Aust. N. Z. J. Stat. **48**(1), 2006, 59–66 doi: 10.1111/j.1467-842X.2006.00425.x

A NEW CALIBRATION METHOD OF CONSTRUCTING EMPIRICAL LIKELIHOOD-BASED CONFIDENCE INTERVALS FOR THE TAIL INDEX

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Summary

Empirical likelihood has attracted much attention in the literature as a nonparametric method. A recent paper by Lu & Peng (2002)[Likelihood based confidence intervals for the tail index. *Extremes* **5,** 337–352] applied this method to construct a confidence interval for the tail index of a heavy-tailed distribution. It turns out that the empirical likelihood method, as well as other likelihood-based methods, performs better than the normal approximation method in terms of coverage probability. However, when the sample size is small, the confidence interval computed using the χ^2 approximation has a serious undercoverage problem. Motivated by Tsao (2004)[A new method of calibration for the empirical loglikelihood ratio. *Statist. Probab. Lett.* **68,** 305–314], this paper proposes a new method of calibration, which corrects the undercoverage problem.

Key words: coverage probability; empirical likelihood method; heavy tail; normal approximation.

1. Introduction

In many fields, such as meteorology, hydrology, climatology, environmental science, telecommunications, insurance and finance, one is faced with a few very large observations on which to base statistical analyses. For instance, in catastrophe insurance the insurance company is concerned with the occurrence of large claims which may lead to large fluctuations in cash-flow. For such data sets, heavy-tailed distributions are recommended to model the underlying distribution functions (e.g. Embrechts, Klüppelberg & Mikosch, 1997).

In recent years the problem of estimating the tail index of a heavy-tailed distribution has attracted much attention from statisticians. Various estimators have been proposed in the literature; see, for example, Hill (1975) and Hall (1982). For more references see Peng & Qi (2004). An important application of tail index estimation is to estimate the probabilities of those rare events beyond the data. This can be done by extrapolating from some intermediate order statistics via equation (1) defined in the next section.

To the best of our knowledge little attention has been paid to constructing confidence intervals for the tail index. Lu & Peng (2002) applied both the empirical likelihood method and the parametric likelihood method to obtain confidence intervals for the tail index of a heavy-tailed distribution, and compared their performance with the normal approximation method

Received September 2004; revised December 2004; accepted January 2005.

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Acknowledgments. Peng's research was supported by NSF grant DMS-04-03443. We thank the editor, an associate editor, a reviewer and Professor Richard Green for their helpful comments.

based on Hill's estimator (Hill, 1975). The simulation study in Lu & Peng (2002) indicates that the empirical likelihood method and the parametric likelihood method are comparable, and both result in better coverage accuracy than the normal approximation method.

The empirical likelihood was introduced by Owen (1988, 1990) for the mean vector for independent identically distributed (i.i.d.) observations, and it has been extended to a wide range of applications. The empirical likelihood method produces confidence regions whose shape and orientation are determined entirely by the data. It possesses some advantages over other methods like the normal approximation method. However, for a small sample size, the asymptotic χ^2 calibrated empirical likelihood-based confidence regions may have a lower coverage probability than the nominal level as indicated by numeric evidence in the literature; see, for example, Owen (1988), Hall & La Scala (1990), and Qin & Lawless (1994). The reason for the undercoverage is that the distribution of the empirical likelihood ratio has an atom at infinity, and the atom can be substantial if the sample size is not large (cf. Tsao, 2004). The same problem exists for the empirical likelihood-based confidence interval for the tail index since only a small proportion of upper-order statistics are employed in the inference. The simulation study in Lu & Peng (2002) exhibits that the coverage probabilities of the empirical likelihood-based confidence interval are below the nominal level in many cases when the sample fraction, k_n/n (to be defined in the next section), is small.

As a remedy for the empirical likelihood method, several alternative methods of calibration have been proposed, for example, the F-calibration (Owen, 2001), the bootstrap calibration (Owen, 2001) and the Bartlett correction (DiCiccio, Hall & Romano, 1991). Recently, Tsao (2004) proposed a new method of calibration for the empirical likelihood-based confidence region for means. This new confidence region is computed by approximating the quantiles of the empirical likelihood ratio by a so-called E-distribution. The E-distribution is defined as the distribution of the empirical likelihood ratio for a normal mean. This method is easy to implement and significantly improves the coverage probabilities for a small sample size. In particular, if the underlying distribution is normal, this new method gives a coverage probability exactly equal to the nominal level.

Motivated by the work of Tsao (2004), we propose a new calibration method for constructing confidence intervals for the tail index of a heavy-tailed distribution; see Section 2. A simulation study is given in section 3 to compare different methods.

2. Methods

Assume $X_1, \ldots, X_n$ are i.i.d. random variables with distribution function F satisfying

$$\lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} = x^{-1/\gamma} \tag{1}$$

for all $x > 0$, where $\gamma > 0$ is an unknown parameter and $1/\gamma$ is called the tail index of the distribution function F.

Let $X_{n,1} \le \cdots \le X_{n,n}$ denote the order statistics based on $X_1, \ldots, X_n$. The well known estimator for γ is the so-called Hill 's estimator defined as

$$\hat{\gamma}_n = \frac{1}{k_n} \sum_{i=1}^{k_n} \log X_{n,n-i+1} - \log X_{n,n-k_n},$$

where $k_n \to \infty$ and $\frac{k_n}{n} \to 0$ as $n \to \infty$ (see Hill, 1975). The condition on k_n means that only some of the upper order statistics can be used in estimating the parameter γ. The choice of k_n will be discussed in Section 3.

In order to make an inference about γ, a condition stronger than (1) is required. Throughout this paper, we assume that there exists a function $A(t) \to 0$ such that

$$\lim_{t\to\infty} \frac{U(tx)/U(t) - x^{\gamma}}{A(t)} = x^{\gamma}\frac{x^{\rho}-1}{\rho} \tag{2}$$

for all $x > 0$, where $U(x)$ is the inverse function of $\frac{1}{1-F(x)}$ and $\rho \le 0$. In addition, if

$$k_n \to \infty,\ \frac{k_n}{n} \to 0 \text{ and } \sqrt{k_n}A(n/k_n) \to 0 \quad \text{as } n \to \infty, \tag{3}$$

we have

$$\sqrt{k_n}(\hat{\gamma}_n - \gamma) \xrightarrow{d} N(0, \gamma^2) \tag{4}$$

(see e.g. De Haan & Peng, 1998) and

$$\sup_x |P\{\sqrt{k_n}(\hat{\gamma}_n/\gamma - 1) \le x\} - \Gamma_{k_n}(k_n + x\sqrt{k_n})| \to 0 \tag{5}$$

(see Cheng & de Haan, 2001), where Γ_{k_n} is a gamma (cumulative) distribution function with shape parameter k_n, that is,

$$\Gamma_{k_n}(t) = \int_0^t \frac{1}{\Gamma(k_n)} x^{k_n-1}e^{-x}dx \text{ for } t > 0.$$

Based on equations (4) and (5), $100(1-\alpha)\%$ confidence intervals are given by

$$I_N(1-\alpha) = \left(\frac{\hat{\gamma}_n}{1 + z_{\alpha/2}/\sqrt{k_n}}, \frac{\hat{\gamma}_n}{1 - z_{\alpha/2}/\sqrt{k_n}}\right)$$

and

$$I_{\Gamma}(1-\alpha) = (k_n\hat{\gamma}_n/\beta_{1-\alpha/2}, k_n\hat{\gamma}_n/\beta_{\alpha/2}),$$

where $z_{\alpha/2}$ denotes the upper $\alpha/2$-level critical value of the standard normal distribution and $\Gamma_{k_n}(\beta_{\alpha/2}) = \alpha/2$. Note that $I_N(1-\alpha) = \{\gamma : -z_{\alpha/2} < \sqrt{k_n}(\hat{\gamma}_n/\gamma - 1) < z_{\alpha/2}\}$ is different from that in Lu & Peng (2002).

By noting that $\{Y_i = i(\log X_{n,n-i+1} - \log X_{n,n-k_n}), i = 1, \ldots, k_n\}$ are asymptotically independent with a common exponential limiting distribution with mean γ for fixed k_n (see e.g. Weissman, 1978), Lu & Peng (2002) proposed the following empirical likelihood-based confidence interval

$$I_e(1-\alpha) = \{\gamma : l(\gamma) < \chi^2_{\alpha}\},$$

where χ^2_α is the upper α-level critical value of a χ^2_1 distribution,

$$l(\gamma) = 2\sum_{i=1}^{k_n} \log(1 + \lambda(Y_i - \gamma)), \tag{6}$$

and λ is determined by

$$\sum_{i=1}^{k_n} \frac{Y_i - \gamma}{1 + \lambda(Y_i - \gamma)} = 0. \tag{7}$$

They proved under conditions (2) and (3) that

$$l(\gamma) \xrightarrow{d} \chi^2_1.$$

Since Y_i/γ in $l(\gamma)$ has an approximate exponential distribution with mean one, we replace $Y_1/\gamma, \ldots, Y_{k_n}/\gamma$ by i.i.d. random variables $E_1, \ldots, E_{k_n}$ with an exponential distribution with mean one. Hence we obtain

$$\mathrm{ELR}(k_n) = 2\sum_{i=1}^{k_n} \log(1 + \lambda'(E_i - 1)),$$

where λ' is the solution to

$$\sum_{i=1}^{k_n} \frac{E_i - 1}{1 + \lambda'(E_i - 1)} = 0.$$

Motivated by Tsao (2004), we approximate the distribution of $l(\gamma)$ by the distribution of ELR(k_n) instead of χ^2_1. This results in the following confidence interval with nominal level $100(1 - \alpha)\%$:

$$I_{\mathrm{new}}(1 - \alpha) = \{\gamma : l(\gamma) < c(k_n, \alpha)\},$$

where $c(k_n, \alpha)$ is the upper α-level critical value of the distribution of ELR(k_n). Note that our definition of ELR(k_n) is different from that in Tsao (2004), where the E_i are taken as standard normal random variables.

Although the exact distribution of ELR(k_n) is hard to calculate, the critical value $c(k_n, \alpha)$ can easily be obtained via Monte Carlo simulation. We have obtained the critical values $c(k_n, \alpha)$ for $\alpha = 10\%, 5\%$ and 1% for all k_n between 10 and 200 based on 1 000 000 random samples. Using the critical values, we compare the above four confidence intervals in the next section.

The critical values $c(k_n, \alpha)$ for $10 \le k_n \le 29$ are listed in Table 1. Interested readers can find the complete table of the critical values in the technical report by Peng & Qi (2005). For $30 \le k_n \le 200$ we fit three linear regression equations as suggested by an Associate Editor. It turns out that the critical values $c(k_n, \alpha)$ we obtained from the simulation are very well approximated by the three regression equations given by

$$c(k, 0.10) = 2.7055 - \frac{0.51269}{\sqrt{k}} + \frac{18.14242}{k},$$

TABLE 1

α-level critical value of the empirical likelihood ratio: ELR(k).

k	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.01$	k	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.01$
10	4.7345	8.1886	∞	20	3.5485	5.3850	11.6495
11	4.4733	7.5047	22.6407	21	3.5085	5.3125	11.2854
12	4.2872	7.0245	19.6575	22	3.4598	5.2159	10.9427
13	4.1499	6.6784	17.9019	23	3.4231	5.1378	10.6890
14	4.0188	6.3691	16.1623	24	3.3868	5.0765	10.3364
15	3.9079	6.1526	15.1273	25	3.3634	5.0220	10.1273
16	3.8185	5.9443	14.0525	26	3.3313	4.9464	9.8952
17	3.7477	5.8020	13.4480	27	3.2988	4.8939	9.7131
18	3.6691	5.6450	12.8207	28	3.2710	4.8531	9.5828
19	3.6006	5.5101	12.1357	29	3.2576	4.8030	9.4580

$$c(k, 0.05) = 3.8415 - \frac{1.12486}{\sqrt{k}} + \frac{32.90613}{k}$$

and

$$c(k, 0.01) = 6.6349 - \frac{4.56941}{\sqrt{k}} + \frac{98.98899}{k}.$$

The three intercepts in the equations are the 10%, 5% and 1% critical values of the χ_1^2 distribution, and the corresponding R^2 values and root MSEs are 0.999, 0.999, 0.9975, and 0.006, 0.01, 0.04, respectively.

3. Simulation study

In this section we compare the four methods described in Section 2 in terms of both coverage probability and interval length by employing the following two cumulative distribution functions (cdfs): (i) the Fréchet cdf given by $F(x) = \exp(-x^{-\alpha_0})$ $(x > 0)$, where $\alpha_0 > 0$ (notation: Fréchet(α_0)); and (ii) the Burr cdf given by $F(x) = 1 - (1 + x^{\alpha_0})^{-\beta_0}$ $(x > 0)$, where $\alpha_0 > 0$, $\beta_0 > 0$ (notation: Burr(α_0, β_0)).

First we drew 10 000 random samples of sample size $n = 500$, 1000 and 2000 from the Fréchet(1), Burr(0.5,1), Burr(1,0.5) and Burr(1,1) distributions, and then computed the coverage probabilities for $I_N(0.95)$, $I_\Gamma(0.95)$, $I_e(0.95)$ and $I_{\text{new}}(0.95)$ for $k = 10, 15, \ldots, 200$. Note that these four methods are independent of α_0 for Fréchet(α_0). Here we report the results for Fréchet(1) and Burr (0.5,1) with $n = 1000$ in Table 2.

Second, we drew 1000 random samples of size $n = 1000$ from the Fréchet(1) and Burr(0.5,1) distributions, and then computed the average lengths of the intervals $I_N(0.95)$, $I_\Gamma(0.95)$, $I_e(0.95)$ and $I_{\text{new}}(0.95)$ for $k = 10, 15, \ldots, 200$; see Table 3.

Our observations from Tables 2 and 3 are as follows. First, the new calibration method gives the best coverage accuracy among the four methods, especially for larger values of k_n. Second, the two empirical likelihood-based methods generate shorter confidence intervals in general. Third, the new calibration method greatly improves on the asymptotic χ^2 calibrated empirical likelihood method given in Lu & Peng (2002) in that it has a much more accurate coverage probability for smaller values of k_n, and has a comparable length of confidence

TABLE 2

Coverage probabilities of the four confidence intervals.

k	Fréchet(1) $I_{\text{new}}(0.95)$	Fréchet(1) $I_e(0.95)$	Fréchet(1) $I_N(0.95)$	Fréchet(1) $I_\Gamma(0.95)$	Burr(0.5,1) $I_{\text{new}}(0.95)$	Burr(0.5,1) $I_e(0.95)$	Burr(0.5,1) $I_N(0.95)$	Burr(0.5,1) $I_\Gamma(0.95)$
10	0.9447	0.8723	0.9561	0.9510	0.9453	0.8731	0.9556	0.9507
15	0.9533	0.8998	0.9550	0.9518	0.9546	0.9015	0.9544	0.9512
20	0.9520	0.9150	0.9521	0.9536	0.9534	0.9162	0.9503	0.9529
25	0.9545	0.9213	0.9521	0.9506	0.9561	0.9234	0.9511	0.9514
30	0.9541	0.9309	0.9533	0.9525	0.9553	0.9330	0.9497	0.9518
35	0.9558	0.9326	0.9506	0.9522	0.9567	0.9338	0.9492	0.9519
40	0.9562	0.9391	0.9528	0.9535	0.9570	0.9381	0.9491	0.9529
45	0.9537	0.9386	0.9490	0.9525	0.9545	0.9390	0.9450	0.9489
50	0.9560	0.9432	0.9505	0.9528	0.9558	0.9426	0.9464	0.9502
55	0.9567	0.9467	0.9523	0.9541	0.9565	0.9447	0.9454	0.9513
60	0.9543	0.9435	0.9495	0.9504	0.9548	0.9434	0.9400	0.9478
65	0.9513	0.9424	0.9479	0.9486	0.9491	0.9405	0.9400	0.9456
70	0.9494	0.9394	0.9438	0.9467	0.9459	0.9366	0.9331	0.9398
75	0.9478	0.9398	0.9433	0.9457	0.9443	0.9354	0.9293	0.9358
80	0.9473	0.9384	0.9381	0.9420	0.9389	0.9307	0.9246	0.9317
85	0.9487	0.9409	0.9406	0.9454	0.9356	0.9278	0.9189	0.9291
90	0.9452	0.9381	0.9362	0.9402	0.9319	0.9216	0.9157	0.9239
95	0.9448	0.9365	0.9351	0.9407	0.9268	0.9173	0.9076	0.9189
100	0.9439	0.9382	0.9333	0.9392	0.9223	0.9146	0.9037	0.9143
105	0.9436	0.9356	0.9311	0.9392	0.9181	0.9094	0.8974	0.9076
110	0.9431	0.9361	0.9331	0.9379	0.9122	0.9044	0.8910	0.9049
115	0.9406	0.9340	0.9320	0.9370	0.9074	0.8994	0.8850	0.9001
120	0.9402	0.9345	0.9283	0.9360	0.9043	0.8954	0.8805	0.8932
125	0.9402	0.9333	0.9285	0.9336	0.8965	0.8891	0.8722	0.8866
130	0.9367	0.9331	0.9263	0.9331	0.8887	0.8814	0.8631	0.8807
135	0.9362	0.9306	0.9233	0.9296	0.8792	0.8722	0.8518	0.8682
140	0.9343	0.9290	0.9211	0.9305	0.8702	0.8613	0.8397	0.8549
145	0.9320	0.9273	0.9196	0.9247	0.8603	0.8511	0.8296	0.8463
150	0.9311	0.9255	0.9157	0.9252	0.8491	0.8398	0.8156	0.8323
155	0.9275	0.9223	0.9119	0.9209	0.8344	0.8262	0.7996	0.8198
160	0.9237	0.9186	0.9081	0.9174	0.8200	0.8105	0.7829	0.8039
165	0.9191	0.9150	0.9054	0.9124	0.8015	0.7921	0.7653	0.7842
170	0.9138	0.9091	0.8962	0.9068	0.7875	0.7786	0.7511	0.7719
175	0.9115	0.9066	0.8918	0.9025	0.7711	0.7635	0.7346	0.7550
180	0.9070	0.9010	0.8856	0.8969	0.7578	0.7509	0.7186	0.7390
185	0.9043	0.9001	0.8845	0.8955	0.7360	0.7288	0.7009	0.7203
190	0.8982	0.8928	0.8790	0.8903	0.7165	0.7091	0.6759	0.6957
195	0.8941	0.8889	0.8723	0.8846	0.6963	0.6882	0.6547	0.6767
200	0.8905	0.8855	0.8692	0.8811	0.6742	0.6657	0.6335	0.6546

interval for larger values of k_n. Since our new calibration method uses larger critical values than that from the asymptotic χ^2 calibrated empirical likelihood method, it generates a wider confidence interval for smaller k_n, but compared with the gain in the coverage accuracy, it is a worthwhile method. To see why our new method works much better than the asymptotic χ^2 calibrated empirical likelihood method one can calculate the size of the atom at infinity for the empirical likelihood ratio $l(\gamma)$ defined in (6). This size is actually equal to the probability of no solution to equation (7) or equivalently of all $(Y_i - \gamma)$s having the same signs. Therefore this probability is close to $e^{-k_n} + (1 - e^{-1})^{k_n}$ when k_n is small. When $k_n = 10$, this probability is as large as 1%, and the actual 5% critical value for $l(\gamma)$ is close to 8.19, much larger than 3.84, the 5% critical value from the χ_1^2 distribution.

TABLE 3

Average lengths of the four confidence intervals.

k	Fréchet(1) $I_{\text{new}}(0.95)$	Fréchet(1) $I_e(0.95)$	Fréchet(1) $I_N(0.95)$	Fréchet(1) $I_\Gamma(0.95)$	Burr(0.5,1) $[I_{\text{new}}(0.95)$	Burr(0.5,1) $I_e(0.95)$	Burr(0.5,1) $I_N(0.95)$	Burr(0.5,1) $I_\Gamma(0.95)$
10	1.53	1.03	2.04	1.52	3.09	2.06	4.10	3.05
15	1.19	0.89	1.39	1.17	2.39	1.78	2.79	2.35
20	0.99	0.81	1.11	0.98	1.99	1.62	2.23	1.98
25	0.86	0.73	0.94	0.86	1.74	1.49	1.90	1.73
30	0.76	0.67	0.83	0.77	1.55	1.36	1.68	1.56
35	0.70	0.62	0.75	0.71	1.41	1.27	1.52	1.43
40	0.65	0.59	0.69	0.66	1.31	1.20	1.40	1.33
45	0.61	0.56	0.65	0.62	1.24	1.14	1.31	1.25
50	0.57	0.53	0.61	0.58	1.16	1.08	1.23	1.18
55	0.54	0.51	0.58	0.55	1.10	1.03	1.17	1.12
60	0.52	0.49	0.55	0.53	1.06	0.99	1.11	1.07
65	0.50	0.47	0.52	0.51	1.01	0.96	1.07	1.03
70	0.47	0.45	0.50	0.49	0.97	0.92	1.02	0.99
75	0.46	0.43	0.48	0.47	0.93	0.89	0.99	0.96
80	0.44	0.42	0.47	0.46	0.90	0.86	0.96	0.93
85	0.43	0.41	0.45	0.44	0.88	0.84	0.93	0.91
90	0.42	0.40	0.44	0.43	0.85	0.82	0.90	0.88
95	0.41	0.39	0.43	0.42	0.83	0.80	0.88	0.86
100	0.40	0.38	0.42	0.41	0.81	0.78	0.86	0.84
105	0.39	0.38	0.41	0.40	0.79	0.76	0.84	0.82
110	0.38	0.37	0.40	0.39	0.77	0.75	0.82	0.80
115	0.37	0.36	0.39	0.38	0.75	0.73	0.80	0.78
120	0.36	0.35	0.38	0.37	0.73	0.71	0.79	0.77
125	0.36	0.35	0.37	0.37	0.72	0.70	0.77	0.76
130	0.35	0.34	0.37	0.36	0.70	0.68	0.76	0.75
135	0.34	0.33	0.36	0.35	0.68	0.66	0.75	0.73
140	0.34	0.33	0.35	0.35	0.66	0.65	0.73	0.72
145	0.33	0.32	0.35	0.34	0.65	0.63	0.72	0.71
150	0.33	0.32	0.34	0.34	0.63	0.61	0.71	0.70
155	0.32	0.31	0.34	0.33	0.61	0.59	0.70	0.69
160	0.31	0.31	0.33	0.33	0.60	0.58	0.69	0.68
165	0.31	0.30	0.33	0.32	0.59	0.57	0.68	0.67
170	0.30	0.30	0.32	0.32	0.57	0.55	0.67	0.67
175	0.30	0.29	0.32	0.31	0.55	0.54	0.67	0.66
180	0.29	0.29	0.31	0.31	0.54	0.53	0.66	0.65
185	0.29	0.28	0.31	0.31	0.53	0.52	0.65	0.64
190	0.29	0.28	0.31	0.30	0.52	0.51	0.64	0.64
195	0.28	0.28	0.30	0.30	0.51	0.49	0.64	0.63
200	0.28	0.27	0.30	0.30	0.48	0.48	0.63	0.62

We observe that the lengths of the confidence intervals from all four methods get smaller and smaller with increasing k_n. This does not make sense when k_n is too large since the coverage probabilities are far below the nominal levels. Condition (3) is imposed to ensure that the coverage probabilities are asymptotically correct. For example, for Fréchet(1), we can verify that $A(t) = (2t)^{-1}$. Then we can choose $k_n = o(n^{2/3})$. If we take $n = 1000$, the range for k_n is actually very limited since $n^{2/3}$ is only 100. If k_n goes beyond this point, a bias appears inevitably, and it leads to undercoverage. As we have observed, the new calibration method is more stable than the others. This is very important since the choice of k_n is a very tough question, both theoretically and practically. For example, Cheng & de Haan (2001) pointed out that the optimal k_n, in terms of coverage probability, depends on the third order regular variation index for a two-sided confidence interval based on the normal approximation method. The second order parameter can be estimated (e.g. Peng & Qi, 2004) but, as far as we know, these estimators are very unstable. In other words, seeking the optimal k_n is quite difficult in practice. Therefore, it is good to have some comparable methods, and to prefer methods which are stable against the choice of k_n.

References

CHENG, S. & DE HAAN, L. (2001). Penultimate approximation for Hill 's Estimator. *Scand. J. Statist.* **28**, 569–575.

CHENG, S. & PENG, L. (2001). Confidence intervals for the tail index. *Bernoulli* **7**, 751–760.

DE HAAN, L. & PENG, L. (1998). Comparison of tail index estimators. *Statistica Neerlandica* **52**, 60–70.

DICICCIO, T.J., HALL, P. & ROMANO, J.P. (1991). Empirical likelihood is Bartlett correctable. *Ann. Statist.* **19**, 1053–1061.

EMBRECHTS, P., KLÜPPELBERG, C. & MIKOSCH, T. (1997). *Modelling Extremal Events for Insurance and Finance*. Berlin: Springer.

HALL, P. (1982). On some simple estimators of an exponent of regular variation. *J. Roy. Statist. Soc. Ser. B* **44**, 37–42.

HALL, P. & LA SCALA, L. (1990). Methodology and algorithms of empirical likelihood. *Internat. Statist. Rev.* **58**, 109–127.

HILL, B.M. (1975). A simple general approach to inference about the tail of a distribution. *Ann. Statist.* **3**, 1163–1174.

LU, J. & PENG, L. (2002). Likelihood based confidence intervals for the tail index. *Extremes* **5**, 337–352.

OWEN, A.B. (1988). Empirical likelihood ratio confidence intervals for a single functional. *Biometrika* **75**, 237–249.

OWEN, A.B. (1990). Empirical likelihood ratio confidence regions. *Ann. Statist.* **18**, 90–120.

OWEN, A.B. (2001). *Empirical Likelihood*. New York: Chapman & Hall/CRC.

PENG, L. & QI, Y. (2004). Estimating the first- and second-order parameters of a heavy-tailed distribution. *Aust. N.Z. J. Stat.* **46**, 305–312.

PENG, L. & QI, Y. (2005). *A new calibration method of constructing empirical likelihood-based confidence intervals for the tail index*. Technical Report TR 2005. Department of Mathematics and Statistics, University of Minnesota Duluth.

QIN, J. & LAWLESS, J. (1994). Empirical likelihood and general estimating equations. *Ann. Statist.* **22**, 300–325.

TSAO, M. (2004). A new method of calibration for the empirical loglikelihood ratio. *Statist. Probab. Lett.* **68**, 305–314.

WEISSMAN, I. (1978). Estimation of parameters and large quantiles based on the k largest observations. *J. Amer. Statist. Assoc.* **73**, 812–815.