A New Approximation Algorithm for Minimum-Weight (1, m)–Connected Dominating Set

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Abstract

Consider a graph with nonnegative node weight. A vertex subset is called a CDS (connected dominating set) if every other node has at least one neighbor in the subset and the subset induces a connected subgraph. Furthermore, if every other node has at least m neighbors in the subset, then the node subset is called a (1, m)CDS. The minimum-weight (1, m)CDS problem aims at finding a (1, m)CDS with minimum total node weight. In this paper, we present a new polynomial-time approximation algorithm for this problem with approximation ratio $2H(\delta_{\max} + m - 1)$, where δ_{\max} is the maximum degree of the given graph and $H(\cdot)$ is the Harmonic function, i.e., $H(k) = \sum_{i=1}^{k} \frac{1}{i}$.

Keywords: minimum-weight connected *m*-fold dominating set, approximation algorithm.

1 Introduction

For a graph G = (V, E), where V is the node set and E is the edge set, a node subset C is a dominating set (DS) of G if any $v \in V \setminus C$ has at least one neighbor in C. A dominating set C is a connected DS (CDS) of G if G[C] is connected, where G[C] is the subgraph of G induced by C. The nodes in C are called *dominators*, and those in $V \setminus C$ are called *dominatees*. The minimum CDS (MinCDS) problem aims to find a CDS with the minimum cardinality/weight.

MinCDS has wide applications in many fields, including computer science, engineering, and operations research. For example, in a wireless sensor network (WSN), CDSs serve as virtual backbones [1, 2], they can save energy and reduce interference while maintaining information sharing.

The sensors in a WSN are prone to failures due to accidental damage or battery depletion. Therefore a fault-tolerant virtual backbone should be maintained. The *minimum k-connected* m-fold CDS (Min(k, m)CDS) problem was proposed for this purpose [3]. A node subset C is a (k, m)CDS if every node in $V \setminus C$ has at least m neighbors in C and the induced subgraph G[C]is k-connected.

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MinCDSs have been extensively studied, especially in unit disk graphs, UDGs, a widely adopted model of homogeneous WSNs. When nodes have nonnegative weights, the node-weighted versions, namely, minimum weight CDSs (MinWCDSs), have also achieved significant progress in UDGs [4, 5, 6, 7]. However, studies on MinWCDSs in general graphs are in a different situation. In 1999, Guha and Khuller [8] designed a $(1.35 + \varepsilon) \ln n$ -approximation algorithm for MinWCDSs, where n is the number of nodes. In real applications, δ_{\max} might be much smaller than n where δ_{\max} is the maximum degree of the input graph. Therefore, one usually expects to replace $\ln n$ by $\ln \delta_{\max}$. However, this expectation became a long-standing open problem. In fact, techniques provided in [8] do not have enough power to do so. Until 2018, with discovery of different techniques, Zhou et al. [9] presented an $(H(\delta_{\max} + m) + 2H(\delta_{\max} - 1))$ -approximation algorithm for the minimum-weight (1, m)CDS (MinW(1, m)CDS) problem, where $H(\cdot)$ is the Harmonic function, i.e., $H(k) = \sum_{i=1}^{k} \frac{1}{i} \leq \ln k + 1$ (however, there is a flaw in this work, please see discussion in Section 4)

In this paper, using a completely new idea of analysis, we design a new algorithm for the MinW(1, m)CDS problem in a general graph to achieve approximation ratio $2H(\delta_{max} + m - 1)$. Note that our ratio is better than that in [9] even if its flaw can be fixed.

2 Preliminaries

We first give a formal definition of the problem and some preliminary results.

Let G = (V, E) be a connected graph and C be a node subset of V. Denote by G[C] the subgraph of G induced by C, $N_C(u)$ the set of neighbors of u in C, $N(u) = N_V(u)$, and deg(u) = |N(u)|. For $C \subseteq V$, $N(C) = (\bigcup_{u \in C} N(u)) \setminus C$ denotes the open neighborhood of C. The formal definition of the MinW(k, m)CDS problem is as follows.

Definition 2.1 (the minimum weight k-connected m-fold dominating set (MinW(k, m)CDS) problem). Let G be a connected graph on node set V and edge set E, k and m be two positive integers, and $c: V \to R^+$ be a cost function on the nodes. A node subset $C \subseteq V$ is a (k, m)CDS if every node in $V \setminus C$ is adjacent to at least m nodes of C, and G[C], the subgraph of G induced by C, is k-connected (that is, G[C] remains connected after removing at most k-1 nodes). The MinW(k,m)CDS problem aims to find a (k,m)CDS with the minimum cost, where the cost of node set C is $c(C) = \sum_{v \in C} c(v)$.

A set function $f: 2^V \to \mathbb{R}^+$ is monotone nondecreasing if $f(A) \leq f(B)$ for any $A \subseteq B \subseteq V$; it is submodular if $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$ for any $A, B \subseteq V$. For node sets $A, B \subseteq V$, let

$$\Delta_A f(B) = f(A \cup B) - f(B)$$

be the marginal profit of A over B. The following results are well-known properties for monotone and submodular functions (see, for example, [10]).

Lemma 2.2. A set function f is monotone nondecreasing if and only if $\Delta_u f(A) \geq 0$ holds for any $A \subseteq V$ and $u \in V$; it is submodular if and only if $\Delta_u f(A) \geq \Delta_u f(B)$ holds for any $A \subseteq B \subseteq E$ and $u \in V \setminus B$; it is monotone nondecreasing and submodular if and only if $\Delta_u f(A) \geq \Delta_u f(B)$ holds for any $A \subseteq B \subseteq E$ and $u \in V$.

The following is a property of a submodular function.

Lemma 2.3. If $f : 2^V \to \mathbb{R}^+$ is a submodular function, then for any subsets $A, B \subseteq V$,

$$\Delta_B f(A) \le \sum_{v \in B \setminus A} \Delta_v f(A).$$

3 Main Results

Let us first describe our algorithm and then give analysis.

3.1 Algorithm

The algorithm uses a greedy strategy. In each iteration, it selects a most cost-effective star. The cost-effectiveness of a star depends on a potential function g designed as follows.

For a node subset $C \subseteq V$ and a node $u \in V$, define

$$q_C(u) = \begin{cases} \max\{0, m - |N_C(u)|\}, & u \in V \setminus C, \\ 0, & u \in C. \end{cases}$$
$$q(C) = \sum_{u \in V \setminus C} q_C(u),$$
$$p(C) = \text{the number of components of } G[C],$$
$$f(C) = p(C) + q(C).$$

For a node set $U \subseteq V \setminus C$, denote by $NC_C(U)$ the set of components of G[C] which are adjacent to U. Every component in $NC_C(U)$ is called a *component neighbor* of U in C (if a component of G[C] has nonempty intersection with U, then it is also viewed as a component neighbor of U). For a node $u \in V$, we use S_u to denote some star with center u, that is, S_u is a subgraph of G induced by some edges between node u and some of u's neighbors. In particular, a single node is a *trivial star*. In the following, we treat S_u as a star as well as the set of nodes in the star. For a node set C, a node $u \in V \setminus C$, and a star S_u , suppose $S_u \setminus \{u\} = \{u_1, \ldots, u_s\}$ has $c(u_1) \leq \cdots \leq c(u_s)$, define

$$b_C^{S_u}(u_i) = \begin{cases} 0, & q_C(u_i) > 0, \\ \min\{1, -\Delta_{u_i} f(C_i)\}, & q_C(u_i) = 0 \end{cases}$$
(1)

where $C_i = C \cup \{u, u_1, ..., u_{i-1}\}$. Let

$$g_C(S_u) = -\Delta_u f(C) + \sum_{i=1}^s b_C^{S_u}(u_i).$$
 (2)

The cost-effectiveness of star S_u with respect to a node set C is defined to be $g_C(S_u)/c(S_u)$.

Pseudo codes of the main algorithm is presented in Algorithm 1. It iteratively adds a most cost-effective star to the current set C. We shall show latter in Lemma 3.3 that such a star can be found efficiently by Algorithm 2.

3.2 Finding a Most Cost-Effective Star

Before showing how to find out a most cost-effective star, we first give some properties for functions p, q and f.

Lemma 3.1. Set functions -q(C), -p(C) and -f(C) satisfy the following properties.

(a) -q(C) is monotone nondecreasing and submodular;

 (b_1) for any node set C and node $u \notin C$, $-\Delta_u p(C) \ge -1$, equality holds if and only if u is not adjacent with C;

 (b_2) for any connected node set C' and any node $u \in V \setminus (C \cup C')$, we have

$$-\Delta_u p(C \cup C') \le -\Delta_u p(C) + 1$$

Algorithm 1

Input: A connected graph G = (V, E). Output: A node set C which is a (1, m)-CDS of G. 1: Set $C \leftarrow \emptyset$. 2: while \exists a star S_u with $g_C(S_u) > 0$ do 3: Use Algorithm 2 to compute a most cost-effective star $S_u = \arg \max_{S_u \subseteq V \setminus C} \frac{g_C(S_u)}{c(S_u)}$. 4: $C \leftarrow C \cup S_u$ 5: end while 6: Output C.

Algorithm 2

Input: A connected graph G = (V, E), a node set $C \subseteq V$. Output: A most cost-effective star S_u with respect to C. 1: for each $u \in V \setminus C$ do 2: $S_u \leftarrow \{u\}$ if $q_C(u) = 0$ then 3: $N_u \leftarrow$ the set of nodes in N(u) satisfying (*iii*) and (*iv*) of Lemma 3.3. 4: Order the nodes in N_u as u_1, \ldots, u_s such that $c(u_1) \leq \cdots \leq c(u_s)$. 5: for j = 1, ..., s do 6: If $b_C^{S_u}(u_j) = 1$ and $\frac{1}{c(u_j)} \ge \frac{g_C(S_u)}{c(S_u)}$, then $S_u \leftarrow S_u \cup \{u_j\}$. 7: end for 8: end if 9: 10: end for 11: Output $S_u \leftarrow \arg \max\{g_C(S_u)/c(S_u) : u \in V \setminus C\}$, giving priority to trivial star.

equality holds only when G[C'] is not adjacent with G[C] and node u is adjacent with G[C']; (c) -f(C) is monotone nondecreasing.

Proof. For any node sets $C_1 \subseteq C_2 \subseteq V$ and node $u \in V$, we have $q_{C_1}(u) \geq q_{C_2}(u)$. So, $q(C_1) = \sum_{u \in V \setminus C_1} q_{C_1}(u) \geq \sum_{u \in V \setminus C_2} q_{C_2}(u) = q(C_2)$ and thus -q is monotone nondecreasing. Furthermore, for any $u \in V \setminus C_2$, we have $-\Delta_u q(C_1) = q_{C_1}(u) + |\{v \in N(u) : q_{C_1}(v) > 0\}| \geq q_{C_2}(u) + |\{v \in N(u) : q_{C_2}(v) > 0\}| = -\Delta_u q(C_2)$, and thus -q is submodular. Property (a) is proved.

Note that adding node u into C will merge those components of G[C] which are adjacent with u into one big component of $G[C \cup \{u\}]$. So,

$$-\Delta_u p(C) = |NC_C(u)| -1 \ge -1.$$
(3)

Inequality becomes equality if and only if $|NC_C(u)| = 0$, which holds if and only if u is not adjacent with C. Property (b_1) is proved.

By equality (3), $-\Delta_u p(C \cup C') - (-\Delta_u p(C)) = |NC_{C \cup C'}(u)| - |NC_C(u)|$. Since C' is a connected node set, adding C' into C will merge those components of G[C] which are adjacent with C' or have nonempty intersection with C' into one component. So, $|NC_{C \cup C'}(u)| > |NC_C(u)|$ happens only when G[C'] is a component of $G[C \cup C']$ and u is adjacent with C', in which case $|NC_{C \cup C'}(u)| = |NC_C(u)| + 1$ and $-\Delta_u p(C \cup C') - (-\Delta_u p(C)) = 1$. In all the other cases, we have $-\Delta_u p(C \cup C') - (-\Delta_u p(C)) \leq 0$. Hence property (b_2) is proved.

By properties $(a), (b_1)$ and Lemma 2.2, we have $-\Delta_u f(C) = -\Delta_u q(C) - \Delta_u p(C) \ge -1$, and equality holds only when $-\Delta_u q(C) = 0$ and $-\Delta_u p(C) = -1$. Since $-\Delta_u p(C) = -1$ implies that u is not adjacent with C and thus $q_C(u) = m$, we have $-\Delta_u q(C) \ge q_C(u) > 0$. So, $-\Delta_u f(C) \ge 0$ holds for any node set C and any $u \in V \setminus C$, which is equivalent to say that -f is monotone nondecreasing.

As a corollary of the above lemma, we have the following result.

Lemma 3.2. Let C be a node set and S_u be a star rooted at u. Suppose $S_u \setminus \{u\} = \{u_1, \ldots, u_s\}$ and $c(u_1) \leq c(u_2) \leq \cdots \leq c(u_s)$. For any $i \in \{1, \ldots, s\}$, denote by $prec(u_i) = \{u, u_1, \ldots, u_{i-1}\}$. Then $-\Delta_{u_i} f(C \cup prec(u_i)) \geq 0$ and equality holds if and only if $-\Delta_{u_i} q(C \cup prec(u_i)) = -\Delta_{u_i} p(C \cup prec(u_i)) = 0$.

Proof. By the monotonicity of -q, we have $-\Delta_{u_i}q(C \cup prec(u_i)) \ge 0$. Since $prec(u_i)$ is a connected set and u_i is adjacent with $prec(u_i)$, by property (b_1) of Lemma 3.1, we have $-\Delta_{u_i}p(C \cup prec(u_i)) \ge 0$. So, $-\Delta_{u_i}f(C \cup prec(u_i)) \ge 0$, and $-\Delta_{u_i}f(C \cup prec(u_i)) = 0$ if and only if $-\Delta_{u_i}q(C \cup prec(u_i)) = -\Delta_{u_i}p(C \cup prec(u_i)) = 0$.

A simple relation will be used in the proof: for four positive real numbers a, b, c, d:

$$\frac{a+b}{c+d} \ge \frac{b}{d} \Longrightarrow \frac{a}{c} \ge \frac{a+b}{c+d} \ge \frac{b}{d}.$$
(4)

This is because $\frac{a+b}{c+d} = \frac{b(\frac{a}{b}+1)}{d(\frac{a}{c}+1)} \ge \frac{b}{d}$ implies $\frac{a}{b} \ge \frac{c}{d}$, and then implies $\frac{a+b}{c+d} = \frac{a(1+\frac{b}{a})}{c(1+\frac{d}{c})} \le \frac{a}{c}$.

The next lemma shows that there exists a most cost-effective star which has some special properties.

Lemma 3.3. Let C be a node set of graph G. There exists a most cost-effective star S_u with respect to C such that for any $v \in V(S_u) \setminus \{u\}$, the following properties hold:

(i) $b_C^{S_u}(v) = 1;$ (ii) $\frac{1}{c(v)} \ge g_C(S_u)/c(S_u);$ (iii) $q_C(v) = 0;$ (iv) $|NC_C(v)| = 1$ and the component of G[C] adjacent with v is not adjacent with u.

Proof. Let S_u be a most cost-effective star. If $V(S_u) = \{u\}$, then S_u satisfies the above conditions. In the following, we assume that there is no trivial most cost-effective star. Suppose $V(S_u) = \{u, u_1, u_2, \ldots, u_s\}$ and $c(u_1) \leq c(u_2) \leq \ldots \leq c(u_s)$.

Proof of property (i). We first show that for any $1 \le i \le s$,

$$b_C^{S_u}(u_j) \le b_C^{S_u-u_i}(u_j).$$
 (5)

If $q_C(u_j) > 0$, then both $b_C^{S_u}(u_j) = b_C^{S_u-u_i}(u_j) = 0$, and (5) trivially holds. So, suppose $q_C(u_j) = 0$. In this case, $b_C^{S_u}(u_j)$ is determined by $-\Delta_{u_j} f(C \cup \{u, u_1, \ldots, u_{j-1}\})$. By Lemma 3.1 and Lemma 2.2, for any $1 \le i \le s$,

$$-\Delta_{u_j} q(C \cup \{u, u_1, \dots, u_{j-1}\}) \le -\Delta_{u_j} q(C \cup \{u, u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_j\}) \text{ and } (6)$$

$$\Delta_{u_j} p(C \cup \{u, u_1, \dots, u_{j-1}\}) \le -\Delta_{u_j} p(C \cup \{u, u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_j\}) + 1.$$
(7)

In fact, (7) can be improved to

 $-\Delta_{u_j} p(C \cup \{u, u_1, \dots, u_{j-1}\}) \le -\Delta_{u_j} p(C \cup \{u, u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_j\}),$

because u_j is adjacent with u and thus the inequality in (b_1) of Lemma 3.1 is strict. Combining this with (6), we have

$$-\Delta_{u_j} f(C \cup \{u, u_1, \dots, u_{j-1}\}) \le -\Delta_{u_j} f(C \cup \{u, u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_j\}),$$
(8)

and thus (5) is proved.

Next, we show that

for any
$$1 \le i \le s, b_C^{S_u}(u_i) \ne 0.$$
 (9)

Suppose (9) is not true, and j is an index with $b_C^{S_u}(u_j) = 0$. Then by (5),

$$g_C(S_u) = -\Delta_u f(C) + \sum_{i=1}^s b_C^{S_u}(u_i)$$

$$\leq -\Delta_u f(C) + \sum_{i=1}^{j-1} b_C^{S_u - u_j}(u_i) + \sum_{i=j+1}^s b_C^{S_u - u_j}(u_i)$$

$$= g_C(S_u - u_j).$$
(10)

Combining this with $c(S_u) = c(S_u - u_j) + c(u_j) > c(S_u - u_j)$, we have

$$\frac{g_C(S_u - u_j)}{c(S_u - u_j)} > \frac{g_C(S_u)}{c(S_u)},\tag{11}$$

and thus $S_u - u_j$ is a more cost-effective star than S_u , contradicting the assumption on S_u . So property (9) is proved.

By the definition of $b_C^{S_u}(u_j)$ in (1), we have $b_C^{S_u}(u_j) \in \{0,1\}$, and thus property (i) follows from (9).

Proof of property (ii). We prove that for any $j = 1, \ldots, s$,

$$\frac{1}{c(u_j)} \ge \frac{g_C(S_u)}{c(S_u)}.$$
(12)

Note that for any i > j, we have $b_C^{S_u}(u_j) = b_C^{S_u-u_i}(u_j)$. For any i < j, by (5) and property (i), we have $1 = b_C^{S_u}(u_j) \le b_C^{S_u-u_i}(u_j) \le 1$, and thus $b_C^{S_u}(u_j) = b_C^{S_u-u_i}(u_j) = 1$. In other words, $b_C^{S_u}(u_j) = b_C^{S_u-u_i}(u_j)$ for any $i \neq j$. Hence

$$g_C(S_u) = g_C(S_u - u_j) + b_C^{S_u}(u_j) = g_C(S_u - u_j) + 1.$$

Since S_u is a most cost-effective star, we have

$$\frac{g_C(S_u - u_j)}{c(S_u - u_j)} \le \frac{g_C(S_u)}{c(S_u)} = \frac{g_C(S_u - u_j) + 1}{c(S_u - u_j) + c(u_j)},$$

Combining this with (4), we have inequality (12). Thus property (ii) is proved.

Proof of property (iii). This property directly follows from property (i) and the definition of $b_C^{S_u}$ in (1).

Before proving property (iv), we first prove

$$-\Delta_{u_j} f(C) = -\Delta_{u_j} q(C) = -\Delta_{u_j} p(C) = 0.$$
(13)

Suppose j is an index with $-\Delta_{u_j} f(C) \neq 0$. Then by the monotonicity of -f (see Lemma 3.1), we have $-\Delta_{u_j} f(C) \geq 1$. Combining this with inequality (12), for the trivial star $\{u_j\}$, we have

$$\frac{g_C(\{u_j\})}{c(u_j)} = \frac{-\Delta_{u_j} f(C)}{c(u_j)} \ge \frac{1}{c(u_j)} \ge \frac{g_C(S_u)}{c(S_u)},$$

which implies that $\{u_j\}$ is a most cost-effective star, contradicting our assumption that there is no trivial most cost-effective star. So, $-\Delta_{u_j} f(C) = 0$.

Notice that $q_C(u_j) = 0$ means u_j is dominated by at least m nodes in C. So, u_j is adjacent with C, and thus by (b_1) of Lemma 3.1, we have $-\Delta_{u_j}p(C) \ge 0$. By the monotonicity of -q, we have $-\Delta_{u_j}q(C) \ge 0$. Hence in order that $-\Delta_{u_j}f(C) = 0$, we must have $-\Delta_{u_j}p(C) = 0$ and $-\Delta_{u_j}q(C) = 0$. Equalities in (13) are proved.

Proof of property (iv). Notice that $-\Delta_{u_j}p(C) = 0$ implies that u_j is adjacent with exactly one component of G[C]. If this component is also adjacent with u, then we also have $-\Delta_{u_j}p(C \cup prec(u_j)) = 0$ (notice that by the star structure, $prec(u_j)$ are in a same component of $G[C \cup prec(u_j)]$). By the monotonicity and submodularity of -q, we have $0 \leq -\Delta_{u_j}q(C \cup prec(u_j)) \leq -\Delta_{u_j}q(C) = 0$, and thus $-\Delta_{u_j}q(C \cup prec(u_j)) = 0$. But then $-\Delta_{u_j}f(C \cup prec(u_j)) = 0$ and thus $b_C^{S_u}(u_j) = 0$, contradicting property (i). Hence the unique component of G[C] adjacent with u_j is not adjacent with u. The proof is completed.

The following lemma shows that a most cost-effective star can be found efficiently.

Lemma 3.4. For a node set C of graph G, a most cost-effective star satisfying Lemma 3.3 can be found in time $O(n^2)$, where n is the number of nodes in G.

Proof. The computation method is described in Algorithm 2. For each $u \in V \setminus C$, the algorithm finds a most cost-effective star centered at u, which satisfies Lemma 3.3 (this will be proved in the following), denote it as S_u . A most cost-effective star with respect to C is the best one of $\{S_u : u \in V \setminus C\}$. The reason why priority is given to trivial star is: if the output is a nontrivial star, then no trivial star is most cost-effective, and property (13) holds, which brings more structural property to be used in the analysis.

The algorithm is illustrated by the example in Fig. 1, and the proof of the correctness is divided into two steps.



Figure 1: An illustration of the execution of Algorithm 2. Every u_i (i = 1, 2, 3, 4) is adjacent with exactly one component of G[C] (indicated by big circle) which is not adjacent with the center u. Suppose $c(u_1) \leq c(u_2) \leq c(u_3) \leq c(u_4)$ and only u_4 has its cost $c(u_4) > \frac{c(S_u^{curr})}{g_C(S_u^{curr})}$. The blackened structure is the final S_u . The reason why node u_2 is not added into S_u is because $b_C^{S_u}(u_2) = 0$. Node u_4 is not added into S_u because its cost is too large to satisfy property (ii).

Claim 1. The star S_u computed by the algorithm satisfies the four properties described in Lemma 3.3.

Notice that if $q_C(u) > 0$, then for any neighbor v of u, we have $-\Delta_v q(C) > 0$, violating property (*iii*). Hence, only when $q_C(u) = 0$, we need to consider a nontrivial star (through line 3 to line 9 of Algorithm 2).

Denote by N_u the set of nodes in N(u) satisfying properties (*iii*) and (*iv*). The feet of S_u can only be taken from N_u . The idea of the algorithm is to start from the trivial star $S_u = \{u\}$, and sequentially check nodes of N_u in increasing order of costs. If properties (i) and (ii) are satisfied, then expand S_u . What needs to be explained is: why $\frac{1}{c(u_j)} \geq \frac{g_C(S_u^{curr})}{c(S_u^{curr})}$ for the current star S_u^{curr} implies $\frac{1}{c(u_j)} \ge \frac{g_C(S_u^{final})}{c(S_u^{final})}$ for the final star S_u^{final} computed by the algorithm. Suppose u_{ℓ} is the last node added into S_u . Denote $S_u^{u_{\ell}} = S_u^{final} - u_{\ell}$. Since u_{ℓ} is eligible to be added by the algorithm, we have $\frac{1}{c(u_\ell)} \ge \frac{g_C(S_u^{u_\ell})}{c(S_u^{u_\ell})}$ and $b_C^{S_u}(u_\ell) = 1$. It follows that $g_C(S_u^{final}) = g_C(S_u^{u_\ell}) + 1$, and thus

$$\frac{g_C(S_u^{final})}{c(S_u^{final})} = \frac{g_C(S_u^{u_\ell}) + 1}{c(S_u^{u_\ell}) + c(u_\ell)}$$

Then by (4), we have

$$\frac{1}{c(u_{\ell})} \ge \frac{g_C(S_u^{final})}{c(S_u^{final})} \ge \frac{g_C(S_u^{u_{\ell}})}{c(S_u^{u_{\ell}})}.$$
(14)

Since $c(u_j) \leq c(u_\ell)$, we also have $\frac{1}{c(u_j)} \geq \frac{g_C(S_u^{j\,ina_\ell})}{c(S_u^{f\,inal})}$.

It should be remarked that similarly to the derivation for the right side of inequality (14), by induction on the feet of S_u in the reverse order of their addition into S_u , it can be seen that $g_C(S_u^i)/c(S_u^i) \leq g_C(S_u^j)/c(S_u^j)$ for i < j, where S_u^i is the current star when u_i is added. As a corollary,

$$\frac{g_C(S_u^{final})}{c(S_u^{final})} \ge \frac{g_C(S_u^{curr})}{c(S_u^{curr})}$$
(15)

throughout the process.

Claim 2. The computed star S_u is indeed most cost-effective.

Let S_u^* be a most cost-effective star centered at u which satisfies those properties in Lemma 3.3. We shall prove that

$$\frac{g_C(S_u)}{c(S_u)} = \frac{g_C(S_u^*)}{c(S_u^*)}.$$
(16)

First consider the case that $S_u^* \subseteq S_u$. If $S_u^* = S_u$, then (16) is obviously true. So, suppose $S_u \setminus S_u^* \neq \emptyset$. Let u_j be a maximum-cost node of $S_u \setminus S_u^*$. By property (ii), $\frac{1}{c(u_j)} \geq \frac{g_C(S_u)}{c(S_u)}$. Notice that any star S_v satisfying property (i) has $g_C(S_v) = -\Delta_v f(C) + |S_v \setminus \{v\}|$. So, $g_C(S_u) = g_C(S_u^*) + | S_u \setminus S_u^* |$. Then by the assumption that u_j has the maximum cost in $S_u \setminus S_u^*$, we have

$$\frac{\mid S_u \setminus S_u^* \mid}{c(S_u \setminus S_u^*)} \ge \frac{1}{c(u_j)} \ge \frac{g_C(S_u)}{c(S_u)} = \frac{g_C(S_u^*) + \mid S_u \setminus S_u^* \mid}{c(S_u^*) + c(S_u \setminus S_u^*)}$$

Then by (4), we have $\frac{g_C(S_u)}{c(S_u)} \ge \frac{g_C(S_u^*)}{c(S_u^*)}$, and thus $\frac{g_C(S_u)}{c(S_u)} = \frac{g_C(S_u^*)}{c(S_u^*)}$ by the optimality of S_u^* . Next, consider the case when $S_u^* \setminus S_u \ne \emptyset$. Consider a node $u_j \in S_u^* \setminus S_u$. By Lemma 3.3 and

observation (15),

$$\frac{1}{c(u_j)} \ge \frac{g_C(S_u^*)}{c(S_u^*)} \ge \frac{g_C(S_u)}{c(S_u)} \ge \frac{g_C(S_u^{curr})}{c(S_u^{curr})}.$$

So, the reason why u_j is not added into S_u is because $b_C^{S_u}(u_j) = 0$, that is, $-\Delta_{u_j} f(C \cup prev(u_j)) = 0$ 0. By Lemma 3.2, we have $-\Delta_{u_i} p(C \cup prev(u_j)) = 0$, which implies that the unique component of G[C] adjacent with u_j is also adjacent with a node u_ℓ with $\ell < j$ which has been added into S_u before. Note that for those nodes in N_u which are adjacent with the same component of G[C], in order that $b_C^{S_u^*}$ has value 1, at most one of them can belong to S_u^* . So, $u_\ell \notin S_u^*$. Let $S'_u = S_u^* + u_\ell - u_j$. Then $g_C(S_u^*) = g_C(S'_u)$. Since u_ℓ is ordered before u_j , we have $c(u_\ell) \leq c(u_j)$. Then

$$\frac{g_C(S_u^*)}{c(S_u^*)} = \frac{g_C(S_u')}{c(S_u') + c(u_j) - c(u_\ell)} \le \frac{g_C(S_u')}{c(S_u')} \le \frac{g_C(S_u^*)}{c(S_u^*)}.$$

It follows that $c(u_{\ell}) = c(u_j)$ and $\frac{g_C(S_u^*)}{c(S_u^*)} = \frac{g_C(S_u')}{c(S_u')}$, which implies that S'_u is also a most costeffective star satisfying Lemma 3.3. Notice that S'_u and S_u have one more common foot. Proceeding like this, by an inductive argument on $|S_u^* \setminus S_u|$, it can be shown that S_u is also most cost-effectiveness.

The correctness of Algorithm 2 follows from Claim 1 and Claim 2. As to the time complexity, note that for any $u \in V \setminus C$, the algorithm only considers u and N(u) at most once, so the time spent by the algorithm is at most $2 \mid E(G) \mid$, which is $O(n^2)$.

3.3 Feasibility and Approximation Ratio

Before proving that the output of Algorithm 1 is a feasible solution with the desired approximation ratio, we first prove two technical lemmas.

Lemma 3.5. Let C be a node set and S_u be a most cost-effective star satisfying the properties in Lemma 3.3. Then $g_C(S_u) = -\Delta_{S_u} f(C)$.

Proof. The lemma holds if S_u is a trivial star. So in the following, we consider nontrivial star. In this case, (13) holds for any node $v \in S_u \setminus \{u\}$. In particular, $-\Delta_v q(C) = 0$. Then by the monotonicity and submodularity of -q, we have

$$0 \le -\Delta_{S_u \setminus \{u\}} q(C \cup \{u\}) \le -\sum_{v \in S_u \setminus \{u\}} \Delta_v q(C \cup \{u\}) \le -\sum_{v \in S_u \setminus \{u\}} \Delta_v q(C) = 0.$$

So, $-\Delta_{S_u \setminus \{u\}} q(C \cup \{u\}) = 0$, and thus $-\Delta_{S_u \setminus \{u\}} f(C \cup \{u\}) = -\Delta_{S_u \setminus \{u\}} p(C \cup \{u\})$. By property (*i*) and (*iv*) (note that these properties imply that those unique component neighbors of distinct feet of S_u are distinct), we have

$$\begin{aligned} -\Delta_{S_u} f(C) &= -\Delta_u f(C) - \Delta_{S_u \setminus \{u\}} f(C \cup \{u\}) \\ &= -\Delta_u f(C) - \Delta_{S_u \setminus \{u\}} p(C \cup \{u\}) \\ &= -\Delta_u f(C) + |S_u \setminus \{u\} | \\ &= -\Delta_u f(C) + \sum_{v \in S_u \setminus \{u\}} b_C^{S_u}(v) \\ &= g_C(S_u). \end{aligned}$$

The lemma is proved.

Lemma 3.6. For two node sets $C, C' \subseteq V(G)$, suppose there is an edge uv with $u \in C' \setminus C$, $v \in V \setminus (C \cup C')$, and $q_C(v) > 0$. Then, $-\Delta_{C'}q(C) + (-\Delta_v q(C)) \ge -\Delta_{C' \cup \{v\}}q(C) + 1$.

Proof. By the assumption $u, v \notin C$, $q_C(v) > 0$, and v is adjacent with u, we have $q_C(v) = q_{C \cup \{u\}}(v) + 1$. Combining this with the submodularity of -q, we have

$$\begin{aligned} -\Delta_v q(C) &= q_C(v) + | \{ x \in N(v) \colon q_C(x) > 0 \} | \\ &\geq q_{C \cup \{u\}}(v) + 1 + | \{ x \in N(v) \colon q_{C \cup \{u\}}(x) > 0 \} | \\ &= -\Delta_v q(C \cup \{u\}) + 1 \\ &\geq -\Delta_v q(C \cup C') + 1. \end{aligned}$$

It follows that

$$-\Delta_{C'}q(C) + (-\Delta_v q(C)) \ge -\Delta_{C'}q(C) - \Delta_v q(C \cup C') + 1 = -\Delta_{C' \cup \{v\}}q(C) + 1.$$

The lemma is proved.

The following result is a folklore for dominating set, which can be found, for example, in Wan et al. [11].

Lemma 3.7. Suppose C is a dominating set of G and G[C] is not connected. Then, the two nearest components of G[C] are at most three hops away.

The next lemma shows that the algorithm outputs a feasible solution.

Lemma 3.8. The output C of Algorithm 1 is a (1,m)-CDS of graph G.

Proof. First, we show that C is an m-DS of G. If not, then there exists a node $u \in V \setminus C$ with $q_C(u) > 0$. If u is adjacent with C, then $-\Delta_u p(C) \ge 0$ and $-\Delta_u q(C) \ge q_C(u) > 0$. In this case, $-\Delta_u f(C) > 0$. If u is not adjacent with C, since G is connected, we may consider such u which is adjacent with a node $v \in (V \setminus C) \cap N(C)$. In this case, $-\Delta_v p(C) \ge 0$ and $-\Delta_v q(C) > 0$ (at least the covering requirement of u is reduced by 1), and thus $-\Delta_u f(v) > 0$. In any case, there is a node x with $-\Delta_x f(C) > 0$ and thus $S_x = x$ is a star with $g_C(S_x) = -\Delta_x f(C) > 0$, which implies that Algorithm 1 will not terminate. So, at the termination, C is an m-DS.

Next, we show that G[C] is connected. If not, then by Lemma 3.7, there exists one node u (or two adjacent nodes u, v) adding which can connect two components of G[C]. Such node u (or adjacent nodes u, v) can be viewed as a star S_u with $g_C(S_u) > 0$. Hence the algorithm will not terminate if G[C] is not connected.

Theorem 3.9. Let C^* be an optimal solution to a MinW(1,m)-CDS instance on graph G, and C be the output of Algorithm 1. Then $c(C) \leq 2H(\delta_{\max} + m - 1)c(C^*)$, where $H(\gamma) = \sum_{i=1}^{\gamma} 1/i$ is the γ th Harmonic number and δ_{\max} is the maximum degree of G.

Proof. Let S_1, S_2, \ldots, S_g be the stars chosen by Algorithm 1 in the order of their selection into set C. For $i = 1, 2, \ldots, g$, denote $C_i = S_1 \cup S_2 \cup \ldots \cup S_i$, and let $C_0 = \emptyset$. Furthermore, let $r_i = g_{C_{i-1}}(S_i)$ and $w_i = \frac{c(S_i)}{r_i}$. By Lemma 3.5, we have

$$r_i = g_{C_{i-1}}(S_i) = -\Delta_{S_i} f(C_{i-1}).$$
(17)

Suppose $|C^*| = t$ and T is a spanning tree of $G[C^*]$. Order nodes in C^* as u_1, \ldots, u_t such that a parent is ordered before its children, and brothers are ordered in non-decreasing order of costs. For $i = 1, 2, \ldots, t$, denote $C_i^* = \{u_1, \ldots, u_i\}$, and let $C_0^* = \emptyset$. Furthermore, let Y_i be the sub-star of T rooted at u_i . Then, T is divided into the union of stars $T = Y_1 \cup Y_2 \cup \cdots \cup Y_t$.

For $i \in \{1, ..., g\}$ and $j \in \{1, ..., t\}$, let $a_{i,j} = g_{C_i}(Y_j)$ and $w_{i,1} = ... = w_{i,r_i} = w_i$. For any

integer $1 \le \ell \le a_{0,j}$, denote $b_{j,\ell} = \frac{c(Y_j)}{\ell}$. **The idea for the following proof.** Let $A = \bigcup_{i=1}^{g} A_i$ with $A_i = \{w_{i,1}, \dots, w_{i,r_i}\}$, and $B = \{b_{1,1}, \dots, b_{1,a_{0,1}}, b_{2,1}, \dots, b_{2,a_{0,2}}, \dots, b_{t,1}, \dots, b_{t,a_{0,t}}\}$. If

there is an injective mapping $h: A \to B$ such that $w \le h(w)$ for any $w \in A$, (18)

then we shall have

$$c(C_g) = \sum_{i=1}^g c(S_i) = \sum_{w \in A} w \le \sum_{w \in A} h(w) \le \sum_{j=1}^t \sum_{\ell=1}^{a_{0,j}} b_{j,\ell} = \sum_{j=1}^t H(a_{0,j})c(Y_j),$$

where the second equality holds because $c(S_i) = w_i r_i = \sum_{w \in A_i} w$, and the second inequality holds because of "injection". Note that $a_{0,j} = g_{\emptyset}(Y_j)$, and $b_{\emptyset}^{Y_j}(v) = 0$ holds for any $v \in Y_j \setminus \{u_j\}$ (since $q_{\emptyset}(v) = m > 0$). So,

$$a_{0,j} = -\Delta_{u_j} f(\emptyset) = m + deg(u_j) - 1 \le m + \delta_{\max} - 1.$$
(19)

Combining this with the fact that every node of C^* appears in at most two Y_j 's, we have

$$c(C_g) \le H(\delta_{\max} + m - 1) \sum_{j=1}^t c(Y_j) \le 2H(\delta_{\max} + m - 1)c(C^*).$$

Constructing a mapping h satisfying (18). The construction is based on the following two claims.

Claim 1. For any $1 \le i \le g$, $\sum_{l=i}^{g} r_l \le \sum_{j=1}^{t} a_{i-1,j}$. Using (17) and the fact $f(C_g) = 1$, the left-hand side can be written as

$$\sum_{l=i}^{g} r_l = \sum_{l=i}^{g} (-\Delta_{S_l} f(C_{l-1})) = \sum_{l=i}^{g} (f(C_{l-1}) - f(C_l))$$
$$= f(C_{i-1}) - f(C_g) = f(C_{i-1}) - 1$$
$$= q(C_{i-1}) + p(C_{i-1}) - 1.$$
(20)

For each $j \in \{2, \ldots, t\}$, denote by $j^{(p)}$ the index for the parent of node u_j in tree T (superscript (p) indicates "parent"). Then the right-hand side can be written as

$$\sum_{j=1}^{t} a_{i-1,j} = \sum_{j=1}^{t} g_{C_{i-1}}(Y_j)$$

$$= \sum_{j=1}^{t} (-\Delta_{u_j} q(C_{i-1}) + |NC_{C_{i-1}}(u_j)| - 1 + \sum_{u \in V(Y_j) \setminus \{u_j\}} b_{C_{i-1}}^{Y_j}(u))$$

$$= \sum_{j=1}^{t} (-\Delta_{u_j} q(C_{i-1})) + \sum_{j=1}^{t} (|NC_{C_{i-1}}(u_j)|) - t + \sum_{j=2}^{t} b_{C_{i-1}}^{Y_j(p)}(u_j)$$
(21)

where the second equality uses expression (3). Let $X_1 = \{u_j \in C^* \setminus \{u_1\} : b_{C_{i-1}}^{Y_j(p)}(u_j) = 1\}, X_2 = \{u_j \in C^* \setminus \{u_1\} : NC_{C_{i-1}}(u_j) \cap NC_{C_{i-1}}(C_{j-1}^*) \neq \emptyset\}, X_3 = \{u_j \in C^* \setminus \{u_1\} : q_{C_{i-1}}(u_j) > 0\}.$ Observe that

$$C^* \setminus \{u_1\} \subseteq X_1 \cup X_2 \cup X_3.$$

$$(22)$$

In fact, for any node $u_j \in C^* \setminus \{u_1\}$, if $u_j \notin X_1 \cup X_3$, then $b_{C_{i-1}}^{Y_{j}(p)}(u_j) = 0$ and $q_{C_{i-1}}(u_j) = 0$. Note that $q_{C_{i-1}}(u_j) = 0$ implies that $NC_{C_{i-1}}(u_j) \neq \emptyset$. In order that $b_{C_{i-1}}^{Y_{j}(p)}(u_j) = 0$, we have $-\Delta_{u_j}f(C_{i-1} \cup prec(u_j)) = 0$. Then by Lemma 3.2, $-\Delta_{u_j}p(C_{i-1} \cup prec(u_j)) = 0$, which implies that any component in $NC_{C_{i-1}}(u_j)$ is adjacent with a node in $prec(u_j) \subseteq C_{j-1}^*$. Hence $u_j \in X_2$, relation (22) is proved.

As a consequence of (22), we have

$$|X_1| + |X_2| + |X_3| \ge t - 1.$$
(23)

Furthermore, by definition, we have

$$\sum_{j=2}^{t} b_{C_{i-1}}^{Y_j(p)}(u_j) = \mid X_1 \mid .$$
(24)

By the submodularity of -q, for any $u_j \in C^* \setminus \{u_1\}$,

$$-\Delta_{C_{j-1}^*}q(C_{i-1}) + (-\Delta_{u_j}q(C_{i-1})) \ge -\Delta_{C_j^*}q(C_{i-1}).$$
(25)

Furthermore, for any node $u_j \in X_3$, by Lemma 3.6 and because u_j is adjacent with $u_{j^{(p)}} \in C_{j-1}^*$, we have

$$-\Delta_{C_{j-1}^*}q(C_{i-1}) + (-\Delta_{u_j}q(C_{i-1})) \ge -\Delta_{C_j^*}q(C_{i-1}) + 1.$$
(26)

Inequalities (25) and (26) can be unified as

$$-\Delta_{C_{j-1}^*}q(C_{i-1}) + (-\Delta_{u_j}q(C_{i-1})) \ge -\Delta_{C_j^*}q(C_{i-1}) + \mathbf{1}_{u_j \in X_3}$$

where $\mathbf{1}_{u_j \in X_3}$ is the indicator of whether $u_j \in X_3$. Then,

$$\sum_{j=1}^{t} (-\Delta_{u_j} q(C_{i-1})) \ge \sum_{j=1}^{t} (-\Delta_{C_j^*} q(C_{i-1}) + \Delta_{C_{j-1}^*} q(C_{i-1})) + |X_3|$$

= $-\Delta_{C_t^*} q(C_{i-1}) + \Delta_{\emptyset} q(C_{i-1}) + |X_3|$
= $q(C_{i-1}) + |X_3|,$ (27)

where the last equality uses the fact $q(C_t^* \cup C_{i-1}) = 0$ and $-\Delta_{\emptyset} q(C_{i-1}) = 0$.

For the second item of expression (21), using the fact $|NC_{C_{i-1}}(C^*)| = p(C_{i-1})$ (since C^* dominates every node of C_{i-1}) and the fact $|NC_{C_{i-1}}(u_j) \cap NC_{C_{i-1}}(C^*_{j-1})| \ge 1$ for any $u_j \in X_2$ (by the definition of X_2), we have

$$\sum_{j=1}^{t} |NC_{C_{i-1}}(u_j)|$$

$$= \sum_{j=1}^{t} (|NC_{C_{i-1}}(u_j) \setminus NC_{C_{i-1}}(C_{j-1}^*)| + |NC_{C_{i-1}}(u_j) \cap NC_{C_{i-1}}(C_{j-1}^*)|)$$

$$\geq |NC_{C_{i-1}}(C^*)| + |X_2| = p(C_{i-1}) + |X_2|.$$
(28)

Combining inequalities (20), (23), (24), (27) and (28), Claim 1 is proved. Claim 2. $a_{i,j} \leq a_{0,j}$ for any $1 \leq i \leq g$ and $1 \leq j \leq t$. Let $N_1(u_j) = \{v \in N(u_j) : q_{C_i}(v) > 0\}, N_2(u_j) = N(u_j) \cap C_i$, and $N_3(u_j) = \{v \in Y_j : b_{C_i}^{Y_j}(v) = 1\}$. Then $-\Delta_{u_j}q(C_i) = q_{C_i}(u_j) + |N_1(u_j)|, |N_{C_i}(u_j)| \le |N_2(u_j)|$, and $\sum_{v \in V(Y_j) \setminus \{u_j\}} b_{C_i}^{Y_j}(v) = |N_3(u_j)|$. Notice that $N_1(u_j), N_2(u_j)$, and $N_3(u_j)$ are mutually disjoint. In fact, since $N_3(u_j) \subseteq Y_j \subseteq V \setminus C_i$, we have $N_2(u_j) \cap N_3(u_j) = \emptyset$. By the definition of $b_C^{S_u}(v)$ in (1), we have $N_1(u_j) \cap N_3(u_j) = \emptyset$. Since any node $v \in C_i$ has $q_{C_i}(v) = 0$, so $N_1(u_j) \cap N_2(u_j) = \emptyset$. Hence $|N_1(u_j)| + |N_2(u_j)| + |N_3(u_j)| \le deg(u_j)$. Then

$$\begin{aligned} a_{i,j} &= g_{C_i}(Y_j) = -\Delta_{u_j} f(C_i) + \sum_{v \in V(Y_j) \setminus \{u_j\}} b_{C_i}^{Y_j}(v) \\ &= -\Delta_{u_j} q(C_i) + (|NC_{C_i}(u_j)| - 1) + \sum_{v \in V(Y_j) \setminus \{u_j\}} b_{C_i}^{Y_j}(v) \\ &\leq (q_{C_i}(u_j) + |N_1(u_j)|) + (|N_2(u_j)| - 1) + |N_3(u_j)| \\ &\leq q_{C_i}(u_j) + deg(u_j) - 1 \\ &\leq m + deg(u_j) - 1 = a_{0,j}, \end{aligned}$$

where the last equality uses (19). Claim 2 is proved.

Finishing the construction of h satisfying (18): For $i \in \{1, \ldots, g\}$, let $B_i = \bigcup_{j=1}^t \{b_{j,1}, b_{j,2}, \ldots, b_{j,a_{i-1,j}}\}$. By Claim 2, every B_i is well defined and $B_i \subseteq B$. By the greedy choice of S_i , we have $w_i = \frac{c(S_i)}{r_i} \leq \frac{c(Y_j)}{a_{i-1,j}}$ ($\forall 1 \leq j \leq t$). So,

$$w_i \le b$$
 holds for any $b \in B_i$. (29)

Next, we show that there exists an injection h on A such that

$$h(A_i) \subseteq B_i \setminus \bigcup_{\ell=i+1}^g h(A_\ell) \text{ for any } i = g, g-1, \dots, 1,$$
(30)

This can be proved by induction on i from g down to 1. First, using Claim 1 for i = g, we have $|B_g| = \sum_{j=1}^t a_{g-1,j} \ge r_g = |A_g|$. So, an injection from A_g into B_g exists. Suppose we have established an injection h from $\bigcup_{x=i+1}^g A_x$ into $\bigcup_{x=i+1}^g B_x$ with $h(A_x) \subseteq B_x \setminus \bigcup_{\ell=x+1}^g h(A_\ell)$ for any $x \in \{i+1,\ldots,g\}$. By $|B_i \setminus \bigcup_{\ell=i+1}^g h(A_\ell) \ge \sum_{j=1}^t a_{i-1,j} - \sum_{\ell=i+1}^g |A_\ell| = \sum_{j=1}^t a_{i-1,j} - \sum_{\ell=i+1}^g r_\ell \ge r_i = |A_i|$, an injection from A_i into $B_i \setminus \bigcup_{\ell=i+1}^g f(A_\ell)$ exists. When i reaches 1, an injection h satisfying (30) is established. Combining (30) with (29), an injection h satisfying (18) is found, and the theorem is proved.

4 Conclusion and Discussion

CDSs were proposed by Das and Bhargharan [1] and Ephremides et al. [2] to serve as virtual backbones in WSNs. There exist many results on CDS in the literature.

In unweighted case, the MinCDS problem in a general graph has received a sequence of efforts [12, 13, 14]. The best approximation ratio is $(\ln \delta_{\max} + 2)$ in [13] or $(1+\varepsilon) \ln(\delta_{\max} - 1)$ in [14], where ε is an arbitrary positive real number. The MinCDS in UDG has polynomial-time approximation sheemes (PTASs) [15, 16]. For the fault-tolerant Min(k, m)CDS problem in general graphs, asymptotically tight approximations have been obtained for k = 1, 2, 3 and $m \ge k$ [17, 18, 19]. For general constants $m \ge k$, a $(2k - 1) \ln \delta_{\max}$ -approximation algorithm was proposed by Zhang et al. [20]. For the Min(k, m)CDS problem in UDGs, constant approximations have been

developed [21, 17, 22, 18]. As for the weighted version of fault-tolerant virtual backbones, Shi et al. [23] and Fukunaga [24] independently presented constant approximation algorithms for the MinW(k, m)CDS problem in UDGs, and Nutov [25] proposed an $O(k \ln n)$ -approximation algorithm for general graphs, where the constant in the big O is at least 10. A question is: can the constant in O be further reduced?

In weighted case, the MinWCDS problem in UDGs has several constant-approximations [4, 6, 3, 5, 26]. However, it is still open whether there exists a PTAS. More information can be found in [27, 28, 29]. For general graphs, progress on the MinWCDS problem is slow. In 1999, Guha and Khuller [8] proposed a $(1.35 + \varepsilon) \ln n$ -approximation algorithm. Until 2018, [9] presented an asymptotic $3 \ln \delta_{\text{max}}$ -approximation algorithm. However, the analysis in [9] contains a flaw. Actually, an inequality in their derivation contains a small error term 1. This small error accumulates to an uncontrollable error in the total weight. Existing methods seem unable to correct this flaw. This is the motivation for the current paper.

In this paper, we presented a $2H(\delta_{\max}+m-1)$ -approximation algorithm for the MinW(1, m)-CDS problem in a general graph. Unlike the algorithm in [9], ours is a one-phase greedy algorithm, where a most cost-effective star is selected in each iteration. The effectiveness of a star is measured by a delicately designed potential function.

There are two difficulties addressed. First, since the number of stars is exponential, identifying a most cost-effective star efficiently is challenging. We showed that under our potential function, a most cost-effective star has a special structure and thus can be found in polynomial time. Second, the potential function is not submodular, it eludes existing techniques used in submodular optimization. Although our previous works [17, 18, 19] successfully dealt with some cases of this problem for the *cardinality* version, those techniques cannot deal with the *weighted* version. A small error in the potential function makes the weight accumulate to an uncontrollable amount. In this paper, we proposed an amortized analysis, showing that although large errors are inevitable in some steps, they can be compensated overall. The crucial part is to establish an injective mapping from *fragments* of the computed solution to the fragments of the optimal solution so that such compensation is possible.

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