thermodynamical metric. The Quevedo metric can be written as [21,22]

$$\mathrm{d}s_Q^2 = \Omega\left(-M_{SS}\mathrm{d}S^2 + M_{QQ}\mathrm{d}Q^2\right), \tag{8}$$

where the (conformal) function Ω has one of the following forms

$$\Omega = \begin{cases} SM_S + QM_Q, & \text{case I} \\ SM_S, & \text{case II} \end{cases}. \tag{9}$$

In order to obtain the curvature singularity of the Quevedo metric, we calculate the Ricci scalar. Although calculation of the Ricci scalar is straightforward, analytically calculated results are too large. So, for the sake of brevity we do not write the long equations of the Ricci scalar; instead, we can use numerical analysis and some plots to investigate the Ricci scalar's behavior. In addition, since we are looking for the divergence points of the Ricci scalars, we can study the denominator of the Ricci scalars with the following explicit forms

$$\mathrm{denom}(\mathcal{R}_{Q1}) = (SM_S + QM_Q)^3 M_{SS}^2 M_{QQ}^2, \tag{10}$$

$$\mathrm{denom}(\mathcal{R}_{Q2}) = S^3 M_S^3 M_{SS}^2 M_{QQ}^2. \tag{11}$$

Due to M_{SS} being in the denominator of both Quevedo Ricci scalars, the divergencies of the heat capacity and Quevedo Ricci scalars coincide. Regarding type one phase transition, although the roots of the heat capacity and divergencies of the Quevedo Ricci scalar of case II coincide, for case I this coincidence takes place only for vanishing M_Q (which is in general a nonzero function). It is worthwhile to mention the fact that there exists an additional function M_{QQ}^2, and its roots provides extra singular points for both cases of Quevedo Ricci scalars. Although M_Q and M_S are independent from each other, generally, for a nonzero M_Q in the case I, it may be possible to set $M_S = -\frac{Q}{S}M_Q$ for special choices of free parameters. In this situation, one finds another divergence point which may not coincide with any phase transition points of the heat capacity.

6 New metric

In order to avoid extra singular points in TRS which do not coincide with phase transitions of any type, and also to ensure all divergencies of the TRS coincide with phase transition points of the both types, we introduce the following new thermodynamical metric

$$\mathrm{d}s_{\mathrm{new}}^2 = S\frac{M_S}{M_{QQ}^3}(-M_{SS}\mathrm{d}S^2 + M_{QQ}\mathrm{d}Q^2). \tag{12}$$

It is worthwhile to mention that the new thermodynamical metric is defined the same as the Quevedo metric with different conformal function. In new metric, we have considered the total mass as thermodynamical potential with entropy and electric charge as extensive parameters. In order to find the geometrical behavior of new thermodynamical metric, we calculate the Ricci scalar. It is a straightforward calculation to show that the numerator and the denominator of TRS for this new metric is, respectively,

$$\begin{aligned}
\mathrm{num}(\mathcal{R}) &= 6S^2M_S^2M_{QQ}M_{SS}^2M_{QQQQ} \\
&- 6SM_S^2M_{QQ}^2M_{SS}M_{SSQQ} + 2SM_{SQQ}^2M_S^2M_{QQ}M_{SS} \\
&+ 2\left[SM_SM_{SSS} - \frac{1}{2}M_{SS}\left(SM_{SS} - M_S\right)\right]SM_{QQ}^2M_SM_{SQQ} \\
&- 9S^2M_{QQQ}^2M_S^2M_{SS}^2 + 4\left[\frac{1}{4}M_{SQ}M_{SS} + M_SM_{SSQ}\right] \\
&\times S^2M_{QQ}M_SM_{QQQ} + \left[S^2M_S^2M_{SSQ} - S^2M_{SQ}M_{SS}M_SM_{SSQ}\right. \\
&\times SM_{QQ}M_S\left(SM_{SS} - M_S\right)M_{SS} - 2\left(S^2M_{SS}^3 + M_S^2M_{SS}\right) \\
&\left.\times M_{QQ} + 2S^2M_{SQ}^2M_{SS}^2\right]M_{QQ}^2,
\end{aligned} \tag{13}$$

and

$$\mathrm{denom}(\mathcal{R}) = S^3 M_S^3 M_{SS}^2. \tag{14}$$

Regarding a typical black hole with nonzero horizon radius, r_+ (and also nonzero entropy), one can find that, in general, the finite mass is an analytic smooth function of its variables. Smooth function is often used technically to mean a function that has derivatives of all orders everywhere in its domain. Therefore, one may take into account that M has regular derivatives of all orders with respect to the extensive parameters (To our knowledge various black holes in gravitational theories have smooth functions of finite mass. Nevertheless, we exclude the possible black holes with non-smooth finite mass and its derivatives).

Taking into account the regular numerator of TRS in Eq. (13) with the denominator of TRS, Eq. (14), we ensure that all the phase transition points of the type one and two coincide with divergencies of the mentioned TRS and there is no extra term that may provide extra divergencies.

We should note that, in general, the derivatives of all orders of M (such as M_{QQ}, M_{SS}, M_{SQ}, M_{SSQ} and so on) are independent from each other. In addition, we should mention that for the case of vanishing M_S or M_{SS}, the numerator of TRS has nonzero value, but denominator of TRS vanishes. In the case of $M_S = M_{SS} = 0$, although it is clear that both numerator and denominator vanish, the denominator approaches zero faster than the numerator. Therefore, one concludes that when M_S and/or M_{SS} go to zero, TRS diverges.

In order to elaborate the efficiency of the newly proposed metric and shortcomings of the previously proposed metrics, we study two cases of the BI black holes in three and four dimensions. We also discuss linear electrodynamics cases and give a comment for neutral solutions.

7 Black hole solutions of Einstein gravity with a BI source

The d-dimensional action of Einstein–BI gravity in the presence of cosmological constant is given by [33]

$$I = -\frac{1}{16\pi}\int d^d x\sqrt{-g}\,[R - 2\Lambda + L(F)], \tag{15}$$

where R is the Ricci scalar and Λ refers to the (negative) cosmological constant. Also, $L(F)$ is the Lagrangian of BI field as follows [34]

$$L(F) = 4\beta^2\left(1 - \sqrt{1 + \frac{F}{2\beta^2}}\right), \tag{16}$$

where β is called the nonlinearity parameter, the Maxwell invariant $F = F_{\mu\nu}F^{\mu\nu}$ in which $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor and A_μ is the gauge potential. The static BI black hole solutions can be obtained with the following d-dimensional metric [33,35–39]

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega^2, \tag{17}$$

where $\mathrm{d}\Omega^2$ represents the line element of $r = constant$ and $t = constant$ hypersurfaces with volume V_{d-2}. For three and four dimensional (3D and 4D) spacetimes, V_{d-2} is equal to 2π and 4π, respectively, and $\mathrm{d}\Omega^2$ can be written with the following explicit forms

$$\mathrm{d}\Omega^2 = \begin{cases} \mathrm{d}\theta^2, & 3D \\ \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2, & 4D \end{cases}. \tag{18}$$

The consistent metric functions $f(r)$ for two cases of three and four dimensional black holes are given by [33,35–39]

$$f(r) = \begin{cases} -m - \Lambda r^2 + 2r^2\beta^2(1-\Gamma) + q^2\left[1 - 2\ln\left(\frac{r(1+\Gamma)}{2l}\right)\right] & 3D \\ 1 - \frac{m}{r} - \frac{\Lambda r^2}{3} + \frac{2\beta^2}{3}r^2(1-\Gamma) + \frac{4q^2}{3r^2}\,{}_2\mathcal{F}_1\left(\left[\frac{1}{2},\frac{1}{4}\right],\left[\frac{5}{4}\right],1-\Gamma^2\right) & 4D \end{cases}, \tag{19}$$

where $\Gamma = \sqrt{1 + \frac{q^2}{r^{2(d-2)}\beta^2}}$, ${}_2\mathcal{F}_1$ is hypergeometric function, m and q are integration constants which are related to mass parameter and the electric charge of the black holes, respectively. The entropy and the electric charge of the mentioned BI black hole solutions were obtained before [33,35–39]

$$S = \frac{V_{d-2}\,r_+^{d-2}}{4}, \tag{20}$$

$$Q = \frac{V_{d-2}\,q}{4\pi}, \tag{21}$$

where r_+ denotes the outer (event) horizon of black holes which is the largest real positive root of metric function, $f(r)|_{r=r_+} = 0$.

Now, we write the quasi-local mass (per unit volume V_{d-2}) as a function of extensive parameters (S and Q) to discuss phase transition. One finds [33,35–39]

$$M=\frac{(d-2)}{16\pi}\times\begin{cases}\frac{4(\pi^2Q^2-\Lambda S^2)+8S^2\beta^2(1-\mathcal{H})-8\pi^2Q^2\ln\left(\frac{S}{\pi l}(1+\mathcal{H})\right)}{\pi^2} & 3D\\ \frac{3S(\pi-\Lambda S)+2S^2\beta^2(1-\mathcal{H})+4\pi^2Q^2\mathfrak{F}}{3\pi^2\sqrt{\frac{S}{\pi}}} & 4D\end{cases}, \tag{22}$$

where $\mathfrak{F}=\mathcal{F}\left(\left[\frac{1}{2},\frac{1}{4}\right],\left[\frac{5}{4}\right],-\frac{\pi^2Q^2}{S^2\beta^2}\right)$ and $\mathcal{H}=\sqrt{1+\frac{\pi^2Q^2}{S^2\beta^2}}$. Regarding Eq. (1), one finds that the heat capacity for three and four dimensional BI black holes can be written as

$$C_Q=\begin{cases}\frac{S^2(1+\mathcal{H})\left[\Lambda\beta^2S^2(1+\mathcal{H})+\pi^2Q^2\left(\Lambda+2\beta^2\mathcal{H}\right)\right]}{2\beta^2S\left(\Lambda S^2-\pi^2Q^2\right)(1+\mathcal{H})+\pi^2Q^2\Lambda(2+\mathcal{H})} & 3D\\ -\frac{2\beta^2\mathcal{H}^2S^3\left\{\left[\Sigma_++4Q^2\pi^2(\mathfrak{F}+4Q^2\mathfrak{F}')\right]\mathcal{H}+2Q^2\pi^2+6\beta^2S^2\right\}}{\beta^2\mathcal{H}^3S^2\left[\Sigma_-+4Q^2\pi^2(3\mathfrak{F}+32Q^2\mathfrak{F}'+16Q^4\mathfrak{F}'')\right]\mathcal{H}+2Q^4\pi^4-6\beta^2S^2(\beta^2S^2+2Q^2\pi^2)} & 4D\end{cases}, \tag{23}$$

where $\Sigma_\pm=-3S\pi\pm 3S^2(\Lambda-2\beta^2)$. It is notable that the prime and double prime are the first and second derivatives with respect to Q^2, respectively.

Here, we regard Eq. (23) and also all obtained TRS in various mentioned methods (Eqs. (2), (5), (8), (12)) to investigate their coincidences. To do so, we plot some figures for three and four dimensional BI black hole solutions (see Figs. 1, 2, 3). We find that for 4-dimensional BI solutions the Ruppeiner, the Weinhold, both methods of the Quevedo and also our new introduced metric have divergencies where the heat capacity diverges (see Fig. 1 and left panel of Fig. 3 for more details). In other words, applying the mentioned methods for these solutions leads to coincidence between divergencies of TRS and phase transitions of the type two, and therefore, we conclude that for these solutions (4-dimensional BI black holes) extra terms in Quevedo metrics, have no contribution to the divergencies of TRS.

Next, we take into account three dimensional solutions. We find that for all methods of the Weinhold, the Ruppeiner and both cases of the Quevedo metrics (Eqs. (2), (5), (8)), there are extra divergencies in plotted figures of TRS which come from the contributions of extra term (see Fig. 2). While for the TRS of our new metric (Eq. (12)) all divergencies coincide with the phase transitions of heat capacity (see Fig. 3 (right panel) for more details).

Another important property of new metric is the behavior of TRS very close to the phase transition points. As Fig. 3 confirms, the sign of the TRS before and after the divergence point for these two types of the phase transitions is different. If the phase transition is related to the vanishing heat capacity (type one), we see a change of sign for TRS before and after of the corresponding singular point. While for the divergence point of the heat capacity (type two), TRS has the same sign for the left and right sides of this point. Therefore, this characteristic behavior enables one to distinguish these two types of phase transitions from each other.

7.1 Linear case: Maxwell solutions

In order to study the effect of electric charge and remove the influence of nonlinearity parameter, we investigate the Maxwell solutions. We show that these linear solutions elaborate the efficiency of our new metric. In order to obtain Maxwell solutions, one can use series expansion of BI solutions for large values of nonlinearity parameter β. Regarding the mentioned results of BI solutions with straightforward series expansion, one can obtain [40–44]

$$f(r)=\begin{cases}-m-\Lambda r^2-2q^2\ln\left(\frac{r}{l}\right) & 3D\\ 1-\frac{m}{r}-\frac{\Lambda r^2}{3}+\frac{q^2}{r^2} & 4D\end{cases}. \tag{24}$$

Since the entropy and electric charge of these black holes do not depend on the nonlinearity parameter, one can obtain the same results. Therefore, the finite mass (per unit volume V_{d-2}) of the Einstein–Maxwell solutions can be written as a function of S and Q with the following explicit form

$$M=\frac{(d-2)}{16\pi}\times\begin{cases}\frac{2\pi^2Q^2\ln\left(\frac{\pi l}{2S}\right)-4\Lambda S^2}{\pi^2} & 3D\\ \frac{3\pi S+3\pi^2Q^2-\Lambda S^2}{3\pi^2\sqrt{\frac{S}{\pi}}} & 4D\end{cases}. \tag{25}$$

Using Eqs. (1) and (25), one can find the heat capacity may be calculated as

$$C_Q=\begin{cases}-\frac{(\pi^2Q^2+\Lambda S^2)S}{\pi^2Q^2-\Lambda S^2} & 3D\\ -\frac{2S(\pi S-\Lambda S^2-\pi^2Q^2)}{\Lambda S^2+\pi S-3\pi^2Q^2} & 4D\end{cases}. \tag{26}$$

Using Eq. (25), we are in a position to study the behavior of TRS for different methods that were mentioned in this paper. We can use Eqs. (2), (5), (8) and (12) with the heat capacity relations of Einstein–Maxwell solutions, Eq. (26), to plot various figures for studying the geometrical behavior of different thermodynamical spacetime (see Figs. 4, 5, 6, 7).

Regarding Figs. 4, 5, 6 and 7, we find that for 3-dimensional Einstein–Maxwell solutions (with considered values for different parameters), there is only one root for the heat capacity. In other words, in this case, we see one phase transition of type one and there is no divergency for the heat capacity. Whereas in 4-dimensional solutions, there are one phase transition of type one and two phase transitions of type

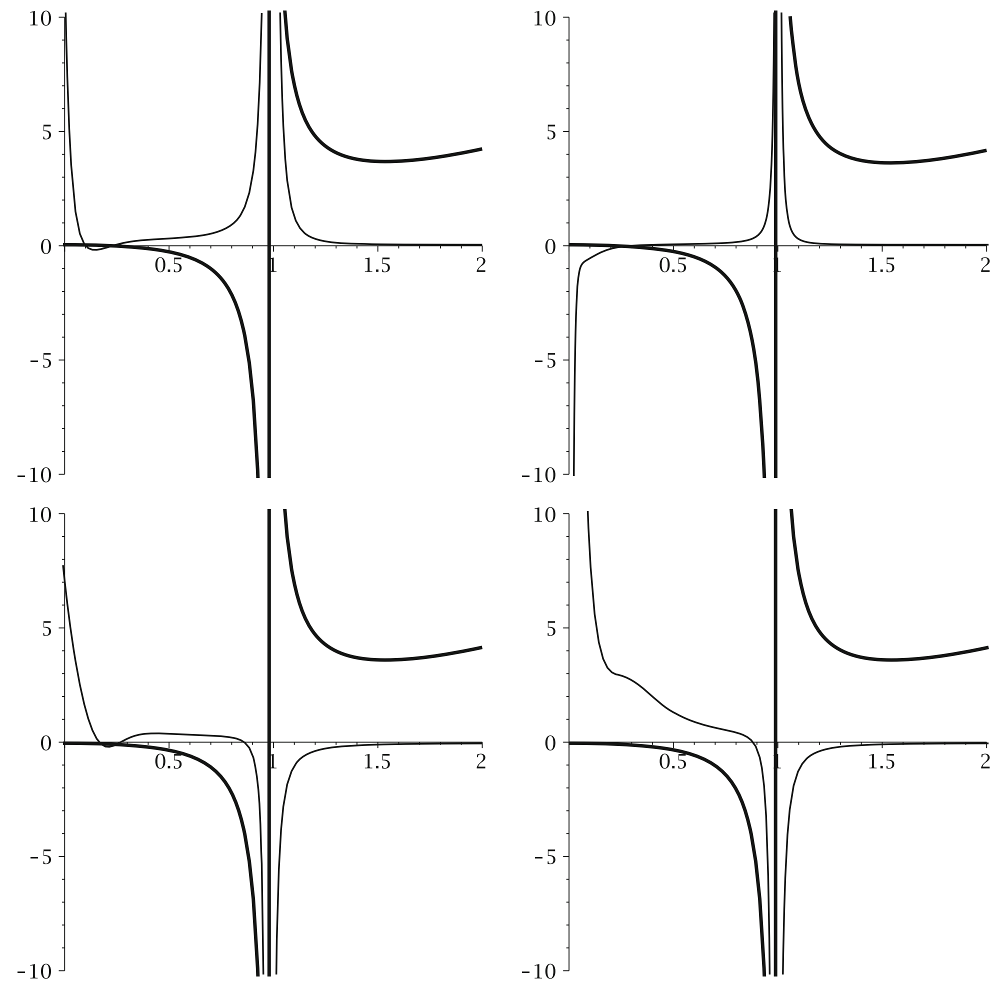

Fig. 1 BI case: $\mathcal{R}$ and C_Q versus r_+ for $l = 1$, $\Lambda = -1$, $\beta = 1$, $d = 4$ and $q = 0.1$. TRS (*continuous line*) and heat capacity (*bold line*) for the Weinhold metric (*left-up panel*), the Ruppeiner metric (*right-up panel*), the Quevedo metric for case I (*left-down panel*) and the Quevedo metric for case II (*right-down panel*)

two. It means that there are three phase transition points in four dimensional Einstein–Maxwell black hole solutions.

Regarding Weinhold approach, one finds both mismatching and extra divergencies in three and four dimensions (see Figs. 4, 6). In other words, in 4-dimensional black holes (Fig. 4), the root and one of the divergencies of the heat capacity are not compatible with any divergency of the Weinhold's TRS. On the other hand, in 3-dimensional solutions (Fig. 6), this approach completely fails to provide an effective machinery for studying the phase transitions. One divergency is observed which does not coincide with any phase transition point and for the root of heat capacity TRS is a smooth function.

As for the Ruppeiner metric, in 4-dimensional Einstein–Maxwell solutions (Fig. 4), one of the divergencies of the heat capacity is not matched with any divergency of the Ruppeiner's TRS. For 3-dimensional case, there are two divergencies for TRS (Fig. 6). One of them coincides with the root of heat capacity whereas the other one is not related to any phase transition point.

Now we investigate both types of Quevedo metrics and their behaviors. Regarding case I in 4-dimensional solutions (Fig. 5 up panels), there is no divergency for TRS in the place of heat capacity vanishing point, whereas divergencies of the heat capacity are matched with divergencies of TRS. For case II (Fig. 5 down panels), we obtain effective results. In other words, all divergence points and vanishing point of the heat capacity are matched with divergencies of the Quevedo's TRS of case II. But for charged BTZ black holes, regardless of case I or II, an extra divergency is seen which is not related to any phase transition point of the heat capacity (Fig. 6 down panels).

Finally, for both 3 and 4 dimensional Einstein–Maxwell black holes (see Fig. 7), new metric is completely successful for describing phase transition points of the heat capacity in context of geometrothermodynamics. In other words,

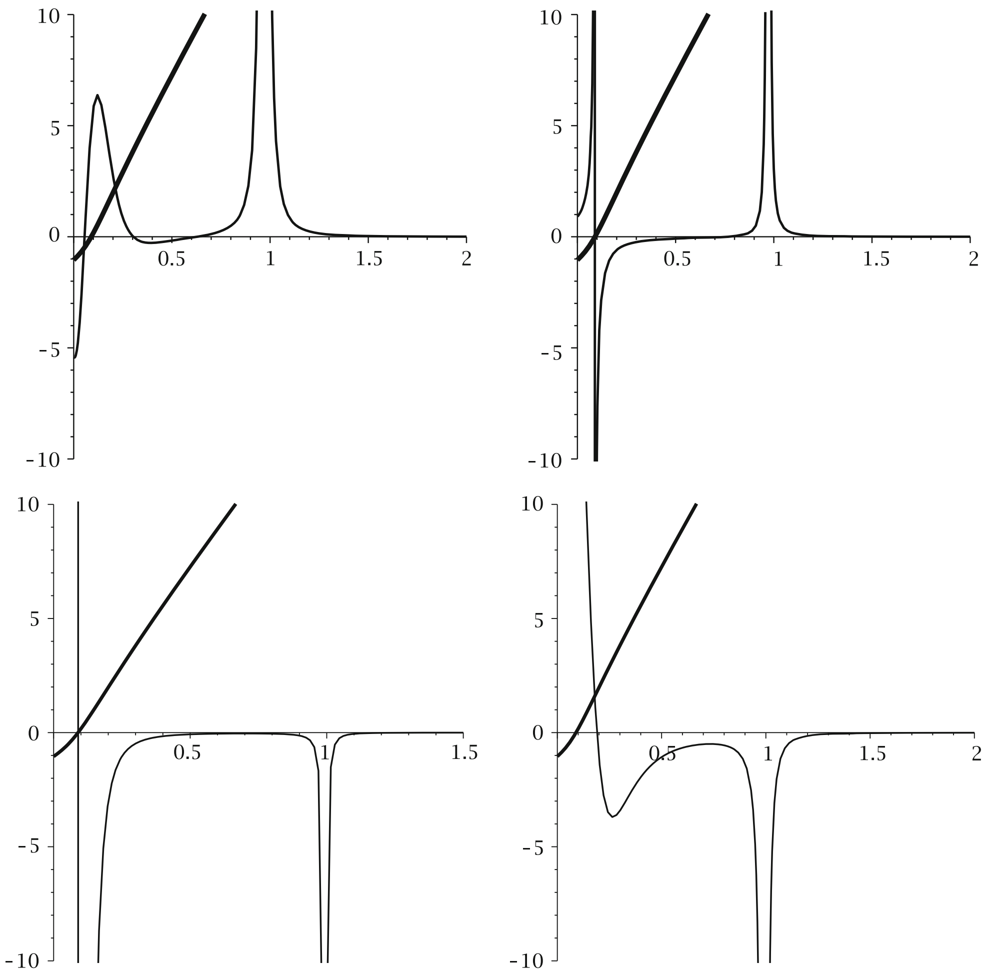

Fig. 2 BI case: $\mathcal{R}$ and C_Q versus r_+ for $l = 1$, $\Lambda = -1$, $\beta = 1$, $d = 3$ and $q = 0.1$. TRS (*continuous line*) and heat capacity (*bold line*) for the Weinhold metric (*left-up panel*), the Ruppeiner metric (*right-up panel*), the Quevedo metric for case I (*left-down panel*) and the Quevedo metric for case II (*right-down panel*)

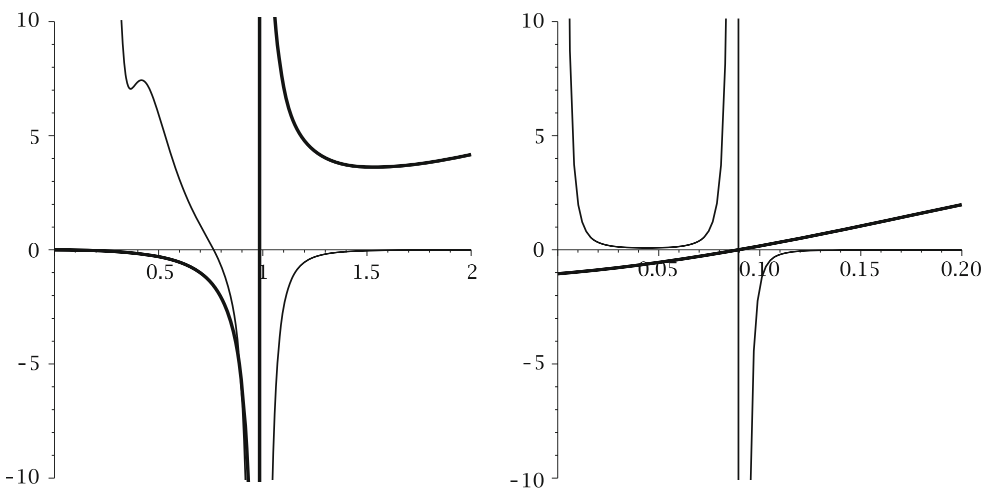

Fig. 3 BI case: $\mathcal{R}$ and C_Q versus r_+ for $l = 1$, $\Lambda = -1$, $\beta = 1$, $q = 0.1$. TRS (*continuous line*) and heat capacity (*bold line*) for the new metric with $d = 4$ (*left panel*) and $d = 3$ (*right panel*)

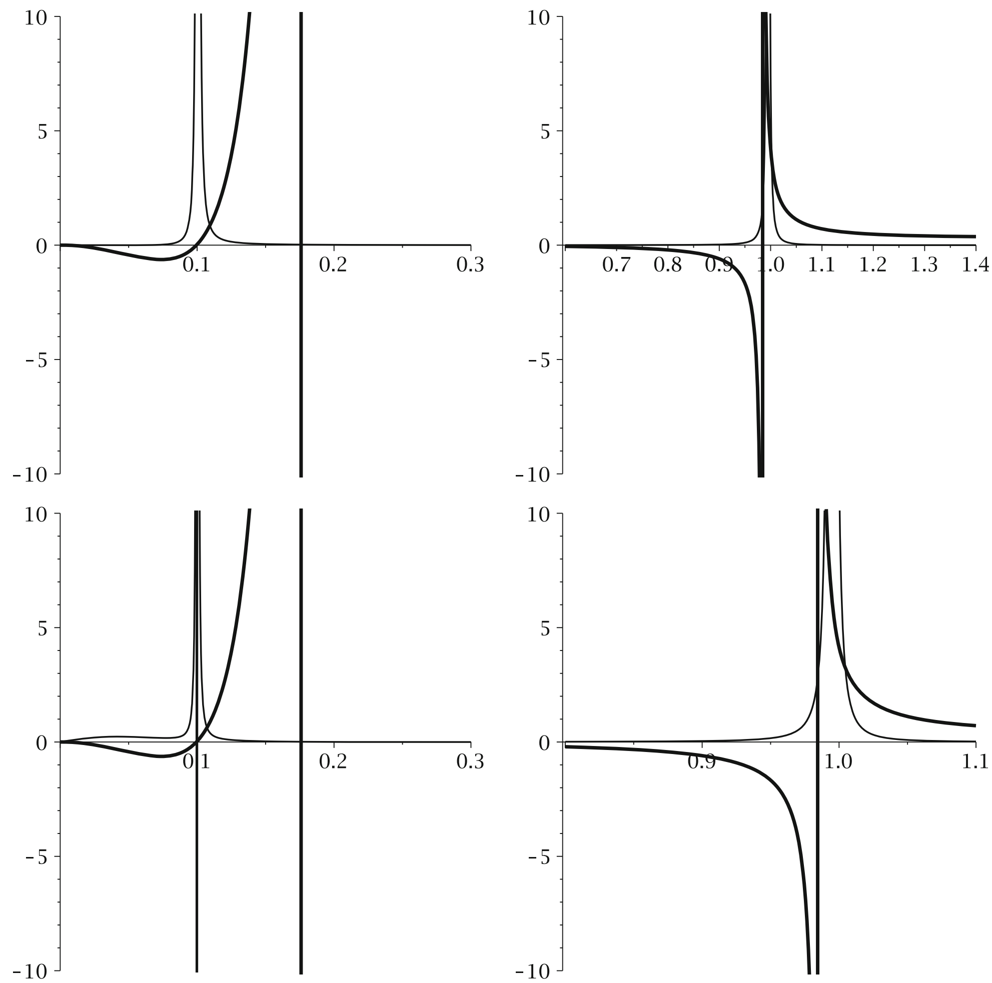

Fig. 4 Maxwell case: $\mathcal{R}$ and C_Q versus r_+ for $l = 1$, $\Lambda = -1$, $d = 4$ and $q = 0.1$. TRS (*continuous line*) and heat capacity (*bold line*) for the Weinhold metric (*up panels*) and the Ruppeiner metric (*down panels*)

regardless of the dimensions, the phase transition points of the heat capacity and divergencies of the Ricci scalar of the constructed spacetime, coincide and this approach is free of extra divergencies for its TRS.

We should note that this new metric enjoys the characteristic behavior which was mentioned in Einstein-BI solutions and one can distinguish two types of phase transitions from each other.

7.2 Neutral (uncharged) solutions

In this section, we study the behavior of the system in case of uncharged solutions ($q = 0$). Considering the uncharged solutions with an extensive parameter (entropy), one can find that the TRS of mentioned methods of geometrothermodynamics vanishes. Therefore, it seems that it is not possible to use these methods for studying the behavior of phase transitions. In other words, the geometrothermodynamics methods are valid for systems containing two or more extensive parameters.

8 Generalization

Now, we are in a position to generalize the new defined metric to the case of more than two extensive parameters. Regarding Weinhold, Ruppeiner and Quevedo approaches, one finds that supplementing additional extensive parameters increases the complexity of the thermodynamical Ricci scalars and also their denominators. The supplemental extensive parameters lead to extra terms which may contribute to the additional number of divergencies of the system under study. Considering the total mass of the system as a function of arbitrary number of extensive parameters χ_i's, one finds that the denominator of TRS for Weinhold, Ruppeiner and Quevedo approaches will be more complicated. Numerical calculations show that denominator of TRS for the mentioned geometrical approaches contain extra terms which may contribute to number of the TRS divergencies. For more clarifications, in addition to S and Q, we consider angular momentum, J, as extensive parameter. Regarding the total mass of the black holes as a function of S, Q and J, we calculate the

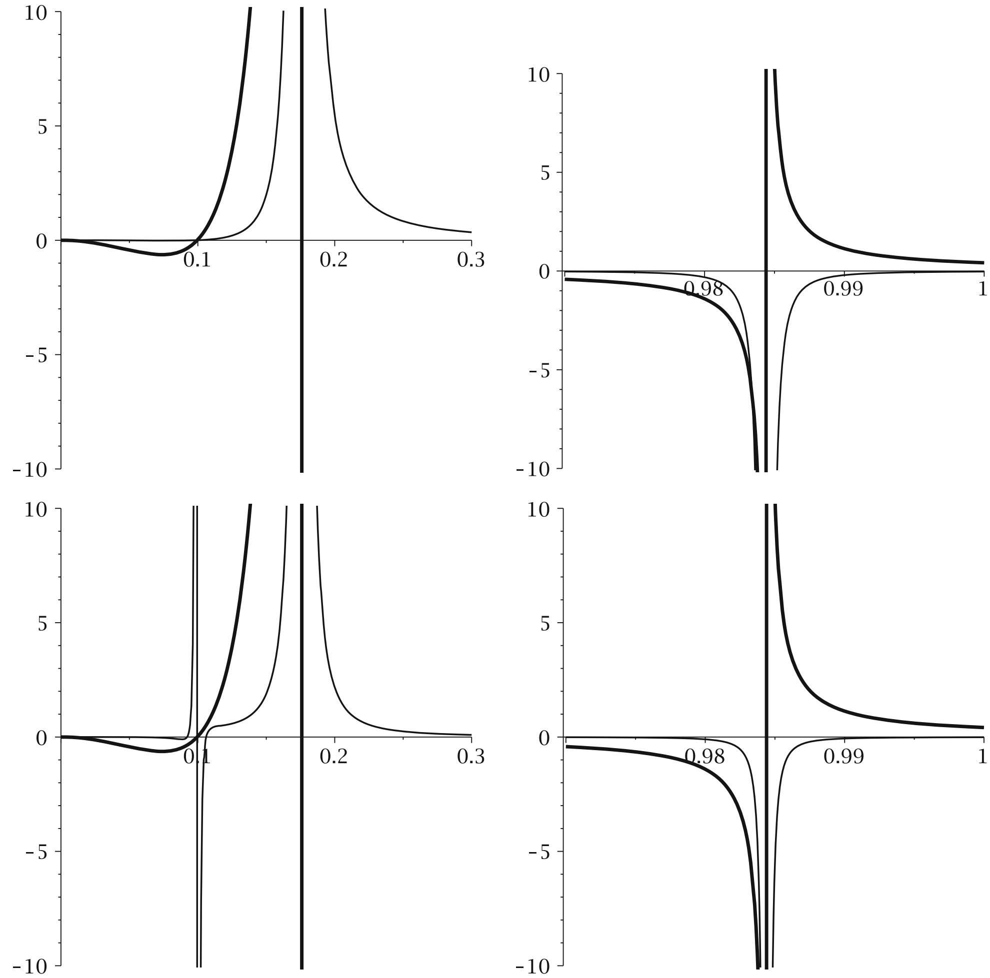

Fig. 5 Maxwell case: $\mathcal{R}$ and C_Q versus r_+ for $l = 1$, $\Lambda = -1$, $d = 4$ and $q = 0.1$. TRS (*continuous line*) and heat capacity (*bold line*) for the Quevedo metric for case I (*up panels*) and the Quevedo metric for case II (*down panels*)

denominator of TRS for the Weinhold, the Ruppeiner and the Quevedo metrics

$$\mathrm{denom}(\mathcal{R}) = \begin{cases} M^3\xi & \text{Weinhold} \\ M^3T^3\xi & \text{Ruppeiner} \\ M_{SS}^2 M_{QQ}^2 M_{JJ}^2 (SM_S + QM_Q + JM_J)^3 & \text{Quevedo Case } I \\ S^3 M_{SS}^2 M_{QQ}^2 M_{JJ}^2 M_S^3 & \text{Quevedo Case } II \end{cases}, \tag{27}$$

where

$$\xi = \Big[M_{SS}\left(M_{QJ}^2 - M_{QQ}M_{JJ}\right) + M_{SQ}^2 M_{JJ} + M_{SJ}^2 M_{QQ} - 2M_{SQ}M_{SJ}M_{QJ} \Big]^2 .$$

Considering Eq. (27), one finds that although vanishing points of heat capacity coincide with related divergence points of TRS in the Ruppeiner case (due to the existence of T), for the Weinhold one the coincidences are not generally observed. Regarding the divergence points of the heat capacity, which are related to vanishing M_{SS}, and in order to match both divergence points of heat capacity and those of the Weinhold and the Ruppeiner cases, one should set last three terms to zero and the coefficient of M_{SS} should be a nonzero finite expression ($M_{QJ}^2 - M_{QQ}M_{JJ} \neq 0$). Since these fine tuning conditions are not hold in general, we encounter mismatch between divergence points and/or extra divergencies in TRS.

As for Quevedo metrics, due to existence of M_{SS}^2, divergencies of both heat capacity and Ricci scalar match with each other. As for the phase transition type one and in order to have coincidence between roots of the heat capacity and divergence points of the TRS, for the case II Quevedo metric, existence of M_S^3 is sufficient to ensure the mentioned matching whereas we have the following restriction for the case I

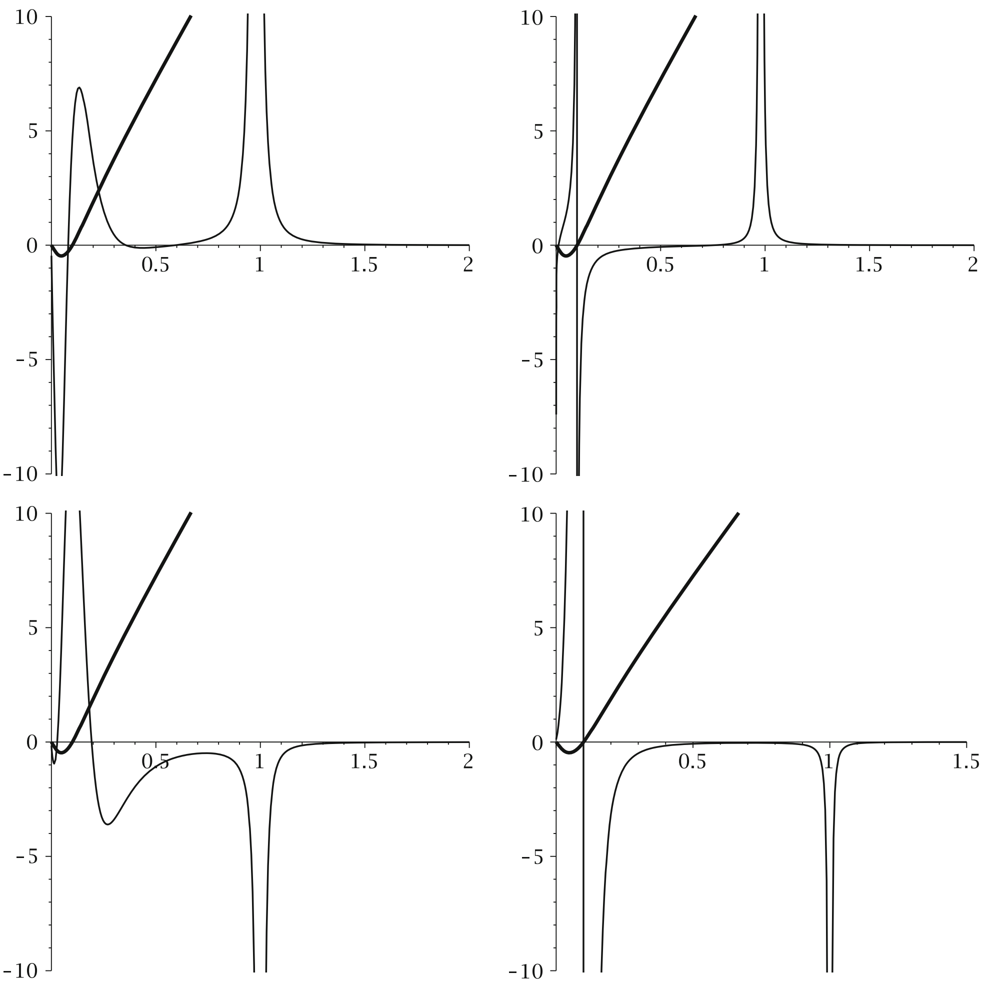

Fig. 6 Maxwell case: $\mathcal{R}$ and C_Q versus r_+ for $l = 1$, $\Lambda = -1$, $d = 3$ and $q = 0.1$. TRS (*continuous line*) and heat capacity (*bold line*) for the Weinhold metric (*left-up panel*), the Ruppeiner metric (*right-up panel*), the Quevedo metric for case I (*left-down panel*) and the Quevedo metric for case II (*right-down panel*)

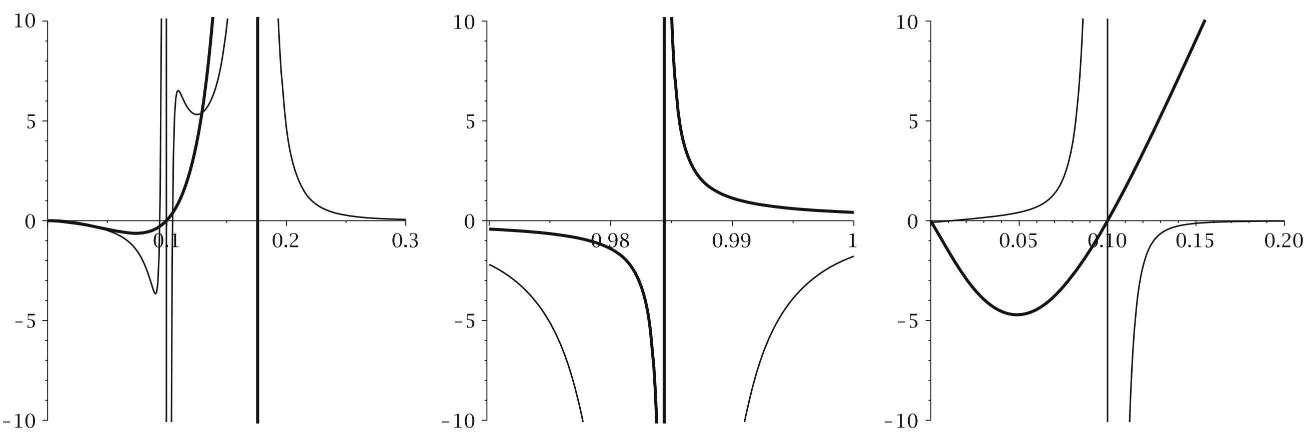

Fig. 7 Maxwell case: $\mathcal{R}$ and C_Q versus r_+ for $l = 1$, $\Lambda = -1$, $q = 0.1$. TRS (*continuous line*) and heat capacity (*bold line*) for the new metric with $d = 4$ (*left* and *middle panels* for different scales) and $d = 3$ (*right panel*)

$$SM_S + QM + JM_J = M_S. \tag{28}$$

In general Eq. (28) is not hold for arbitrary metric function parameters and therefore, there will be extra divergencies for the Ricci scalar which are not matched with any phase transition point. Regarding both of Quevedo metrics, M_{QQ}^2 and M_{JJ}^2, may contribute to additional divergencies of the Ricci scalar which was seen in case of 3 -dimensional solutions.

In order to solve these problems we generalize the new introduced thermodynamical metric to the case of n extensive parameters ($n \geq 2$) with following form

$$ds_{\mathrm{New}} = \frac{SM_S}{\left(\Pi_{i=2}^{n} \frac{\partial^2 M}{\partial \chi_i^2}\right)^3} \left(-M_{SS} dS^2 + \sum_{i=2}^{n} \left(\frac{\partial^2 M}{\partial \chi_i^2}\right) d\chi_i^2\right), \tag{29}$$

where $\chi_i \neq S$. In order to obtain the curvature singularity of the new generalized metric, we should calculate the Ricci scalar. Since analytical calculations are too large, for the sake of brevity, we study the denominator of the Ricci scalar. Calculations show that the too long expression of the numerator of TRS (generalization of Eq. (13)) is divergence free (To our knowledge various black holes in gravitational theories have smooth functions of finite mass. Nevertheless, we exclude the possible black holes with non-smooth finite mass and its derivatives), while its denominator is

$$\mathrm{denom}(\mathcal{R}) = S^3 M_S^3 M_{SS}^2. \tag{30}$$

Equation (30) confirms that all singular points of TRS coincide with both types of heat capacity phase transition points without any extra divergency.

9 Conclusions

Motivated by a surge of study of geometrical concept of black hole thermodynamics, we introduced a new metric regarding the matter. Considering the various approaches toward geometrical study of black hole thermodynamics, we first discussed the shortcomings of the mentioned methods. It was believed that the divergencies of TRS indicate the thermodynamical phase transitions, and therefore we focused on the roots of denominator of TRS (since we regarded smooth function of mass and its derivatives, the numerator of TRS is a regular function and therefore, divergencies of TRS is equivalent to the roots of denominator of TRS). Taking into account the Weinhold, Ruppeiner and different types of Quevedo metrics, we showed that divergencies of TRS may not coincide with roots and divergencies of the corresponding heat capacity.

In order to avoid this problem, we introduced a new metric. We showed that the denominator of its TRS contains terms which are only the product of numerator and denominator of the corresponding heat capacity. In other words, all divergencies of TRS in this approach coincide with phase transition points of the heat capacity. Moreover, for more clarifications, we regarded two known examples to show the shortcomings of the other previous metrics and the efficiency of the new devised metric.

Next, we generalized these metrics to contain more than two extensive parameters. As it was seen, in the cases of Weinhold, Ruppeiner and different types of Quevedo metrics, (denominator of) TRS contained complicated expressions and extra terms that may increase the number of TRS divergencies and shift the place of these divergencies in a way that they may not coincide with phase transition points arisen from the heat capacity. We showed that the divergencies of TRS related to the generalized new introduced metric are compatible to the phase transition points of heat capacity.

Another important property of the new metric is the different behavior of TRS before and after its divergence points. It was seen that the behavior of TRS for divergence points related to two types of the phase transition is different. Therefore, considering this approach also enable us to distinguish these two types of phase transition from one another.

Recently, it was seen and proposed that different constants (such as cosmological constant, BI nonlinearity parameter, Gauss-Bonnet parameter, Newton constant and etc.) may vary and have contribution to thermodynamical structure of the system [45–54]. In other words, in case of black holes, the total mass of the black hole is a function of these parameters as extensive parameters. It will be interesting to reconsider these constants as thermodynamical variables and modify TRS of the mentioned geometrothermodynamical methods. Although this modification changes TRS of various methods, it does not cause inconsistent results for new generalized metric.

The approach that we introduced in this paper enable one to map the divergence points of its TRS with phase transition points without any concern regarding contribution of other terms and extra divergence points. This method may also be employed to study phase transition of other non gravitating systems.

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