Int. Journal of Math. Analysis, Vol. 2, 2008, no. 5, 235 - 253

A New Class of Bivariate Gompertz Distributions and its Mixture[1]

A. Al-Khedhairi and A. El-Gohary[2]

Department of Statistics and O.R.
College of Science, King Saud University
P.O. Box 2455 Riyadh 11451, Saudi Arabia
elgohary0@yahoo.com

Abstract

A new class of bivariate Gompertz distributions is presented in this paper. The model introduced here is of Marshall-Olkin type. The used procedure is based on a latent random variable with exponential distribution. A mixture of the suggested bivariate distributions is also derived. The obtained results in this paper generalize those of Marshall-Olkin bivariate exponential distribution and other present in the literature.

Keywords: Bivariate Gompertz distribution, Moment generating function, Mixture distributions

1 Introduction

The Gompertz distribution plays an important role in modelling human mortality and fitting actuarial tables. This distribution was first introduced by Gompertz [1]. It has been used as a growth model and also used to fit the tumor growth. The Gompretz distribution is related by a simple transformation to certain distribution in the family of distributions obtained by Pearson. Applications and more recent survey of the Gompertz distribution can be found in [2].

[1]This research was supported by the College of Science Research Center at King Saud University under project No. Stat/2008/17.

[2]Permanent address: Department of Mathematics, College of Science, Mansoura University, Mansoura 35516, Egypt.

In many practical situation, multivariate lifetime data arise frequently, and in these situations it is important to consider different multivariate models that could be used to model such multivariate lifetime data. The model introduced in this paper is of some interest, in reliability theory, for example, sometimes failure rate can occur for more than one reason and a mixture distribution is nice tool for modelling such situation.

In fact, shock models are used in reliability to describe different applications. Shocks can refer for example to damage caused to biological organs by illness or environmental causes of damage acting on a technical system,see for example El-Gohary [4, 5, 6], and A-hameed and Proschan [7]. Also El-Gohary and Al-Ruzaiza have obtained a new class of bivariate distribution with pareto of Marshall-Olkin type [3].

The objective of this paper is to introduce a new class of bivariate Gompertz distributions of Marsall-Olkin type. It is considered as a distribution of the life times of two dependent components each has a Gompertz distribution. Also the mixture of the proposed Gompertz distributions will be derived.

The paper is organized as follows. Section *2* presents the shock model yielding the bivariate Gompertz distribution. The joint survival and probability density function of bivariate proposed Gompertz distribution is derived. Section *3* presents the joint moment generating function of this bivariate distribution and its marginal moment generating functions. Section *4* discusses the mixture of proposed bivariate Gompertz distributions and its moment generating function.

2 The new class of bivariate Gompertz distributions

In this section, we define a new class of bivariate Gompertz distribution using shock models. We start with the joint survival function of the proposed bivariate distribution and so used it to derive the corresponding joint probability density function. The marginal probability density functions and conditional probability density functions of this distribution are also derived. Finally this contains mathematical expectation of this distribution.

2.1 The joint survival function

Assume that there exists a three independent sources of shocks are presented in the environment of a system consists of two components [7]. A shock from source *1* destroys the component *1*; it occurs at a random time T_1. A shock from source *2* destroys component *2*; it occurs at a random time T_2. A shock from source *3* destroys both the components; it occurs at a random time T_3. Thus the random lifetime of the component *1*, say X_1, satisfies $X_1 = \min(T_1, T_3)$. While the random lifetime of component *2*, say X_2, satisfies $X_2 = \min(T_2, T_3)$ [9].

Let us assume that the random variables T_1 and T_2 having Gompertz distribution with parameters α_1, β_1 and α_2, β_2 respectively while the random variable T_3 has an exponential distribution with parameter θ, shortly we say $\mathrm{Gomp}(\alpha_i, \beta_i)$, $i = 1, 2$ and T_3 has $\mathrm{Exp}(\theta)$ [3].

That is, the probability density function of the random lifetime T_i, $i = 1, 2$, takes the following form

$$g_i(t) = \alpha_i\, e^{\beta_i t} \bar{G}_i(t),\ t \geq 0,\ \alpha_i > 0,\ \beta_i > 0,\ (i = 1, 2) \tag{2.1}$$

where $\bar{G}_i(t)$ is the survival function of T_i, $i = 1, 2$, which is given by

$$\bar{G}_i(t) = \exp\Big\{ - k_i \left(e^{\beta_i t} - 1\right) \Big\},\ \alpha_i = k_i \beta_i,\ (i = 1, 2) \tag{2.2}$$

The probability density function of T_3 takes the following form

$$g_3(t) = \theta \bar{G}_3(t),\ t \geq 0,\ \theta > 0. \tag{2.3}$$

where $\bar{G}_3(t)$ is survival function of T_3 which given by

$$\bar{G}_3(t) = e^{-\theta t}. \tag{2.4}$$

Obviously, the random variables X_1 and X_2 are dependent because of the common source of shock 3.

Now we proceed to investigate the joint survival function of the random variables X_1 and X_2. The following lemma presents the joint survival function $\bar{F}_{X_1,X_2}(x_1, x_2)$ of these variables.

Lemma 2.1 *The joint survival function of* X_1 *and* X_2 *is*

$$\bar{F}_{X_1,X_2}(x_1, x_2) = \bar{G}_3(z) \prod_{i=1}^{2} \bar{G}_i(x_i) = \exp\left[-k_1 \Big(e^{\beta_1 x_1} - 1\Big) - k_2 \Big(e^{\beta_2 x_2} - 1\Big) - \theta z \right] \tag{2.5}$$

where $z = \max(x_1, x_2)$.

Proof. Since the joint survival function of X_1 and X_2 is defined as

$$\bar{F}_{X_1,X_2}(x_1,x_2) = P\Big(X_1 > x_1,\, X_2 > x_2\Big)$$

Then

$$\begin{aligned}
\bar{F}_{X_1,X_2}(x_1,x_2) &= P\Big(\Big\{\min(T_1,\, T_3) > x_1\Big\},\, \Big\{\min(T_2,\, T_3) > x_2\Big\}\Big) \\
&= P\Big(\Big\{T_1 > x_1,\, T_3 > x_2\Big\},\, \Big\{T_2 > x_1,\, T_3 > x_2\Big\}\Big) \\
&= P\Big(T_1 > x_1,\, T_2 > x_2,\, T_3 > \max(x_1,\, x_2)\Big)
\end{aligned}$$

As the random variables $T_j, (j = 1, 2, 3)$ are mutually independent, we directly obtain

$$\begin{aligned}
\bar{F}_{X_1,X_2}(x_1,x_2) &= P(T_1 > x_1)\, P(T_2 > x_2)\, P(T_3 > \max(x_1,\, x_2)) \\
&= \bar{G}_1(x_1)\, \bar{G}_2(x_2)\, \bar{G}_3(z), \quad z = \max(x_1, x_2)
\end{aligned}$$

Substituting from (2.2) and (2.4) into the above relation, we can reach the form (2.5) that completes the proof. □

The following Corollary gives the marginal survival functions of the random variable X_1 and X_2.

Corollary 2.1 The marginal survival functions of bivariate Gompertz distribution are given by

$$\bar{F}_{X_i}(x_i) = \bar{G}_i(x_i)\bar{G}_3(x_i) = \exp\left[-k_i(e^{\beta_i x_i} - 1) - \theta x_i\right],\ i = 1, 2 \tag{2.6}$$

Proof. The proof of this Corollary can be done in a similar manner as in the proof of Lemma 2.1.

The following Corollary presents the joint distribution of X_1 and X_2.

Corollary 2.2 The joint distribution function $F_{X_1,X_2}(x_1,x_2)$ of X_1 and X_2 is given by

$$\begin{aligned}
F_{X_1,X_2}(x_1,x_2) &= 1 - \bar{G}_1(x_1)\bar{G}_3(x_1) - \bar{G}_2(x_2)\bar{G}_3(x_2) + \bar{G}_3(z)\prod_{i=1}^{2}\bar{G}_i(x_i) \\
&= 1 - \exp\left[-k_1(e^{\beta_1 x_1} - 1) - \theta x_1\right] - \exp\left[-k_2(e^{\beta_2 x_2} - 1) - \theta x_2\right] + \\
&\quad \exp\left[-k_1(e^{\beta_1 x_1} - 1) - k_2(e^{\beta_2 x_2} - 1) - \theta z\right]
\end{aligned} \tag{2.7}$$

Proof. The proof of this Corollary can be reached by using (2.5) and (2.6) with the help of the following relation:

$$F_{X_1,X_2}(x_1,x_2) = 1 - \bar{F}_{X_1}(x_1) - \bar{F}_{X_2}(x_2) + \bar{F}_{X_1,X_2}(x_1,\, x_2).$$

We can easily observe that the joint survival function of Marshall-Olkin bivariate exponential distribution [8, 9, 10] can be obtained by setting β_i, $(i = 1, 2)$ tend to zero.

2.2 The joint probability density function

The following theorem provides the joint probability density function of the new bivariate Gompertz distribution.

Theorem 2.1 *If the joint survival function of X_1 and X_2 is given by*

$$\bar{F}_{X_1,\,X_2}(x_1, x_2) = \exp\left[-k_1\left(e^{\beta_1 x_1} - 1\right) - k_2\left(e^{\beta_2 x_2} - 1\right) - \theta z\right] \tag{2.8}$$

Then, then joint probability density function $f_{X_1,X_2}(x_1, x_2)$ of X_1 and X_2 takes the form

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} f_1(x_1, x_2) & \text{if } x_1 > x_2 > 0, \\ f_2(x_1, x_2) & \text{if } x_2 > x_1 > 0, \\ f_0(x, x) & \text{if } x_1 = x_2 = x > 0 \end{cases} \tag{2.9}$$

where

$$f_1(x_1, x_2) = k_2\beta_2 e^{\beta_2 x_2}(\theta + k_1\beta_1 e^{\beta_1 x_1}) \exp\left[-k_1\left(e^{\beta_1 x_1} - 1\right) - k_2\left(e^{\beta_2 x_2} - 1\right) - \theta x_1\right]$$

$$f_2(x_1, x_2) = k_1\beta_1 e^{\beta_1 x_1}(\theta + k_2\beta_2 e^{\beta_2 x_2}) \exp\left[-k_1\left(e^{\beta_1 x_1} - 1\right) - k_2\left(e^{\beta_2 x_2} - 1\right) - \theta x_2\right]$$

$$f_0(x, x) = \theta \exp\left[-k_1\left(e^{\beta_1 x} - 1\right) - k_2\left(e^{\beta_2 x} - 1\right) - \theta x\right] \tag{2.10}$$

Proof. The proof of this Theorem is based on the obtaining forms of $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ by differentiating the joint survival function $\bar{F}_{X_1,X_2}(x, y)$ with respect to x_1 and x_2, that is

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} \frac{\partial^2 \bar{F}(x_1,x_2)}{\partial x_1 \partial x_2} & \text{if } x_1 > x_2 > 0, \\ \\ \frac{\partial^2 \bar{F}(x_1,x_2)}{\partial x_1 \partial x_2} & \text{if } x_2 > x_1 > 0, \end{cases}$$

But $f_0(x, x)$ can not be derived in a similar method. Instead we use the following identity to derive $f_0(x, x)$.

$$\int_0^\infty \int_0^{x_1} f_1(x_1, x_2)dx_1dx_2 + \int_0^\infty \int_0^{x_2} f_2(x_1, x_2)dx_1dx_2 + \int_0^\infty f_0(x, x)dx = 1 \tag{2.11}$$

One can find out that

$$\int_0^\infty \int_0^{x_1} f_1(x_1, x_2)dx_2dx_1 = 1 - \int_0^\infty \left(\theta + k_1\beta_1 e^{\beta_1 x_1}\right)\bar{F}_{X_1}(x_1)dx_1 \tag{2.12}$$

and

$$\int_0^\infty \int_0^{x_2} f_2(x_1, x_2)dx_1dx_2 = 1 - \int_0^\infty \left(\theta + k_2\beta_2 e^{\beta_2 x_2}\right)\bar{F}_{X_2}(x_2)dx_2 \tag{2.13}$$

Substituting from (2.10) and (2.11) into (2.9) we obtain

$$\int_0^\infty f_0(x,x)dx = \theta \int_0^\infty \prod_{i=1}^{3} \bar{G}_i(x)\, dx$$

Thus, $f_0(x,x)$ is

$$f_0(x,x) = \theta \exp\Big[-k_1\Big(e^{\beta_1 x}-1\Big) - k_2\beta_2\Big(e^{\beta_2 x}-1\Big) - \theta x\Big],\ x>0.$$

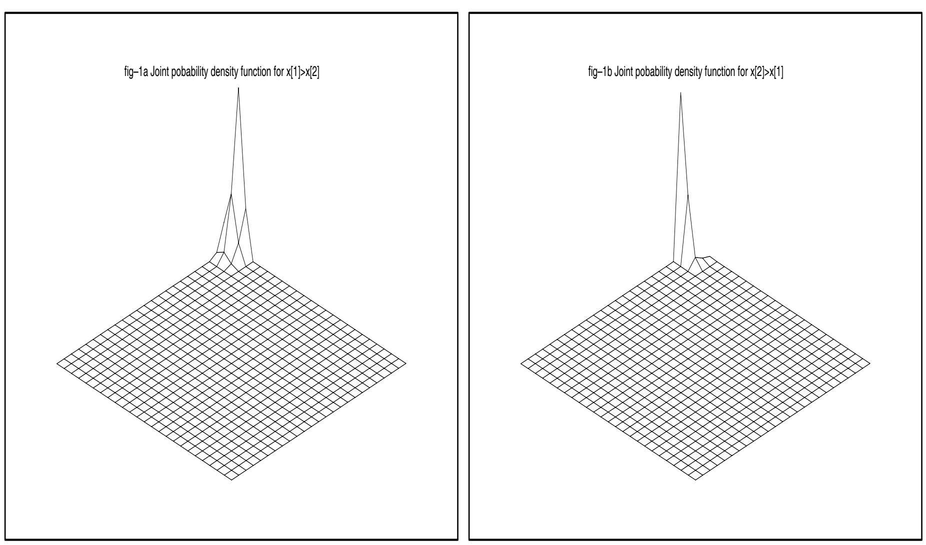
which completes the proof. □

This class of bivariate distributions generalizes the bivariate distribution of Marshall-Olkin [8]. This result shows that the results of this paper generalize those of Marshall-Olkin bivariate exponential distribution. Further the survival function of bivariate Weibull distributions can be derived using the nonlinear transformation $X_i = \ln(1+Y_i^{\beta_i)}/\beta_i$, $(i=1,2)$ [12]. That is

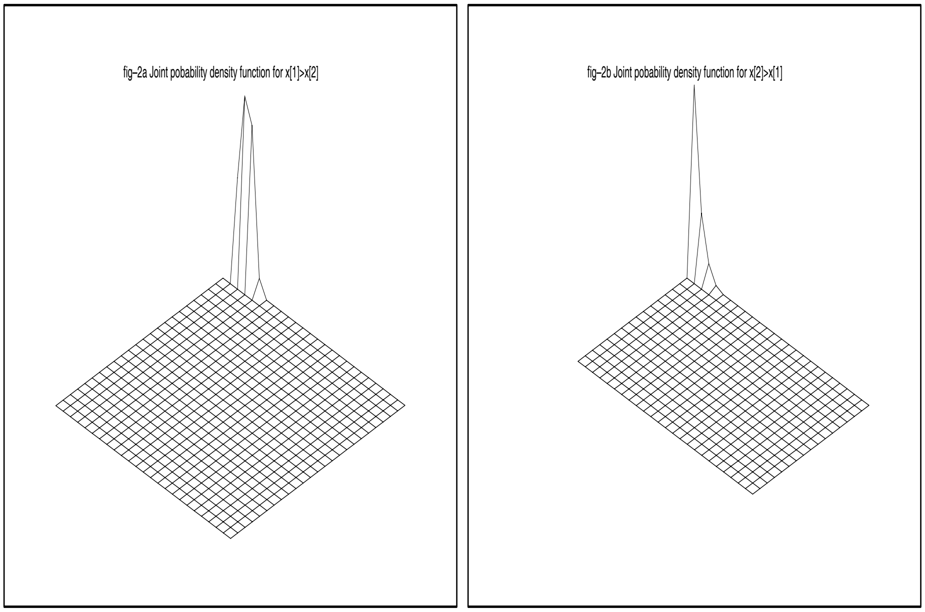
$$\bar{F}(y_1,y_2) = e^{-(k_1 y_1^{\beta_1} + k_1 y_2^{\beta_2} + \theta z)},\ z = \max(y_1,y_2).$$

Therefore this survival function can be used to derive the probability density function of bivariate Weibull distribution in [12].

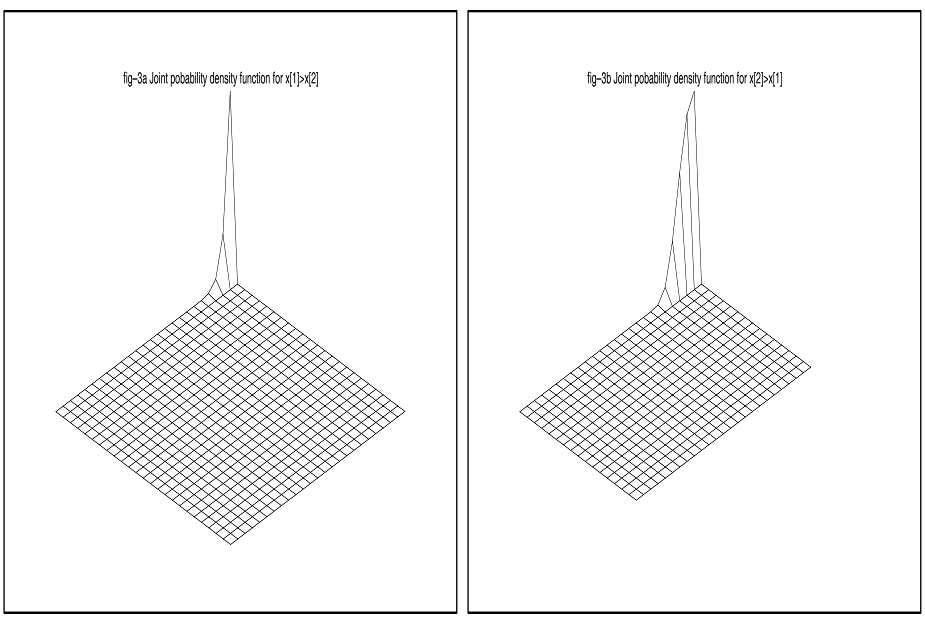
Next, plots of the joint density function (2.9) for some selected values of distribution parameters are shown in figures $1-4$.



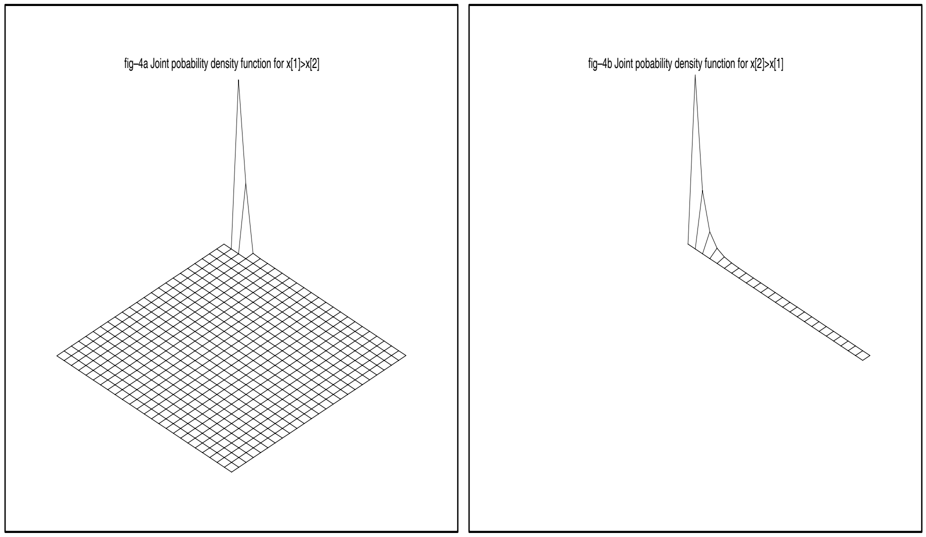
Figures 1a and 1b. A plot of the joint density function $f(x_1,x_2)$, Equation (2.9) for $\alpha_2 = 2\alpha_1 = 0.4, \beta_1 = \beta_2 = 10$ and $\theta = 5$.



Figures 2a and 2b. A plot of the joint density function $f(x_1, x_2)$, Equation (2.9) for $\alpha_1 = \alpha_2 = 0.4, \beta_1 = 40\beta_2 = 200$ and $\theta = 5$.



Figures 3a and 3b. A plot of the joint density function $f(x_1, x_2)$, Equation (2.9) for $\alpha_1 = \alpha_2 = 2, \beta_2 = 100\beta_1 = 200$ and $\theta = 5$.



Figures 4a and 4b. A plot of the joint density function $f(x_1, x_2)$, Equation (2.9) for $\alpha_1 = \alpha_2 = 2, \beta_2 = 1500, \beta_1 = 2$ and $\theta = 25$.

A comparison of Figures 3*b* and 4*b* shows the relative rate at which the density tail of.

Lemma 2.2 *The joint probability density function of the Marshall-Olkin bivariate exponential distribution is [8, 11]*

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} \alpha_2(\alpha_1+\theta)\ \exp\Big[-(\alpha_1+\theta)x_1 - \alpha_2 x_2\Big], & x_1 > x_2 > 0 \\ \\ \alpha_1(\alpha_2+\theta)\ \exp\Big[-k_1\beta_1 x_1 - (\alpha_2+\theta)x_2\Big], & x_2 > x_1 > 0 \end{cases}$$

Also, this distribution has a mass of $\theta/(\theta+\alpha_1+\alpha_2)$ along the diagonal $x_1 = x_2$.
Proof. The result of this lemma can be obtained immediately from theorem (2.1) upon setting β_1 and β_2 tend to zero. □

2.3 Marginal probability density functions

The following Corollary gives the marginal probability density functions of X_1 and X_2.

Theorem 2.2 *The marginal pdf of* X_i, $(i = 1, 2)$ *is given by*

$$f_{X_i}(x_i) = (\theta + k_i\beta_i e^{\beta_i x_i}) \exp\left[-k_i\left(e^{\beta_i x_i} - 1\right) - \theta x_i\right],\ x_i > 0,\ (i = 1, 2) \quad (2.14)$$

Proof. First we derive $f_{X_1}(x_1)$ using the fact that

$$f_{X_1}(x_1) = \int_0^\infty f_{X_1,X_2}(x_1,x_2)\,dx_2 = \int_0^{x_1} f_1(x_1,x_2)\,dx_2 + \int_{x_1}^\infty f_2(x_1,x_2)\,dx_2 + f_0(x_1,x_1)$$

Using the expressions of $f_1(x_1,x_2)$, $f_2(x_1,x_2)$ and $f_0(x_1,x_1)$ given in Theorem (2.1), we can get $f_{X_1}(x_1)$ of the form (2.14). Proceeding similarly, we can derive $f_{X_2}(x_2)$ as given in (2.14), which completes the proof of the theorem. □

Lemma 2.3 *The marginal probability density function of X_i, $(i = 1,2)$ of Marshall-Olkin bivariate exponential distribution is*

$$f_{X_i}(x_i) = (\theta + \alpha_i)e^{-(\theta+\alpha_i)x_i},\ x_i > 0,\ (i = 1,2)$$

Proof. The proof of this lemma can be obtained immediately from theorem (2.2) upon setting β_i, $(i = 1,2)$ tend to zero. □

2.4 Conditional probability density functions

Theorem 2.3 *The conditional probability density function of X_i given $X_j = x_j$, $(i,j = 1,2,\ i \neq j)$ is given by*

$$f_{X_i|X_j}(x_i|x_j) = \begin{cases} f^{(1)}_{X_i|X_j}(x_i|x_j) & \text{if} \quad x_i > x_j > 0, \\ f^{(2)}_{X_i|X_j}(x_i|x_j) & \text{if} \quad x_j > x_i > 0, \\ f^{(0)}_{X_i|X_j}(x_i|x_j) & \text{if} \quad x_i = x_j > 0 \end{cases} \tag{2.15}$$

where

$$f^{(1)}_{X_i|X_j}(x_i|x_j) = k_j\beta_j\big(\theta + k_i\beta_i e^{\beta_i x_i}\big)\exp\big\{-k_i\big(e^{\beta_i x_i} - 1\big) - \theta x_i + (\theta + \beta_j)x_j)\big]\big\} / \big(\theta + k_j\beta_j e^{\beta_j x_j}\big)$$

$$f^{(2)}_{X_i|X_j}(x_i|x_j) = k_i\beta_i \exp\Big\{-\Big[k_i\big(e^{\beta_i x_i} - 1\big) - \beta_i x_i\Big]\Big\},$$

$$f^{(0)}_{X_i|X_j}(x_i|x_j) = \theta\exp\big\{-k_i\big(e^{\beta_i x_i} - 1\big)\big\} / \big(\theta + k_j\beta_j\, e^{\beta_j x_j}\big),$$

Proof. The theorem follows readily upon substituting for the joint probability density function of (X_1, X_2) in (2.10) and the marginal probability density function of X_i, $(i = 1,2)$ in (2.14), the following relation

$$f_{X_i|X_j}(x_i|x_j) = \frac{f_{X_i,X_j}(x_i,x_j)}{f_{X_i}(x_i)},\ (i = 1,2) \tag{2.16}$$

which completes the proof. □

Lemma 2.1 *For the case of Marshall-Olkin bivariate exponential distribution, we can obtain by setting* β_1 *and* β_2 *tend to zero in (2.15).*

$$f_{X_i|X_j}(x_i|x_j) = \begin{cases} \alpha_j\, e^{-(\theta+\alpha_i)x_i+\theta x_j} & \text{if} \quad x_i > x_j > 0, \\ \alpha_i\, e^{-\alpha_i x_i} & \text{if} \quad x_j > x_i > 0, \\ \theta\, e^{-\alpha_i x_i}/(\theta+\alpha_j) & \text{if} \quad x_i = x_j > 0 \end{cases} \tag{2.17}$$

Proof. The proof of this lemma can be reach readily by setting β_i, $(i = 1, 2)$ tend to zero in (2.15) which completes the proof. □

2.5 Mathematical expectation

This subsection presents the exact forms of mathematical expectation of X_i, $(i = 1, 2)$, and the second moments of X_i^2, $(i = 1, 2)$ and X_1X_2.

Theorem 2.4 *The expectation of* X_i, $(i = 1, 2)$ *is given*

$$E\big(X_i\big) = k_i^{\theta/\beta_i}\, e^{k_i}\Gamma\big(-\theta/\beta_i, k_i\big)/\beta_i,\ (i = 1, 2) \tag{2.18}$$

where $\Gamma(\mu, \nu)$ *is the incomplete gamma function that defined by*

$$\Gamma(\mu, \nu) = \int_{\nu}^{\infty} t^{\mu-1}\, e^{-t}\, dt,\ t > 0$$

which for fixed ν *is an entire function of* μ*. For non-integer* μ *this function is a multi-valued function of* ν *with a branch point at* $\nu = 0$*.*

Proof. Starting with

$$E(X_i) = \int_0^{\infty} x_i\, f_{X_i}(x_i)\, dx_i$$

and substituting $f_{X_i}(x_i)$ from (2.14), we get

$$\begin{aligned} E(X_i) &= \int_0^{\infty} x_i\,(\theta+k_i\beta_i\, e^{\beta_i x_i})\exp\left[-k_i\big(e^{\beta_i x_1}-1\big)-\theta x_i\right]\, dx_i \\ &= \int_0^{\infty} \exp\left[-k_i\big(e^{\beta_i x_1}-1\big)-\theta x_i\right]\, dx_i = e^{k_i}\int_0^{\infty} e^{-\theta x_i}\exp\big[-k_i\, e^{\beta_i x_i}\big]\, dx_i \end{aligned}$$

and using the incomplete gamma function definition (see Gradshteyn and Ryzhik 3.331 p.308 and 8.350 p.940 [11]) we derive the expression in (2.18). which completes the proof. □

Theorem 2.5 *The expectation of X_i^2 for $i = 1, 2$ is given*

$$E(X_i^2) = 2\, k_i^{\theta/\beta_i}\, e^{k_i} \left[\partial\Gamma\big(-\theta/\beta_i, k_i\big)/\partial\theta - \ln k_i\, \Gamma\big(-\theta/\beta_i, k_i\big)\right] / \beta_i^2,\ (i = 1, 2) \tag{2.19}$$

Proof. Starting with

$$E\big(X_i^2\big) = \int_0^\infty x_i^2\, f_{X_i}(x_i)\, dx_i$$

and substituting $f_{X_i}(x_i)$ from (2.14), and putting $u = k_i e^{\beta_i x_i}$, we get

$$E\big(X_i^2\big) = 2\, e^{k_i}\, k_i^{\theta/\beta_i} \left[\int_{k_i}^\infty \frac{\ln u\, e^{-u}}{u^{(\theta/\beta_i+1)}}\, du - \ln k_i \int_{k_i}^\infty \frac{e^{-u}}{u^{(\theta/\beta_i+1)}}\, du\right] / \beta_i^2$$

and using the incomplete gamma function definition and its differentiation with respect to one of their parameters (see Gradshteyn and Ryzhik 4.358 p.578 and 8.350 p.940 [11]) we derive the expression in (2.19). which completes the proof. □

Theorem 2.6 *The expectation of X_1X_2 is given*

$$\begin{aligned} E\big(X_1X_2\big) = e^{k_1+k_2} \Big\{ & k_1^{\theta/\beta_1} \Big[\Gamma(0,k_2)\Gamma\big(-\theta/\beta_1, k_1\big) - \big(\Psi(1) + \ln k_2\big)\Gamma\big(-\theta/\beta_1, k_1\big) + \\ & 2\beta_2 \ln k_1 \Gamma\big(-\theta/\beta_1, k_1\big)/\beta_1 + 2\beta_2 \partial\Gamma\big(-\theta/\beta_1, k_1\big)/\partial\theta - \beta_1\beta_2\Gamma\big(1-\theta/\beta_1, k_1\big)/\theta \\ & -\beta_2 \ln k_1 \Gamma\big(1-\theta/\beta_1, k_1\big)\Big] + k_2^{\theta/\beta_2}\Gamma(0,k_1)\Gamma(-\theta/\beta_2) + \sum_{n=1}^{\infty}(-1)^n k_1^{(\theta-n\beta_2)/\beta_1} k_2^n \times \\ & \Big[n\beta_1\partial\Gamma\Big((n\beta_2-\theta)/\beta_1+1, k_1\Big)/\partial\theta - (n\beta_2-\theta)\beta_2\partial\Gamma\Big((n\beta_2-\theta)/\beta_1+1, k_1\Big)/\partial\theta + \\ & n\ln k_1\Gamma\Big((n\beta_2-\theta)/\beta_1+1, k_1\Big)\beta_1 - (n\beta_2-\theta)(\beta_1 - n\beta_2\ln k_1)\Gamma\Big((n\beta_2-\theta)/\beta_1 + \\ & 1, k_1\Big)/\beta_1\Big] / \Big(n! n(n\beta_1-\theta)\Big)\Big\} / \big(\beta_1\beta_2\big), \end{aligned} \tag{2.20}$$

where $\Psi(x)$ is the digamma function.

Proof. Starting with

$$\begin{aligned} E(X_1X_2) = & \int_0^\infty \int_0^{x_1} x_1x_2\, f_1(x_1,x_2)dx_2dx_1 + \\ & \int_0^\infty \int_{x_1}^\infty x_1x_2\, f_2(x_1,x_2)dx_2dx_1 + \int_0^\infty x^2\, f_0(x,x)dx \end{aligned}$$

and substituting $f_1(x_1, x_2), f_2(x_1, x_2)$ and $f_0(x_1, x_2)$ from (2.10) and using the integral properties we get

$$E(X_1X_2) = \int_0^\infty \int_0^{x_1} x_1\big(\theta+k_1\beta_1\, e^{\beta_1 x_1}\big)\exp\Big\{-k_1\, e^{\beta_1 x_1}-k_2\, e^{\beta_2 x_2}-\theta x_1\Big\}dx_2dx_1+$$

$$k_1\beta_1 \int_0^\infty \int_{x_1}^\infty x_1\, e^{\beta_1 x_1}\exp\Big\{-k_1\, e^{\beta_1 x_1}-k_2\, e^{\beta_2 x_2}-\theta x_2\Big\}dx_2dx_1$$

using the definition of the incomplete gamma function and its series form (see Gradshteyn and Ryzhik 8.212 p.925, 8.214 p.927 and 8.354 p.941 [11]) and after a lengthly algebraic manipulation we derive the expression in (2.20). which completes the proof. □

Lemma 2.2 *For the Marshall-Olkin bivariate exponential, we have*

$$E(X_i) = \frac{1}{\theta+\alpha_i}, \qquad E(X_i^2) = \frac{2}{(\theta+\alpha_i)^2}, \qquad (i=1,2) \tag{2.21}$$

and

$$E\big(X_1X_2\big) = \frac{1}{\alpha_2(\alpha_1+\theta)} + \frac{\alpha_1}{(\theta+\alpha_2)(\theta+\alpha_1+\alpha_2)^2} - \frac{\theta+\alpha_1}{\alpha_2(\theta+\alpha_1+\alpha_2)^2} \tag{2.22}$$

By taking limits of the expressions (2.18), (2.19) and (2.20) as β_1 and β_2 tend to zero we can get the expressions (2.21) and (2.22).

3 Moment generating functions

In this section we present the joint moment generating function of (X_1, X_2) and marginal moment generating function of X_i, $(i=1,2)$.

Theorem 3.1 *The joint moment generating function of (X_1, X_2) is given by:*

$$M_{X_1,X_2}(t_1,t_2) = 1-e^{k_1+k_2}\Big\{t_1k_1^{(t_1+\theta)/\beta_1}\Gamma\big(-t_1/\beta_1-\theta/\beta_1,k_1\big)/\beta_1+t_2k_2^{t_2/\beta_2}\Big[\Gamma\big(-t_2/\beta_2,k_2\big)-$$
$$\Gamma\big(-t_2/\beta_2\big)\Big]\Big[e^{-k_1}-t_1k_1^{(\theta+t_1)/\beta_1}\Gamma\big(-(t_1+\theta)/\beta_1,k_1\big)/\beta_1\Big]+t_2k_2^{(\theta+t_2)/\beta_2}\Gamma\big(-(t_1+\theta))/\beta_1\big)\Gamma\big(1-$$
$$t_1/\beta_1,k_1\big)/\beta_2+t_2\sum_{n=0}^{\infty}(-1)^nk_2^n\Big[\theta e^{-k_1}\Big((n\beta_2-t_2)(n\beta_2-t_2-\theta)\Big)^{-1}-k_1^{(t_1+t_2+\theta-n\beta_2)/\beta_1}\big(t_1+$$
$$t_2+\theta-n\beta_2)\Gamma\big(1-(t_1+t_2+\theta-n\beta_2)/\beta_1,k_1\big)\Big(\beta_1(n\beta_2-\theta-t_2)\Big)^{-1}+k_1^{(t_1+t_2-n\beta_2)/\beta_1}\times$$
$$\big(t_1+t_2-n\beta_2)\Gamma\big(-(t_1+t_2+\theta-n\beta_2)/\beta_1,k_1\big)\big(\beta_1(n\beta_2-t_2)\big)^{-1}\Big]/\big(n!n\big)\Big\} \tag{3.1}$$

Proof. Starting with, the definition moment generating function of bivariate distribution of (X_1, X_2) we have

$$M_{X_1,X_2}(t_1,t_2) = E\left(e^{-(t_1X_1+t_2X_2)}\right)$$

That is,

$$\begin{aligned} M_{X_1,X_2}(t_1,t_2) &= \int_0^\infty \int_0^{x_1} e^{-(t_1x_1+t_2x_2)}\, f_1(x_1,x_2)\, dx_2\, dx_1 \\ &\quad + \int_0^\infty \int_0^{x_2} e^{-(t_1x_1+t_2x_2)} f_2(x_1,x_2)\, dx_1\, dx_2 \\ &\quad + \int_0^\infty e^{-(t_1+t_2)x}\, f_0(x,x) dx \end{aligned}$$

Substituting from (2.9) into the above relation and after a lengthly algebraic manipulation we can reach the form (3.1).
The following lemma presents the marginal moment generating function of X_i, $(i = 1, 2)$.

Lemma 3.1 *The marginal moment generating function of* X_i, $(i = 1, 2)$ *is given by:*

$$M_i(t_i) = \left[1 - t_i\, e^{k_i}\, k_i^{(t_i+\theta)/\beta_i} \Gamma\big(-(t_i+\theta)/\beta_i, k_i\big)/\beta_i\right],\ (i = 1, 2) \qquad (3.2)$$

Proof. The proof of this lemma can be done by using either: the relation between the joint and marginal moment generating functions, or the definition of the marginal moment generating function of the random variable. Note that the moment generating function of the bivariate Gompertz can be used to derive the mathematical expectations X_i, X_i^2, $i = 1, 2$ and X_1X_2.

Next, we will discuss a form of the bivariate mixture of Gompertz distribution of Marshall-Olkin type where the dependence among the components is characterized by a latent variable independently distributed of the individual components.

4 Mixture of Bivariate Gompertz Distributions

In this section, we present the mixture of independent Gompertz distributions. Then we derive a mixture of bivariate Gompertz distributions where the dependence among the components is characterized by a latent random variable which is independently exponentially distributed of the individual component.

Consider a system of two components where the lifetime of component i, $i = 1, 2$, say, X_i, is a mixture of two independent Gompertz distributions. That is,

$$X_1 \sim a_1 \, \mathrm{Gomp}(\alpha_{11}, \beta_{11}) + (1 - a_1) \, \mathrm{Gomp}(\alpha_{12}, \beta_{12}) \, , \; 0 \le a_1 \le 1 \, ,$$

and

$$X_2 \sim a_2 \, \mathrm{Gomp}(\alpha_{21}, \beta_{21}) + (1 - a_2) \, \mathrm{Gomp}(\alpha_{22}, \beta_{22}) \, , \; 0 \le a_2 \le 1 \, .$$

The notation $\mathrm{Gomp}(\alpha_{ij}, \beta_{ij})$ means a random variable, say X_{ij}, having a Gompertz distribution with the parameters α_{ij}, β_{ij}. That is the probability density function of X_{ij} takes the following form

$$f_{X_{ij}}(x) = \alpha_{ij} e^{\beta_{ij} x} \exp\Big\{ - k_{ij}(e^{\beta_{ij} x} - 1)\Big\}, \; x > 0, \; \alpha_{ij} = k_{ij}\beta_{ij} > 0, \; \beta_{ij} > 0, \; \forall \, i, j \, .$$

Consider also an exponentially distributed random variable, say Z, with parameter θ which is independent of X_{ij} for all i, j. The random variable Z will be used as a latent variable to introduce dependence among $X's$. The density function of Z is

$$f_Z(z) = \theta \, e^{-\theta z}, \;\; z > 0, \; \theta > 0 \, .$$

Using the independence assumption in the above model, we can see that Z is also independent of X_1 and X_2.
Define $S_i = \mathrm{Min}(X_i, Z)$ for $i = 1, 2$. Then the vector $\underline{S} = (S_1, S_2)$ follows a bivariate distribution and obviously they are dependent as they commonly share the influence of the latent random variable Z.
In what follows we present the joint probability density function of (S_1, S_2). Firstly, we derive the joint survival function of (S_1, S_2). Then we use it to derive the joint probability density function of the mixture of bivariate Gompertz distribution.

Corollary 4.1 *The joint survival function of S_1, S_2 is*

$$\begin{aligned} \bar{F}_{S_1,S_2}(s_1, s_2) = p_{11} \; \exp\Big[- k_{11}(e^{\beta_{11} s_1} - 1) - k_{21}(e^{\beta_{21} s_2} - 1) - \theta s\Big] \\ + p_{12} \; \exp\Big[- k_{11}(e^{\beta_{11} s_1} - 1) - k_{22}(e^{\beta_{22} s_2} - 1) - \theta s\Big] \\ + p_{12} \; \exp\Big[- k_{12}(e^{\beta_{12} s_1} - 1) - k_{21}(e^{\beta_{21} s_2} - 1) - \theta s\Big] \\ + p_{22}\Big[- k_{12}(e^{\beta_{12} s_1} - 1) - k_{22}(e^{\beta_{22} s_2} - 1) - \theta s\Big] \end{aligned} \tag{4.1}$$

where $s = max(s_1, s_2) > 0$ and for $i, j \in \{1, 2\}$:

$$p_{ij} = a_1^{2-i} \, a_2^{2-j} \, (1 - a_1)^{i-1} \, (1 - a_2)^{j-1} \, .$$

Proof. Since

$$\bar{F}_{S_1,S_2}(s_1,s_2) = P(S_1 > s_1, S_2 > s_2)$$

Then using the definitions of S_1 and S_2 we have

$$\begin{aligned}\bar{F}_{S_1,S_2}(s_1,s_2) &= P(X_1 > s_1)\,P(X_2 > s_2)\,P(Z > s_0) = \\ &= e^{-\theta z}\prod_{i=1}^{2}\left\{a_i \exp\left[-k_{i1}(e^{\beta_{i1}s_i}-1)\right]\right. \\ &\quad \left.+(1-a_i)\exp\left[-k_{i2}(e^{\beta_{i2}s_i}-1)\right]\right\} \qquad (4.2)\end{aligned}$$

One can write the above relation as given by (3.1), that completes the proof. □

Note that:

1. For $i,j \in \{1,2\}$, $p_{ij} \geq 0$ and $p_{11}+p_{12}+p_{21}+p_{22}=1$.
2. Each term function in the right hand side of (4.1) is a survival function of the new bivariate Gompertz distribution.

This means that, survival function given in (4.1) is a joint survival function of a mixture of four bivariate Gompertz distributions.
The following Theorem gives the joint probability density function of (S_1,S_2).

Theorem 4.1 *The joint pdf of S_1, S_2, say $f(s_1,s_2)$, is*

$$f(s_1,s_2) = \begin{cases} f_1(s_1,s_2) & \text{if } s_1 > s_2 > 0\,, \\ f_2(s_1,s_2) & \text{if } s_2 > s_1 > 0\,, \\ f_0(s,s) & \text{if } s_1 = s_2 = s > 0\,, \end{cases} \qquad (4.3)$$

where

$$\begin{aligned} f_1(s_1,s_2) &= p_{11}k_{21}\beta_{21}e^{\beta_{21}s_2}\left(\theta + k_{11}\beta_{11}e^{\beta_{11}s_1}\right)\exp\left\{-k_{11}(e^{\beta_{11}s_1}-1) - k_{21}(e^{\beta_{21}s_2}-1) - \theta s_1\right\} \\ &+p_{12}k_{22}\beta_{22}e^{\beta_{22}s_2}(\theta + k_{11}\beta_{11}e^{\beta_{11}s_1})\exp\left\{-k_{11}(e^{\beta_{11}s_1}-1) - k_{22}(e^{\beta_{22}s_2}-1) - \theta s_1\right\} \\ &+p_{21}k_{21}\beta_{21}e^{\beta_{21}s_2}(\theta + k_{21}\beta_{12}e^{\beta_{12}s_1})\exp\left\{-k_{12}(e^{\beta_{12}s_1}-1) - k_{21}(e^{\beta_{21}s_2}-1) - \theta s_1\right\} \\ &+p_{22}k_{22}\beta_{22}e^{\beta_{22}s_2}(\theta + k_{12}\beta_{12}e^{\beta_{12}s_1})\exp\left\{-k_{12}(e^{\beta_{12}s_1}-1) - k_{22}(e^{\beta_{22}s_2}-1) - \theta s_1\right\} \end{aligned} \qquad (4.4)$$

$$\begin{aligned} f_2(s_1,s_2) &= p_{11}k_{11}\beta_{11}e^{\beta_{11}s_1}(\theta + k_{21}\beta_{21}e^{\beta_{21}s_2})\exp\left\{-k_{11}(e^{\beta_{11}s_1}-1) - k_{21}(e^{\beta_{21}s_2}-1) - \theta s_2\right\} \\ &+p_{12}k_{11}\beta_{11}e^{\beta_{11}s_1}(\theta + k_{22}\beta_{22}e^{\beta_{22}s_2})\exp\left\{-k_{11}(e^{\beta_{11}s_1}-1) - k_{22}(e^{\beta_{22}s_2}-1) - \theta s_2\right\} \\ &+p_{21}k_{12}\beta_{12}e^{\beta_{12}s_1}(\theta + k_{21}\beta_{21}e^{\beta_{21}s_2})\exp\left\{-k_{12}(e^{\beta_{12}s_1}-1) - k_{21}(e^{\beta_{21}s_2}-1) - \theta s_2\right\} \\ &+p_{22}k_{12}\beta_{12}e^{\beta_{12}s_1}(\theta + k_{22}\beta_{22}e^{\beta_{22}s_2})\exp\left\{-k_{12}(e^{\beta_{12}s_1}-1) - k_{22}(e^{\beta_{22}s_2}-1) - \theta s_2\right\} \end{aligned} \qquad (4.5)$$

$$f_0(s,s) = \theta \left\{ p_{11} \exp\left[-k_{11}(e^{\beta_{11}s}-1) - k_{21}(e^{\beta_{21}s}-1) - \theta s\right] + p_{12} \exp\left[-k_{11}(e^{\beta_{11}s}-1) - k_{22}(e^{\beta_{22}s}-1) - \theta s\right] + p_{21} \exp\left[-k_{12}(e^{\beta_{12}s}-1) - k_{21}(e^{\beta_{21}s}-1) - \theta s\right] + p_{22} \exp\left[-k_{12}(e^{\beta_{12}s}-1) - k_{22}(e^{\beta_{22}s}-1) - \theta s\right] \right\} \tag{4.6}$$

Proof. The proof of this Theorem follows along the same lines as of theorem (2.1). □

The following Corollary gives the marginal pdf's of S_1 and S_2.

Lemma 4.1 *The marginal density functions of S_1 and S_2 are respectively,*

$$f_{S_1}(s_1) = a_1(\theta + k_{11}\beta_{11}e^{\beta_{11}s_1}) \exp\left\{-k_{11}(e^{\beta_{11}s_1}-1) - \theta s_1\right\} +(1-a_1)(\theta + k_{12}\beta_{12}e^{\beta_{12}s_1}) \exp\left\{-k_{12}(e^{\beta_{12}s_1}-1) - \theta s_2\right\},\ s_1 > 0, \tag{4.7}$$

and

$$f_{S_2}(s_2) = a_2(\theta + k_{21}\beta_{21}e^{\beta_{21}s_2}) \exp\left\{-k_{21}(e^{\beta_{21}s_2}-1) - \theta s_2\right\} +(1-a_2)(\theta + k_{22}\beta_{22}e^{\beta_{22}s_2}) \exp\left\{-k_{12}(e^{\beta_{22}s_2}-1) - \theta s_2\right\},\ s_2 > 0. \tag{4.8}$$

From the marginal densities, we can derive the marginal moment generating functions of S_i, $(i = 1,2)$ as follows.

Lemma 4.2 *The moment generating functions of X_1 and X_2 are respectively,*

$$M_{S_1}(t_1) = 1 - t_1 \Big\{ a_1\, e^{k_{11}} k_{11}^{(\theta+t_1)/\beta_{11}} \Gamma\Big(-(\theta+t_1)/\beta_{11}, k_{11}\Big)/\beta_{11} + (1-a_1)\, e^{k_{12}}\, k_{12}^{(\theta+t_1)/\beta_{12}} \Gamma\Big(-(\theta+t_1)/\beta_{12}, k_{12}\Big)/\beta_{12} \Big\} \tag{4.9}$$

and

$$M_{S_2}(t_2) = 1 - t_2 \Big\{ a_2\, e^{k_{21}} k_{21}^{(\theta+t_2)/\beta_{21}} \Gamma\Big(-(\theta+t_2)/\beta_{21}, k_{21}\Big)/\beta_{21} + (1-a_2)\, e^{k_{22}}\, k_{22}^{(\theta+t_2)/\beta_{22}} \Gamma\Big(-(\theta+t_2)/\beta_{22}, k_{22}\Big)/\beta_{22} \Big\} \tag{4.10}$$

Lemma 4.3 *From (4.9) and (4.10), we readily have*

$$E(S_1) = a_1\, e^{k_{11}}\, k_{11}^{\theta/\beta_{11}} \Gamma\Big(-\theta/\beta_{11}, k_{11}\Big)/\beta_{11} + (1-a_1)\, e^{k_{12}}\, k_{12}^{\theta/\beta_{12}} \Gamma\Big(-\theta/\beta_{12}, k_{12}\Big)/\beta_{12} \tag{4.11}$$

and

$$E(S_2) = a_2\, e^{k_{21}}\, k_{21}^{\theta/\beta_{21}}\Gamma\Big(-\theta/\beta_{21}, k_{21}\Big)/\beta_{21} + (1-a_2)\, e^{k_{22}}\, k_{22}^{\theta/\beta_{22}}\Gamma\Big(-\theta/\beta_{22}, k_{22}\Big)/\beta_{22} \tag{4.12}$$

Lemma 4.4 *From (4.9) and (4.10), we readily have*

$$\begin{aligned} E(S_1^2) = & -2\, a_1\, e^{k_{11}} \Big[\beta_{11}\partial\Gamma\Big(-\theta/\beta_{11}, k_{11}\Big)/\partial\theta + \ln k_{11}\Gamma\Big(-\theta/\beta_{11}, k_{11}\Big)\Big]/\beta_{11}^2 \\ & -2(1-a_1)\, e^{k_{12}} \Big[\beta_{12}\partial\Gamma\Big(-\theta/\beta_{12}, k_{12}\Big)/\partial\theta + \ln k_{12}\Gamma\Big(-\theta/\beta_{12}, k_{12}\Big)\Big]/\beta_{12}^2 \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} E(S_2^2) = & -2\, a_2\, e^{k_{21}} \Big[\beta_{21}\partial\Gamma\Big(-\theta/\beta_{21}, k_{21}\Big)/\partial\theta + \ln k_{21}\Gamma\Big(-\theta/\beta_{21}, k_{21}\Big)\Big]/\beta_{21}^2 \\ & -2(1-a_2)\, e^{k_{22}} \Big[\beta_{22}\partial\Gamma\Big(-\theta/\beta_{22}, k_{22}\Big)/\partial\theta + \ln k_{22}\Gamma\Big(-\theta/\beta_{22}, k_{22}\Big)\Big]/\beta_{22}^2 \end{aligned} \tag{4.14}$$

Lemma 4.5 *The joint moment generating function of (S_1, S_2) is given by*

$$M_{S_1,S_2}(t_1,t_2) = \textstyle\sum_i^4 b_i M_i(t_1,t_2) \tag{4.15}$$

where

$$\left.\begin{array}{l} b_1 = p_{11}, b_2 = p_{12}, b_3 = p_{21}, b_4 = p_{22} \\ M_1 = M_{X_{11},X_{21}}(t_1,t_2), M_2 = M_{X_{11},X_{22}}(t_1,t_2), \\ M_3 = M_{X_{12},X_{21}}(t_1,t_2), M_4 = M_{X_{12},X_{22}}(t_1,t_2) \end{array}\right\} \tag{4.16}$$

and the functions $M_{X_{11},X_{21}}(t_1,t_2), M_{X_{11},X_{22}}(t_1,t_2), M_{X_{12},X_{21}}(t_1,t_2)$ and $M_{X_{12},X_{22}}(t_1,t_2)$ can be derived from (2.14) respectively, by replacing $\alpha_1, \alpha_2, \beta_1, \beta_2$ with $\alpha_{11}, \alpha_{21}, \beta_{11}, \beta_{21}$; $\alpha_{11}, \alpha_{22}, \beta_{11}, \beta_{22}$; $\alpha_{12}, \alpha_{21}, \beta_{12}, \beta_{21}$ and $\alpha_{12}, \alpha_{22}, \beta_{12}, \beta_{22}$.

Proof. One can establish this lemma from (3.1) and (4.15).

5 Conclusion

Finally we conclude that a new class of bivariate Gompertz distribution of Marshall-Olkin type is derived. The used procedure is based on a latent random variable with exponential distribution. The moment generating functions of bivariate new class is derived. A mixture of the suggested bivariate distributions is derived. The present results generalize those of Marshall-Olkin bivariate exponential distribution and other present in the literature [5, 8, 10].

References

[1] B. Gompertz, (1824) *On the nature of the function expressive of the law of human mortality and on the new mode of determining the value of life contingencies*, *Philosophical Transactions of Royal Society A115, pp. 513-580.*

[2] Ahuja, J. C. and Nash, S. W. *The generalized Gompertz verhulst family distriutions*, *Sankhya Part A. 29 (1979), pp. 141-156.*

[3] Al-Ruzaiza, A. S. and El-Gohary, A. (2007) *A New Class of Positively Quadrant Dependent Bivariate Distributions with Pareto*, *International Mathematical Fourm (In Press).*

[4] El-Gohary, A. (2005) *A multivariate mixture of linear failure rate distribution in reliability models*, *International Journal of Reliability and Applications, In Press.*

[5] El-Gohary, A. and Sarhan, A. (2005) *The Distributions of sums, rroducts, ratios and differences for Marshall-Olkin Bivariate exponential distribution*, *International Journal of Applied Mathematics (In Press).*

[6] El-Gohary, A. (2006), *A Mixture of Multivariate Distributions with Pareto in Reliability Models*, *International Journal of Reliability and Applications, In Press.*

[7] A-Hameed, M. S. and Proshan, F. (1973). *Nonstationary shock models*, *Stoch. Proc. Appl., 1:333-404.*

[8] Marshall, A. W. and Olkin, I. A. (1967b). *A generalized bivariate exponential distribution*, *J. Appl. Prob., 4:291-302.*

[9] Barlow, R. E. and Proschan, F. (1981). *Statistical Theory of Reliability and Life Testing, Probability Models*, *To Begin With Silver Spring, MD.*

[10] Marshall, A. W. and Olkin, I. A. (1967a). *A multivariate exponential distribution*, *J. Amer. Statist. Assoc., 30-44.*

[11] Gradshteyn, I. S. and Ryzhih, I. M. *Table of integrals, Series and Products*, *Academic press New York (2000).*

[12] Patra, K. and Dey, D. (1999), *A multivariate mixture of Weibull distrbutions in reliability modeling*, *Statistics & Probability Letters, 45, pp. 225-235.*

Received: December 8, 2007