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## A NEW CLASS OF ENTROPY SOLUTIONS OF THE BUCKLEY–LEVERETT EQUATION\*

C. J. VAN DUIJN<sup>†</sup>, L. A. PELETIER<sup>‡</sup>, AND I. S. POP<sup>†</sup>

**Abstract.** We discuss an extension of the Buckley–Leverett (BL) equation describing two-phase flow in porous media. This extension includes a third order mixed derivatives term and models the dynamic effects in the pressure difference between the two phases. We derive existence conditions for traveling wave solutions of the extended model. This leads to admissible shocks for the original BL equation, which violate the Oleinik entropy condition and are therefore called *nonclassical*. In this way we obtain nonmonotone weak solutions of the initial-boundary value problem for the BL equation consisting of constant states separated by shocks, confirming results obtained experimentally.

**Key words.** conservation laws, dynamic capillarity, two-phase flows in porous media, shock waves, pseudoparabolic equations

**AMS subject classifications.** 35L65, 35L67, 35K70, 76S05, 76T05

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**1. Introduction.** We consider the first order initial-boundary value problem

$$(1.1) \quad \text{(BL)} \quad \begin{cases} \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 & \text{in } Q = \{(x, t) : x > 0, t > 0\}, \\ u(x, 0) = 0 & x > 0, \\ u(0, t) = u_B & t > 0, \end{cases}$$

where  $u_B$  is a constant such that  $0 \leq u_B \leq 1$ . The nonlinearity  $f : \mathbf{R} \rightarrow \mathbf{R}$  is given by

$$(1.2) \quad f(u) = \frac{u^2}{u^2 + M(1-u)^2} \quad \text{if } 0 \leq u \leq 1,$$

whilst  $f(u) = 0$  if  $u < 0$  and  $f(u) = 1$  if  $u > 1$ . Here,  $M > 0$  is a fixed constant. The function  $f(u)$  is shown in Figure 1.

Equation (1.1), with the given flux function  $f$ , arises in two-phase flow in porous media, and problem (BL) models oil recovery by water-drive in one-dimensional horizontal flow. In this context,  $u : \overline{Q} \rightarrow [0, 1]$  denotes water saturation,  $f$  the water fractional flow function, and  $M$  the water/oil viscosity ratio. In petroleum engineering, (1.1) is known as the *Buckley–Leverett (BL) equation* [5]. It is a prototype for first order conservation laws with convex-concave flux functions.

It is well known that first order equations such as (1.1) may have solutions with discontinuities, or shocks. The value ( $u_\ell$ ) to the left of the shock, the value ( $u_r$ ) to

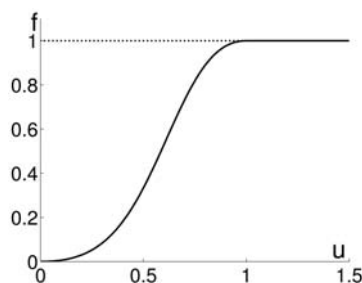
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FIG. 1. Nonlinear flux function for Buckley–Leverett ( $M = 2$ ).

the right, and the speed  $s$  of the shock with trace  $x = x(t)$  are related through the Rankine–Hugoniot condition,

$$(1.3) \quad (\text{RH}) \quad \frac{dx}{dt} = s = \frac{f(u_\ell) - f(u_r)}{u_\ell - u_r}.$$

We will denote shocks by their values to the left and to the right:  $\{u_\ell, u_r\}$ .

If a function  $u$  is such that (1.1) is satisfied away from the shock curve, and the Rankine–Hugoniot condition is satisfied across the curve, then  $u$  satisfies the identity

$$(1.4) \quad \int_Q \left\{ u \frac{\partial \varphi}{\partial t} + f(u) \frac{\partial \varphi}{\partial x} \right\} = 0 \quad \text{for all } \varphi \in C_0^\infty(Q).$$

Functions  $u \in L^\infty(Q)$  which satisfy (1.4) are called weak solutions of (1.1). Clearly, for any  $u_B \in [0, 1]$ , a weak solution of problem (BL) is given by the shock wave

$$(1.5) \quad u(x, t) = S(x, t) \stackrel{\text{def}}{=} \begin{cases} u_B & \text{for } x < st \\ 0 & \text{for } x > st \end{cases} \quad \text{where } s = \frac{f(u_B)}{u_B}.$$

Experiments of two-phase flow in porous media reveal complex infiltration profiles, which may involve *overshoot*; i.e., profiles may not be monotone [13]. Our main objective is to understand the shape of these profiles and to determine how the shape depends on the boundary value  $u_B$  and the flux function  $f(u)$ .

Equation (1.1) usually arises as the limit of a family of extended equations of the form

$$(1.6) \quad \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \mathcal{A}_\varepsilon(u), \quad \varepsilon > 0,$$

in which  $\mathcal{A}_\varepsilon(u)$  is a singular regularization term involving higher order derivatives. It is often referred to as a *viscosity term*. Weak solutions of problem (BL) are called *admissible* when they can be constructed as limits, as  $\varepsilon \rightarrow 0$ , of solutions  $u_\varepsilon$  of (1.6), i.e., for which  $\mathcal{A}_\varepsilon(u_\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  in some weak sense. We return to this limit in section 6. This raises the question of which of the shock waves  $S(x, t)$  defined in (1.5) are admissible. We shall see that this depends on the operator  $\mathcal{A}_\varepsilon$ . To obtain criteria for admissibility we shall use families of traveling wave solutions.

A classical viscosity term is

$$\mathcal{A}_\varepsilon(u) = \varepsilon \frac{\partial^2 u}{\partial x^2},$$

and with this term, (1.6) becomes

$$(1.7) \quad \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}.$$

Seeking a traveling wave solution, we put

$$(1.8) \quad u = u(\eta) \quad \text{with} \quad \eta = \frac{x - st}{\varepsilon},$$

and we find that  $u(\eta)$  satisfies the following two-point boundary value problem:

$$(1.9a) \quad \begin{cases} -su' + (f(u))' = u'' & \text{in } \mathbf{R}, \\ u(-\infty) = u_\ell, \quad u(\infty) = u_r, \end{cases}$$

where primes denote differentiation with respect to  $\eta$ . An elementary analysis shows that problem (1.9) has a solution if and only if  $f$  and the limiting values  $u_\ell$  and  $u_r$  satisfy (i) the Rankine–Hugoniot condition (1.3), and (ii) the *Oleinik entropy condition* [29]:

$$(1.10) \quad (E) \quad \frac{f(u_\ell) - f(u)}{u_\ell - u} \geq \frac{f(u_\ell) - f(u_r)}{u_\ell - u_r} \quad \text{for } u \text{ between } u_\ell \text{ and } u_r.$$

Shocks  $\{u_\ell, u_r\}$  which satisfy (E) are called *classical shocks*.

Note that in the limit as  $\varepsilon \rightarrow 0^+$ , traveling waves converge to the shock  $\{u_\ell, u_r\}$ .

Applying (RH) and (E) to the flux function (1.2) we find that the function  $S(x, t)$  defined in (1.5) is an admissible shock wave if and only if

$$(1.11) \quad s = \frac{f(u_B)}{u_B} \quad (RH) \quad \text{and} \quad u_B \leq \alpha \quad (E),$$

where  $\alpha$  is the unique root of

$$f'(u) = \frac{f(u)}{u}.$$

It is found to be given by

$$\alpha = \sqrt{\frac{M}{M + 1}} \in (0, 1).$$

If  $u_B > \alpha$ , then the weak solution is composed of a rarefaction wave in the region where  $u > \alpha$  and a shock which spans the range  $0 < u < \alpha$ . Thus, for any  $u_B \in (0, 1]$  the weak solution  $u(x, t)$  is, at any given time  $t$ , a nonincreasing function of  $x$ , in contrast to the experimental data for infiltration in porous media.

For gaining a better understanding of the data, it is natural to go back to the origins of (1.1). With  $S_i$  ( $i = o, w$ ) being the saturations of the two phases, oil and water, conservation of mass yields

$$(1.12) \quad \phi \frac{\partial S_i}{\partial t} + \frac{\partial q_i}{\partial x} = 0, \quad i = o, w,$$

where  $q_i$  denotes the specific discharge of oil/water and  $\phi$  the porosity of the medium. By Darcy's law,  $q_i$  is proportional to the gradient of the phase pressure  $P_i$ :

$$(1.13) \quad q_i = -k \frac{k_{ri}(S_i)}{\mu_i} \frac{\partial P_i}{\partial x},$$

where  $k$  denotes the absolute permeability and  $k_{ri}$  and  $\mu_i$  the relative permeability and the viscosity of water, respectively, oil. The capillary pressure  $P_c$  expresses the difference in the pressures of the two phases:

$$(1.14) \quad P_c = P_o - P_w.$$

This quantity is commonly found to depend on one phase saturation, say  $S_w$ . In addition to this, studies like [28] and [30] show that  $P_c$  does not only depend on  $S_w$ , but also involves hysteretic and dynamic effects. Hassanizadeh and Gray [19, 20] have defined the dynamic capillary pressure as

$$(1.15) \quad P_c = p_c(S_w) - \phi\tau \frac{\partial S_w}{\partial t},$$

where  $p_c(S_w)$  is the *static* capillary pressure and  $\tau$  a positive constant. Assuming that the medium is completely saturated,

$$S_w + S_o = 1,$$

and we obtain, upon combining (1.12)–(1.15), the single equation for the water saturation  $u = S_w$ :

$$(1.16) \quad \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = -\frac{\partial}{\partial x} \left\{ H(u) \frac{\partial}{\partial x} \left( J(u) - \tau \frac{\partial u}{\partial t} \right) \right\},$$

in which the functions  $f$ ,  $H$ , and  $J$  are related to  $k_{ri}$  and  $p_c$ . Other nonequilibrium models are considered in [3]. Restricting, for simplicity, to linear terms on the right-hand side of (1.16), we obtain, after a suitable scaling, the pseudoparabolic equation

$$(1.17) \quad \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2} + \varepsilon^2 \tau \frac{\partial^3 u}{\partial x^2 \partial t}.$$

Thus, in addition to the classical second order term  $\varepsilon u_{xx}$ , we find a third order term  $\varepsilon^2 \tau u_{xxt}$ , its relative importance being determined by the parameter  $\tau$ . We show that the value of  $\tau$  is critical in determining the type of profile the solution of problem (BL) will have.

The right-hand side of (1.17) resembles the regularization  $\mathcal{A}_\varepsilon(u) = \varepsilon u_{xx} + \varepsilon^2 \delta u_{xxx}$ , which has received considerable attention (cf. [23] and the monograph [26] and the references cited therein). We mention in particular the seminal paper [24] in which  $f(u) = u^3$ . There, for  $\delta > 0$  an explicit function  $\varphi(u; \delta)$  is derived such that the shock  $\{u_\ell, u_r\}$  is admissible if and only if  $u_r = \varphi(u_\ell; \delta)$ . Properties of this *kinetic function*  $\varphi$ , such as monotonicity with respect to  $u_\ell$ , have been studied in a series of papers (see [6] and the references cited there).

Other regularizations have been studied in [7] and [8] where a fourth order viscosity term was introduced motivated by thin film flow ( $\mathcal{A}_\varepsilon(u) = -(u^3 u_{xxx})_x$ , and the flux function is  $f(u) = u^2 - u^3$ ) and in [18], where fourth order regularizations are used, motivated by problems in image processing. Traveling waves for dynamic capillarity models, but for a convex flux function, are investigated in [11].

In this paper we focus on the relation between  $u_\ell$  and the parameter  $\tau$ . With  $\beta$  being defined in Proposition 1.1 (see also Figure 2) we establish the existence of a function  $\tau(u_\ell)$  defined for  $\alpha < u_\ell < \beta$  such that (1.17) has a traveling wave solution with  $u_r = 0$ , if and only if  $\tau = \tau(u_\ell)$ . We shall show that this function is *monotone*, *continuous*, and has limits

$$\tau(u_\ell) \rightarrow \tau_* > 0 \quad \text{as} \quad u_\ell \searrow \alpha \quad \text{and} \quad \tau(u_\ell) \rightarrow \infty \quad \text{as} \quad u_\ell \nearrow \beta.$$

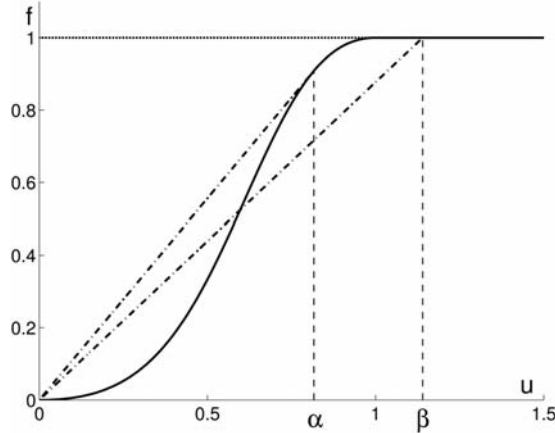


FIG. 2. Critical values of  $u$  when  $M = 2$ :  $\alpha \approx 0.816$  and  $\beta \approx 1.147$ .

Thus  $\tau$  serves as a bifurcation parameter: for  $0 < \tau \leq \tau_*$  the situation will be much like in the classical case (E), but for  $\tau > \tau_*$  the situation changes abruptly and new types of shock waves become admissible. Note that in the framework of [26] we have  $0 = \varphi(u_\ell; \tau(u_\ell))$ .

The properties of the function  $\tau(u_\ell)$  will be based on three existence, uniqueness, and nonexistence theorems, Theorems 1.1, 1.2, and 1.3, for traveling waves of (1.17). Substituting (1.8) into (1.17) we obtain

$$-su' + (f(u))' = u'' - s\tau u''' \quad \text{in } \mathbf{R}.$$

When we integrate this equation over  $(\eta, \infty)$ , we obtain the second order boundary value problem

$$(1.18a) \quad \text{(TW)} \begin{cases} -s(u - u_r) + \{f(u) - f(u_r)\} = u' - s\tau u'' & \text{in } \mathbf{R}, \\ u(-\infty) = u_\ell, \quad u(\infty) = u_r, \end{cases}$$

where  $s = s(u_\ell, u_r)$  is given by the Rankine–Hugoniot condition (1.3).

We consider two cases:

$$(I) \quad u_r = 0, \quad u_\ell > 0 \quad \text{and} \quad (II) \quad u_r > u_\ell > 0.$$

Case I.  $u_r = 0$ . We first establish an upper bound for  $u_\ell$ .

PROPOSITION 1.1. Let  $u$  be a solution of problem (TW) such that  $u_r = 0$ . Then,  $u_\ell < \beta$ , where  $\beta$  is the value of  $u$  for which the equal area rule holds:

$$(1.19) \quad \int_0^\beta \left\{ f(u) - \frac{f(\beta)}{\beta}u \right\} du = 0.$$

In Figure 2 we indicate the different critical values of  $u$  in a graph of  $f(u)$  when  $M = 2$ .

Proof. When we put  $u_r = 0$  into (1.18a), multiply by  $u'$ , and integrate over  $\mathbf{R}$ , we obtain the inequality

$$\int_0^{u_\ell} \left\{ f(u) - \frac{f(u_\ell)}{u_\ell}u \right\} du = - \int_{\mathbf{R}} (u')^2(\eta) d\eta < 0,$$

from which it readily follows that  $u_\ell < \beta$ .  $\square$

Next, we turn to the questions of existence and uniqueness. Note that if  $u_\ell \in (\alpha, \beta)$ , then

$$s = s(u_\ell, 0) = \frac{f(u_\ell)}{u_\ell} > f'(u_\ell) \geq f'(0) \quad \text{for} \quad u_\ell > \alpha,$$

and traveling waves, if they exist, lead to an admissibility condition for *fast under-compressive waves*. For convenience we write  $s(u_\ell, 0) = s(u_\ell)$ .

In the theorems below we first show that for each  $\tau > 0$ , there exists a unique value of  $u_\ell \geq \alpha$ , denoted by  $\bar{u}(\tau)$ , for which there exists a solution of problem (TW) such that  $u_r = 0$ .

**THEOREM 1.1.** *Let  $M > 0$  be given. Then there exists a constant  $\tau_* > 0$  such that the following hold:*

- (a) *For every  $0 \leq \tau \leq \tau_*$ , problem (TW) has a unique solution with  $u_\ell = \alpha$  and  $u_r = 0$ .*
- (b) *For each  $\tau > \tau_*$  there exists a unique constant  $\bar{u}_\ell(\tau) \in (\alpha, \beta)$  such that problem (TW) has a unique solution with  $u_\ell = \bar{u}_\ell(\tau)$  and  $u_r = 0$ .*
- (c) *The function  $\bar{u} : [0, \infty) \rightarrow [\alpha, \beta)$  defined by*

$$(1.20) \quad \bar{u}(\tau) = \begin{cases} \alpha & \text{for } 0 \leq \tau \leq \tau_*, \\ \bar{u}_\ell(\tau) & \text{for } \tau > \tau_* \end{cases}$$

*is continuous, strictly increasing for  $\tau \geq \tau_*$ , and  $\bar{u}(\infty) = \beta$ .*

*The solutions in parts (a) and (b) are strictly decreasing.*

We shall refer to  $\bar{u} = \bar{u}(\tau)$  as the *plateau value* of  $u$ . In what follows, we shall often denote the speed  $s(\bar{u})$  of the shock  $\{\bar{u}, 0\}$  by  $\bar{s}$ .

Next, suppose that  $u_\ell \neq \bar{u}(\tau)$ . To deal with this case we need to introduce another critical value of  $u$ , which we denote by  $\underline{u}(\tau)$ .

- For  $\tau \in [0, \tau_*]$  we put  $\underline{u}(\tau) = \alpha$ .
- For  $\tau > \tau_*$  we define  $\underline{u}(\tau)$  as the unique zero in the interval  $(0, \bar{u}(\tau))$  of

$$f(r) - \frac{f(\bar{u})}{\bar{u}}r = 0, \quad 0 < r < \bar{u}.$$

Plainly, if  $\tau > \tau_*$ , then

$$0 < \underline{u}(\tau) < \alpha < \bar{u}(\tau) < \beta \quad \text{for} \quad \tau > \tau_*.$$

In Figure 3, we show graphs of the functions  $\bar{u}(\tau)$  and  $\underline{u}(\tau)$ . They are computed numerically for  $M = 2$  by means of a shooting technique that is explained in section 3. In this case we found

$$\tau_* \approx 0.61.$$

The following theorem states that if  $u_r = 0$  and  $u_\ell \in (0, \bar{u})$ , then traveling waves exist if and only if  $u_\ell < \underline{u}(\tau)$ .

**THEOREM 1.2.** *Let  $M > 0$  and  $\tau > 0$  be given, and let  $\underline{u} = \underline{u}(\tau)$  and  $\bar{u} = \bar{u}(\tau)$ .*

- (a) *For any  $u_\ell \in (0, \underline{u})$ , there exists a unique solution of problem (TW) such that  $u_r = 0$ . We have  $s(u_\ell) < \bar{s}$ .*
- (b) *Let  $\tau > \tau_*$ . Then for any  $u_\ell \in (\underline{u}, \bar{u})$ , there exists no solution of problem (TW) such that  $u_r = 0$ .*

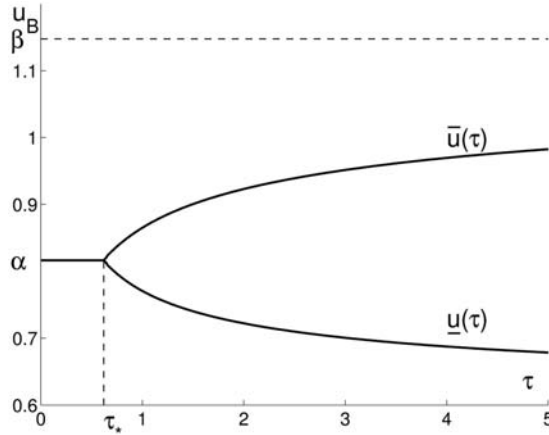


FIG. 3. The functions  $\bar{u}(\tau)$  and  $\underline{u}(\tau)$  computed for  $M = 2$ .

The solution in part (a) may exhibit a damped oscillation as it tends to  $u_\ell$ .

Case II.  $u_r > 0$ . The results of Case I raise the question as to how to deal with solutions of problem (BL) when  $u_B \in (\underline{u}, \bar{u})$ , and by Theorem 1.2 there is no traveling wave solution with  $u_r = 0$ . In this situation we use two traveling waves in succession: one from  $u_B$  to the plateau value  $\bar{u}$ , and one from  $\bar{u}$  down to  $u = 0$ . The existence of the latter has been established in Theorem 1.1. In the next theorem we deal with the former, in which  $u_r = \bar{u}$ .

THEOREM 1.3. Let  $M > 0$  and  $\tau > \tau_*$  be given, and let  $\underline{u} = \underline{u}(\tau)$  and  $\bar{u} = \bar{u}(\tau)$ .

- (a) For any  $u_\ell \in (\underline{u}, \bar{u})$ , there exists a unique solution of problem (TW) such that  $u_r = \bar{u}$ . We have  $s(u_\ell, \bar{u}) < \bar{s}$ .
- (b) For any  $u_\ell \in (0, \underline{u})$ , there exists no solution of problem (TW) such that  $u_r = \bar{u}$ .

The solution in part (a) may exhibit a damped oscillation as it tends to  $u_\ell$ .

In section 2 we show how these theorems can be used to construct weak solutions of problem (BL), i.e., weak solutions, which are admissible within the context of the regularization proposed in (1.17), and which involve shocks which may be either classical or nonclassical. In section 3 we solve the Cauchy problem for (1.17) numerically, starting from a smoothed step function, i.e.,  $u(x, 0) = u_B \tilde{H}(-x)$ , where  $\tilde{H}(x)$  is a regularized Heaviside function and  $M = 2$ . We find that for different values of the parameters  $u_B$ ,  $\tau$ , and  $\varepsilon$  the solution converges to solutions constructed in section 2 as  $t \rightarrow \infty$ . In sections 4 and 5 we prove Theorems 1.1, 1.2, and 1.3. The proofs rely on phase plane arguments. We conclude this paper with a discussion of the dissipation of the entropy function  $u^2/2$  when  $u$  is the solution of the Cauchy problem for (1.17) (cf. section 6).

In this paper we have seen that nonmonotone traveling waves such as those observed in [13] may be explained by a regularization that takes into account properties of two-phase flow. It will be interesting to determine to what extent such results as derived in this paper for the simplified equation (1.17), continue to hold for the full equation (1.16) when realistic functions  $H(u)$  and  $J(u)$  are used. Such equations may be degenerate at  $u = 0$  as well as at  $u = 1$ , and singular behavior, as in the porous media equation [2, 4, 27] may be expected. In this connection it is interesting to mention a numerical study of traveling waves of the original, fully nonlinear equations of this model in [14, 15].



**2. Entropy solutions of problem (BL).** In this section we give a classification of admissible solutions of problem (BL) based on the “extended viscosity model” (1.17), using the results about traveling wave solutions formulated in Theorems 1.1, 1.2, and 1.3. Before doing that we make a few preliminary observations, and we recall the construction based on the classical model (1.7).

Because (1.1) is a first order partial differential equation and  $u_B$  is a constant, any solution of problem (BL) depends only on the combination  $x/t$ , with shocks, constant states, and rarefaction waves as building blocks [29]. The latter are continuous solutions of the form

$$(2.1) \quad u(x, t) = r(\zeta) \quad \text{with} \quad \zeta = \frac{x}{t}.$$

After substitution into (1.1) this yields

$$(2.2) \quad \frac{dr}{d\zeta} \left( -\zeta + \frac{df}{du}(r(\zeta)) \right) = 0.$$

Hence, the function  $r(\zeta)$  satisfies

$$\text{either } r = \text{constant} \quad \text{or} \quad \frac{df}{du}(r(\zeta)) = \zeta.$$

When solving problem (BL), we will combine solutions of (2.2) with admissible shocks, i.e., shocks  $\{u_\ell, u_r\}$  in which  $u_\ell$  and  $u_r$  are such that (1.6), with the a priori selected and physically relevant viscous extension  $\mathcal{A}_\varepsilon$ , has a traveling wave solution  $u(\eta)$  such that  $u(\eta) \rightarrow u_\ell$  as  $\eta \rightarrow -\infty$  and  $u(\eta) \rightarrow u_r$  as  $\eta \rightarrow +\infty$ . Although in the physical context in which the viscous extension employed in (1.17) was derived,  $0 \leq u_B \leq 1$ , we shall drop this restriction. It will be convenient to first assume that  $0 \leq u_B \leq \beta$ . At the end of this section we discuss the case that  $u_B > \beta$ .

All solution graphs shown in this section and the next are numerically obtained solutions of (1.17). They are expressed in terms of the independent variable  $\zeta$  and  $t$ , i.e.,

$$u(x, t) = w(\zeta, t),$$

and considered for fixed  $\varepsilon > 0$  ( $= 1$ ) and for large times  $t$ . We return to the computational aspects in section 3.

Before discussing the implications of the viscous extension in (1.17), we recall the construction of *classical entropy solutions* of problem (BL). It uses (RH) and the entropy condition (E), which was derived for the diffusive viscous extension used in (1.8). We distinguish two cases:

$$(a) \quad 0 \leq u_B \leq \alpha \quad \text{and} \quad (b) \quad \alpha < u_B \leq \beta.$$

*Case (a).*  $0 \leq u_B \leq \alpha$ . This case was discussed in the introduction, where we found that the entropy solution is given by the shock  $\{u_B, 0\}$ .

*Case (b).*  $\alpha < u_B \leq \beta$ . In the introduction we saw that in this case, the shock  $\{u_B, 0\}$  is no longer a classical entropy solution. Instead, in this case the entropy solution is a composition of three functions:

$$(2.3) \quad u(x, t) = v(\zeta) = \begin{cases} u_B & \text{for } 0 \leq \zeta \leq \zeta_B, \\ r(\zeta) & \text{for } \zeta_B \leq \zeta \leq \zeta_*, \\ 0 & \text{for } \zeta_* \leq \zeta < \infty, \end{cases}$$

where  $\zeta_B$  and  $\zeta_*$  are determined by

$$\zeta_B = \frac{df}{du}(u_B) \quad \text{and} \quad \zeta_* = \frac{df}{du}(\alpha) = \frac{f(\alpha)}{\alpha} = s(\alpha),$$

and  $r : [\zeta_B, \zeta_*] \rightarrow [\alpha, u_B]$  by the relation

$$(2.4) \quad \frac{df}{du}(r(\zeta)) = \zeta \quad \text{for} \quad \zeta_B \leq \zeta \leq \zeta_*.$$

Since  $f''(u) < 0$  for  $u \in [\alpha, u_B]$ , (2.4) has a unique solution, and hence  $r(\zeta)$  is well defined. Note that if  $u_B \geq 1$ , then  $\zeta_B = 0$ , because  $f'(u) = 0$  if  $u \geq 1$ .

Solutions corresponding to Case (b) are shown in Figure 4.

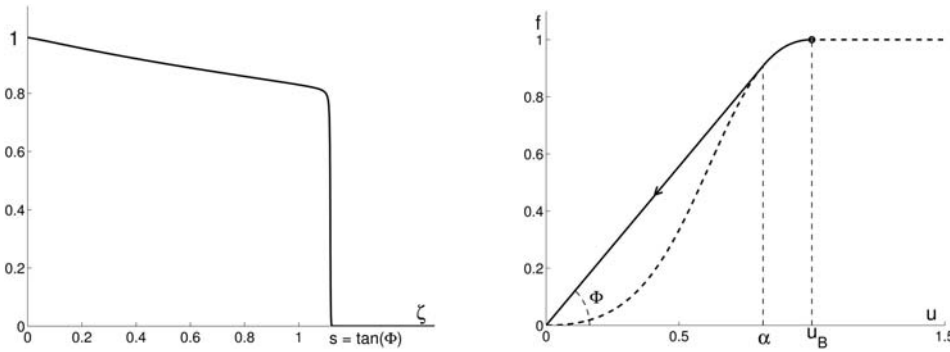


FIG. 4. Case (b). Solution graph (left) and flux function with transitions from  $u_B$  to  $\alpha$  and from  $\alpha$  to 0 (right).

We now turn to the pseudoparabolic equation (1.17) that arises in the context of the two-phase flow model of Hassanizadeh and Gray [19, 20]. For this problem, we define a class of nonclassical entropy solutions in which shocks are admissible if problem (TW) has a traveling wave solution with the required limit conditions.

For given  $M > 0$  and  $\tau > 0$ , the relative values of  $u_B$  and  $\bar{u}(\tau)$  and  $\underline{u}(\tau)$  are now important for the type of solution we are going to get. It is easiest to represent them in the  $(u_B, \tau)$ -plane. Specifically, we distinguish three regions in this plane:

$$\begin{aligned} \mathcal{A} &= \{(u_B, \tau) : \tau > 0, \bar{u}(\tau) \leq u_B < \beta\}, \\ \mathcal{B} &= \{(u_B, \tau) : \tau > \tau_*, \underline{u}(\tau) < u_B < \bar{u}(\tau)\}, \\ \mathcal{C} &= \{(u_B, \tau) : \tau > 0, 0 < u_B < \underline{u}(\tau)\}. \end{aligned}$$

These three regions are shown in Figure 5.

Case I.  $(u_B, \tau) \in \mathcal{A}$ . If  $0 \leq \tau \leq \tau_*$ , i.e.,  $(u_B, \tau) \in \mathcal{A}_1$ , the construction is as in the classical case described above. After a plateau, where  $u = u_B$  and  $0 \leq \zeta = x/t \leq \zeta_B$ , we find a rarefaction wave  $r(\zeta)$  from  $u_B$  down to  $\alpha$  followed by a classical shock connecting  $\alpha$  to the initial state  $u = 0$ .

If  $\tau > \tau_*$ , i.e.,  $(u_B, \tau) \in \mathcal{A}_2$ , the solution starts out as before, with a plateau where  $u = u_B$  and  $0 \leq \zeta \leq \zeta_B$  and a rarefaction wave  $r(\zeta)$  which now takes  $u$  down from  $u_B$  to  $\bar{u} > \alpha$ . This takes place over the interval  $\zeta_B \leq \zeta \leq \bar{\zeta}$ . By (2.2),

$$\bar{\zeta} = \frac{df}{du}(\bar{u}(\tau)).$$

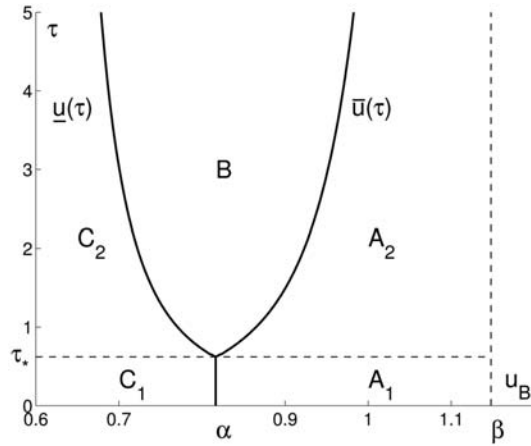


FIG. 5. The regions  $A$ ,  $B$ , and  $C$  in the  $(u_B, \tau)$ -plane.

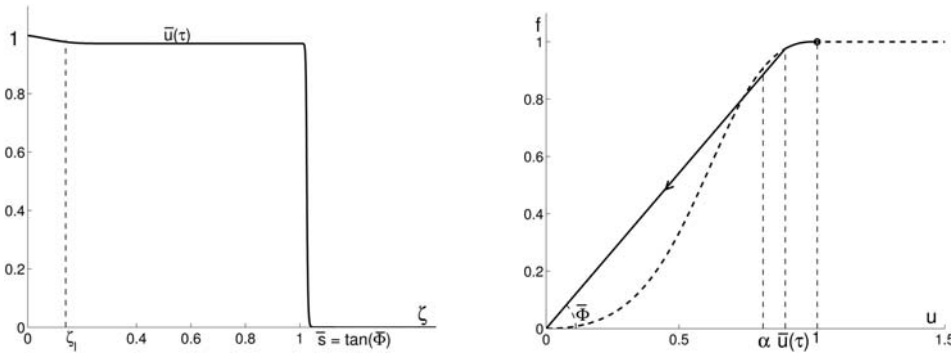


FIG. 6. Case I. Solution graph (left) and flux function (right), with transitions from  $u_B = 1$  to  $\bar{u}(\tau)$  and from  $\bar{u}(\tau)$  to 0.

Subsequently,  $u$  drops down to the initial state  $u = 0$  through a shock,  $\{\bar{u}, 0\}$ , which is admissible by Theorem 1.1. By (RH) the shock moves with speed

$$s = \bar{s} = \frac{f(\bar{u})}{\bar{u}} > \frac{df}{du}(\bar{u}) = \bar{\zeta},$$

because  $f$  is concave on  $(\alpha, \infty)$ . Therefore, the shock outruns the rarefaction wave and a second plateau develops between the rarefaction wave and the shock in which  $u = \bar{u}$ . Summarizing, we find that the (nonclassical) entropy solution has the form

$$(2.5) \quad u(x, t) = v(\zeta) = \begin{cases} u_B & \text{for } 0 \leq \zeta \leq \zeta_B, \\ r(\zeta) & \text{for } \zeta_B \leq \zeta \leq \bar{\zeta}, \\ \bar{u}(\tau) & \text{for } \bar{\zeta} \leq \zeta \leq \bar{s}, \\ 0 & \text{for } \bar{s} \leq \zeta < \infty. \end{cases}$$

A graph of  $v(\zeta)$  is given in Figure 6.

Note that if  $u_B \geq 1$ , then  $\zeta_B = 0$ . At this point  $v$  shocks to the maximum of  $\bar{u}(\tau)$  and 1. If  $\bar{u}(\tau) \geq 1$ , then the rarefaction wave disappears and for  $\zeta > 0$  the solution is continued by the shock  $\{\bar{u}(\tau), 0\}$ .

*Case II.*  $(u_B, \tau) \in \mathcal{B}$ . It follows from Theorem 1.2 that there are no traveling wave solutions with  $u_\ell = u_B$  and  $u_r = 0$ , so that the shock  $\{u_B, 0\}$  is now not admissible. However, in Theorem 1.3 we have shown that there does exist a traveling wave solution, and hence an admissible shock, with  $u_\ell = u_B$  and  $u_r = \bar{u}(\tau)$ , and speed  $s = s(u_B, \bar{u}(\tau))$ . This shock is then followed by a second shock from  $u = \bar{u}(\tau)$  down to  $u = 0$ , which is admissible because by Theorem 1.1 there does exist a traveling wave solution which connects  $\bar{u}$  and  $u = 0$  with speed  $\bar{s} > s(u_B, \bar{u}(\tau))$ . Thus

$$(2.6) \quad u(x, t) = v(\zeta) = \begin{cases} u_B & \text{for } 0 \leq \zeta \leq s(u_B, \bar{u}), \\ \bar{u}(\tau) & \text{for } s(u_B, \bar{u}) \leq \zeta \leq \bar{s}, \\ 0 & \text{for } \bar{s} \leq \zeta < \infty. \end{cases}$$

An example of this type of solution is shown in Figure 7. The undershoot in the solution graph is due to oscillations which are also present in the traveling waves.

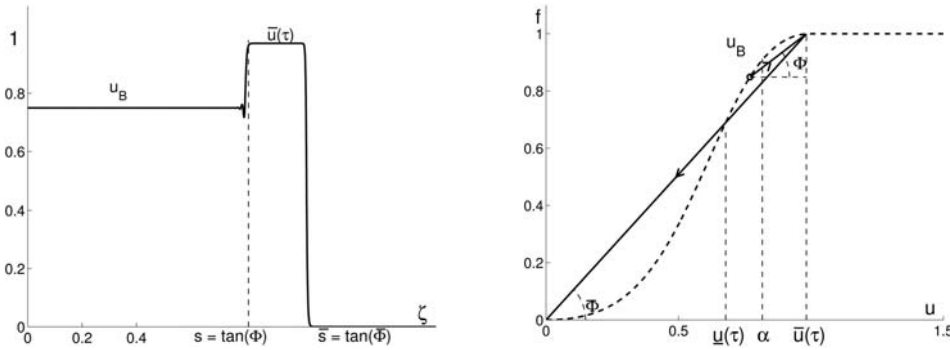


FIG. 7. *Case II.* Solution graph (left) and flux function (right), with transitions from  $u_B = 0.75$  to  $\bar{u}(\tau)$  and from  $\bar{u}(\tau)$  to 0.

*Remark 2.1.* It is readily seen that

$$s(u_B, \bar{u}(\tau)) \nearrow \bar{s} \quad \text{as} \quad u_B \searrow \underline{u},$$

while the plateau level  $\bar{u}$  remains the same. Thus, in this limit, the plateau

$$\left\{ \left( u, \frac{x}{t} \right) : u = \bar{u}(\tau), s(u_B, \bar{u}(\tau)) < \frac{x}{t} < \bar{s} \right\}$$

becomes thinner and thinner and eventually disappears when  $u_B = \underline{u}$ .

*Remark 2.2.* If  $u_B = 1$  and  $\bar{u}(\tau) > 1$ , then the first shock degenerates in the sense that

$$s(u_B, \bar{u}(\tau)) = 0 \quad \text{and} \quad u(x, t) = \bar{u}(\tau) \quad \text{for all } 0 < \frac{x}{t} < \bar{s}.$$

*Case III.*  $(u_B, \tau) \in \mathcal{C}$ . We have seen in Theorem 1.2 that in this case there exists a traveling wave solution with  $u_\ell = u_B$  and  $u_r = 0$ . It may exhibit oscillatory behavior near  $u = u_\ell$ , and it leads to the classical entropy shock solution  $\{u_B, 0\}$ . An example of such a solution is shown in Figure 8. Note the overshoot in the solution graph, reflecting oscillations also present in the traveling waves.

We conclude with a remark about the case when  $u_B > \beta$ . It is readily verified that for such values of  $u_B$  the situation is completely analogous to the one for  $(u_B, \tau) \in \mathcal{A}$ .

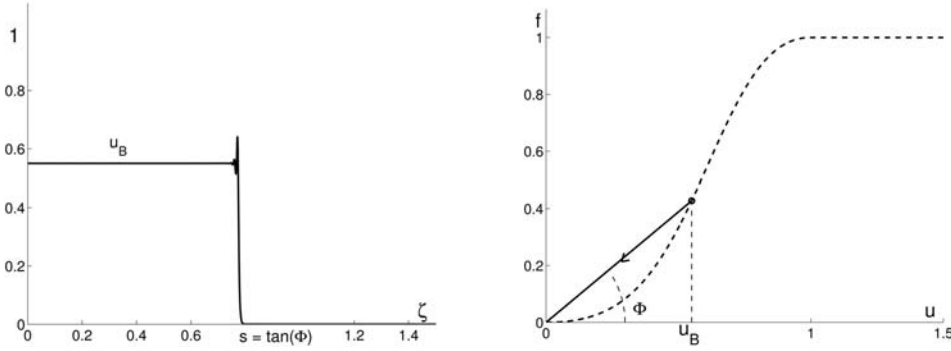


FIG. 8. Case III. Solution graph (left) and flux function (right), with transition from  $u_B = 0.55$  to 0.

**3. Numerical experiments for large times.** In this section we report on the computations carried out for obtaining the numerical results presented in this paper. All computations are done for  $M = 2$ . We start with the calculation of the diagram in Figure 3. For determining the graphs of  $\bar{u}$  and  $\underline{u}$  as functions of  $\tau$  we fix  $u_r = 0$ . Then, given a  $\tau > 0$  and a left state  $u_\ell \geq 0$ , we look for a strictly decreasing solution  $u(\eta)$  of the problem (1.18a) and (1.18b). If such a solution exists, we can invert the function  $u(\eta)$  and define the new dependent variable  $z(u) = -u'(\eta(u))$ , which satisfies

$$s\tau z z' + z = su - f(u)$$

on the open interval  $(0, u_\ell)$ . Moreover, we have  $z > 0$  on  $(0, u_\ell)$ , and  $z(0) = z(u_\ell) = 0$ .

Following Theorem 1.1, an  $\tau_* > 0$  exists so that solutions  $z$  to the given first order equation and boundary conditions are possible for any  $\tau \leq \tau_*$ , and with  $u_\ell = \alpha$ . To compute  $\tau_*$  we fix  $u_\ell = \alpha$  and solve the equation in  $z$  with  $z(0) = 0$ . We start with a sufficiently small  $\tau > 0$  and increase its value until  $z(u_\ell)$  becomes strictly positive. This gives

$$\tau_* \approx 0.61.$$

Further, for  $\tau > \tau_*$  there is a unique  $u_\ell = \bar{u}(\tau) \in (\alpha, \beta)$  yielding a solution  $z$  with the required properties. Moreover,  $\bar{u}$  is strictly increasing in  $\tau$ . For finding the corresponding  $u_\ell$  we solve numerically the equation in  $z$  with the initial value  $z(0) = 0$ . We repeat this procedure for different values of  $\tau$ , starting close to  $\tau_*$  and increasing gradually the difference between two successive values of  $\tau$  as the corresponding  $u_\ell$  approaches  $\beta$ . Accurate computations with different ODE solvers have led to negligible differences in the resulting diagrams. Finally, the function  $\underline{u}(\tau)$  follows from a simple construction involving  $f(u)$ .

Nonstandard shock solutions of a hyperbolic conservation are computed numerically in [21] and [22]. The schemes considered there are applied to the hyperbolic problem, but they actually solve more accurately a regularized problem involving a  $\partial_{xxx}$  term. This term vanishes as the discretization parameters are approaching 0.

Here we consider the regularized initial value problem for (1.17) in the domain  $S = \mathbf{R} \times \mathbf{R}^+$ :

$$\begin{aligned} (3.1a) \quad & \begin{cases} \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2} + \varepsilon^2 \tau \frac{\partial^3 u}{\partial x^2 \partial t} & \text{in } S, \\ u(x, 0) = u_B \tilde{H}(-x) & \text{for } x \in \mathbf{R}. \end{cases} \end{aligned}$$

Here  $\tilde{H}(x)$  is a smooth monotone approximation of the Heaviside function  $H$ . We use  $\tilde{H}$  instead of  $H$  because discontinuities in the initial conditions will persist for all  $t > 0$ , as shown in [12]. This would require an adapted and more complicated numerical approach for ensuring the continuity in flux and pressure (see, for example, [10], or [9, Chapter 3]). By the above choice we avoid this unnecessary complication.

Important parameters in this problem are  $M, \varepsilon, \tau > 0$ , and  $u_B \in (0, 1]$ . The scaling

$$(3.2) \quad x \rightarrow \frac{x}{\varepsilon}, \quad t \rightarrow \frac{t}{\varepsilon}$$

removes the parameter  $\varepsilon$  from (3.1a). Therefore, we fix  $\varepsilon = 1$  and show how for different values of  $\tau$  and  $u_B$  the solution  $u(x, t)$  of problem (3.1) converges as  $t \rightarrow \infty$  to qualitatively different final profiles.

For solving (3.1) numerically we consider a first order time stepping, combined with the finite difference discretization of the terms involving  $\partial_{xx}$ . To deal with the first order term we apply a minmod slope limiter method that is based on first order upwinding and Richtmyer’s scheme. Specifically, with  $k > 0$  and  $h > 0$  being the discretization parameters, we define  $x_i = ih$  ( $i \in \mathbb{Z}$ ) and  $t_n = nk$  ( $n \in \mathbb{N}$ ), and let  $u_i^n$  stand for the numerical

approximation of  $u(x_i, t_n)$ . With

$$\Delta_{h,i}(u) := \frac{1}{h^2}(u_{i+1} - 2u_i + u_{i-1}),$$

the fully discrete counterpart of (3.1a) in  $(x_i, t_k)$  reads

$$u_i^{n+1} - u_i^n + \frac{k}{h}(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n) = k\varepsilon\Delta_{h,i}(u^{n+1}) + \varepsilon^2\tau\Delta_{h,i}(u^{n+1} - u^n),$$

where  $F_{i+\frac{1}{2}}^n$  is the numerical flux at  $x_{i+\frac{1}{2}} = x_i + \frac{h}{2}$  and  $t_n$ . As mentioned above,  $F_{i+\frac{1}{2}}^n$  is a convex combination of the first order upwind flux and the second order Richtmyer flux:

$$F_{i+\frac{1}{2}}^n = (1 - \Theta_i^n)F_{i+\frac{1}{2}}^{n,low} + \Theta_i^n F_{i+\frac{1}{2}}^{n,high},$$

where

$$\begin{aligned} F_{i+\frac{1}{2}}^{n,low} &= f(u_i^n), \\ F_{i+\frac{1}{2}}^{n,high} &= f(w_i^n), \quad \text{for } w_i^n = \frac{u_i^n + u_{i+1}^n}{2} - \frac{k}{2h}(f(u_{i+1}^n) - f(u_i^n)), \end{aligned}$$

and

$$\Theta_i^n = \max(0, \min(1, \theta_i^n)), \quad \text{for } \theta_i^n = \begin{cases} 0 & \text{if } u_i^n = u_{i+1}^n, \\ \frac{u_i^n - u_{i-1}^n}{u_{i+1}^n - u_i^n} & \text{otherwise.} \end{cases}$$

To compute the numerical solution, we restrict (3.1) to the sufficiently large spatial interval  $(-1000, 5000)$ , and define the artificial boundary conditions  $u(-1000, t) = u_B$  and  $u(5000, t) = 0.0$ . The computations are performed for large times ( $t > 2000$ ), as long as the results are not affected by the presence of the boundaries. We apply the discretization scheme mentioned above, yielding a linear tridiagonal system that is

solved at each time step. A convergence proof for the numerical scheme is beyond the scope of the present work. Similar numerical schemes for problems of pseudoparabolic type are considered, for example, in [1] and [16], where also the convergence is proven.

In the figures below, we show graphs of solutions at various times  $t$ , appropriately scaled in space. Specifically, we show graphs of the function

$$(3.3) \quad w(\zeta, t) = u(x, t), \quad \text{where} \quad \zeta = \frac{x}{t},$$

so that a front with speed  $s$  will be located at  $\zeta = s$ .

We recall that the numerical results are obtained for  $M = 2$ . In this case  $\tau_* \approx 0.61$  (see also Figures 3 and 5). We begin with a simulation where  $(u_B, \tau) = (1, 0.2) \in \mathcal{A}_1$ . In Figure 9 we show the resulting solution  $w(\zeta, t)$  at time  $t = 1000$ . It is evident that  $w$  converges to the classical entropy solution constructed in section 2.

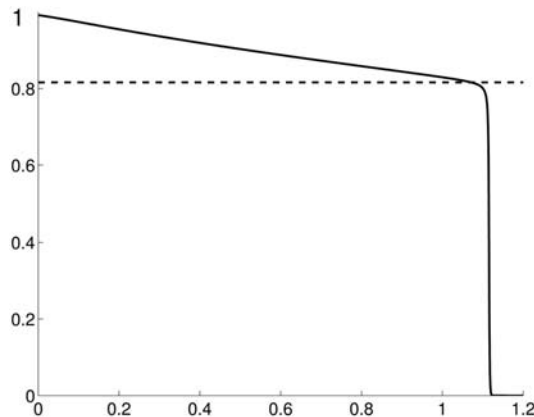


FIG. 9. Graph of  $w(\zeta, t)$  at  $t = 1000$  when  $(u_B, \tau) = (1, 0.2) \in \mathcal{A}_1$ . In this case  $\bar{u}(\tau) = \alpha \approx 0.816$  and  $s \approx 1.11$ .

In the simulations that we present in the remainder of this section we take  $\tau$  to be fixed above  $\tau_*$ :  $\tau = 5$ . Correspondingly, by the ODE method involved in computing the diagram in Figure 3 we obtained  $\bar{u}(\tau = 5) \approx 0.98$  and  $\underline{u}(\tau = 5) \approx 0.68$ . In the first of these experiments, in which we keep  $u_B = 1$ , we see that for large time the graph consists of three pieces: one in which  $w$  gradually decreases from  $w = u_B = 1$  to the “plateau” value  $w = \bar{u}$ , one in which  $w$  is constant and equal to  $\bar{u}$ , and one in which it drops down to  $w = 0$ ; see Figure 10(a). It is clear from the graph that  $\bar{u} > \alpha$ . The plateau value  $\bar{u} \approx 0.98$  computed here is in excellent agreement with the value obtained by the ODE method; see also Figures 3 and 5.

In the next experiment we decrease  $u_B$  to  $u_B = 0.9$ . We are then in the region  $\mathcal{B}$ . For large times the solution  $w(\zeta, t)$  develops two shocks, one where it jumps up from  $u_B$  to the plateau at  $\bar{u} \approx 0.98$  (the same value as in the previous experiment), and one where it jumps down from  $\bar{u}$  to  $w = 0$ ; see Figure 10(b).

In the next experiments we decrease the value of  $u_B$  to values around the value  $\underline{u} \approx 0.68$ . The results are shown in Figure 11, where we have zoomed into the front. We see that, as  $u_B$  decreases and approaches the boundary between the regions  $\mathcal{B}$  and  $\mathcal{C}_2$  in Figure 5, the part of the graph where  $w \approx u_B$  grows at the expense of the part where  $w \approx \bar{u}$ .

Finally, in Figure 12 we show the graph of  $w(\zeta, t)$  when  $\tau = 5$  and  $u_B$  is further

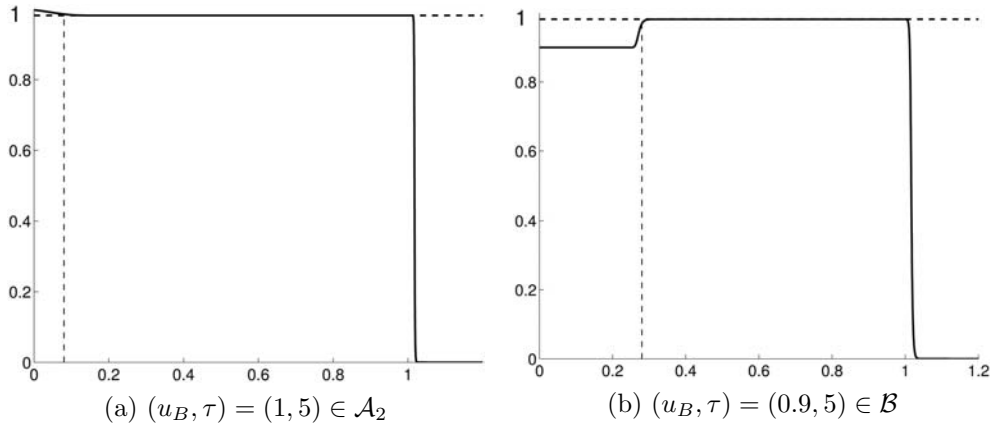


FIG. 10. Graphs of  $w(\zeta, t)$  at  $t = 1000$  when  $(u_B, \tau) = (1, 5) \in \mathcal{A}_2$  (left) and  $(u_B, \tau) = (0.9, 5) \in \mathcal{B}$  (right). Here  $\bar{u}(\tau) \approx 0.98$  and  $s \approx 1.02$ , while  $\zeta_\ell \approx 0.08$  (left) and  $s_B \approx 0.28$  (right).

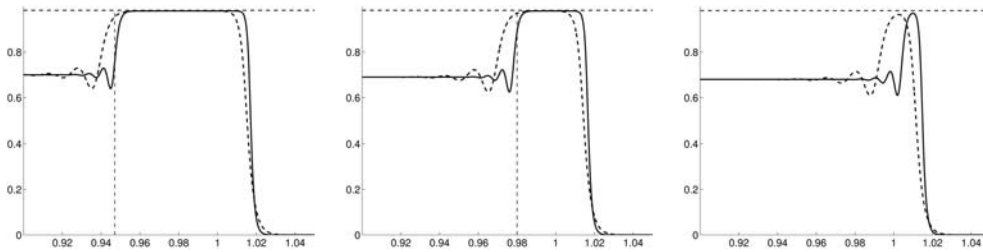


FIG. 11. Graphs of  $w(\zeta, t)$  with  $\tau = 5$  at  $t = 1000$  (dashed) and  $t = 2000$  (solid); zoomed view:  $0.9 \leq \zeta \leq 1.05$ . Here  $\bar{u}(\tau) \approx 0.68$  and  $u_B$  approaches  $\bar{u}(\tau)$  from above through 0.70 (left), 0.69 (middle), and 0.68 (right). Then  $s_B$  increases from 0.95 (left) to 0.98 (middle) up to 1.02 (right). The other values are  $\bar{u}(\tau) \approx 0.98$  and  $s \approx 1.02$ .

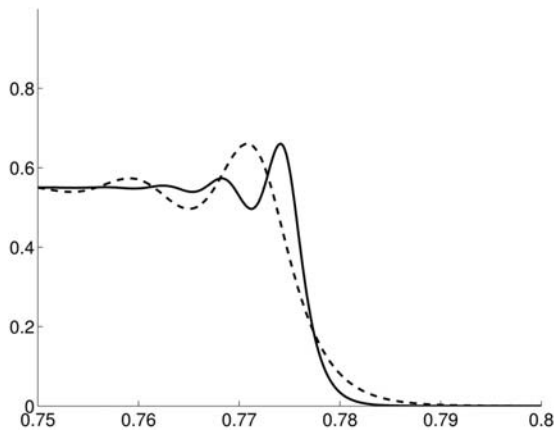


FIG. 12. Graphs of  $w(\zeta, t)$  at  $t = 1000$  (dashed) and  $t = 2000$  (solid) when  $(u_B, \tau) = (0.55, 5) \in \mathcal{C}_2$ ; zoomed view:  $0.75 \leq \zeta \leq 0.8$ . Then  $s \approx 0.78$ .

reduced to 0.55, so that we are now in  $\mathcal{C}_2$ . We find that the solution no longer jumps up to a higher plateau, but instead jumps right down after a small oscillation.

Note that the oscillations in Figures 11 and 12 contract around the shock as time



progresses. This is due to the scaling, since we have plotted  $w(\zeta, t)$  versus  $\zeta = x/t$  for different values of time  $t$ .

We conclude from these simulations that the entropy solutions constructed in section 2 emerge as limiting solutions of the Cauchy problem (3.1). This suggests that these entropy solutions enjoy certain stability properties. It would be interesting to see whether these same entropy solutions would emerge if the initial value were chosen differently. We leave this question to a future study.

**4. Proof of Theorem 1.1.** In Theorem 1.1 we considered traveling wave solutions  $u(\eta)$  of (1.17) in which the limiting conditions had been chosen so that  $u(-\infty) = u_\ell \geq \alpha$  and  $u(\infty) = u_r = 0$ . Putting  $u_r = 0$  in (1.18a) and (1.18b) we find that they are solutions of the problem

$$\begin{aligned} (4.1a) \quad & s\tau u'' - u' - su + f(u) = 0 \quad \text{for} \quad -\infty < \eta < \infty, \\ (4.1b) \quad & \text{(TW}_0) \begin{cases} u(-\infty) = u_\ell, & u(+\infty) = 0, \end{cases} \end{aligned}$$

in which the speed  $s$  is a priori determined by  $u_\ell$  through

$$(4.2) \quad s = s(u_\ell) \stackrel{\text{def}}{=} \frac{f(u_\ell)}{u_\ell}.$$

The proof proceeds in a series of steps.

*Step 1.* We choose  $u_\ell \in (\alpha, \beta)$  and prove that there exists a unique  $\tau > 0$  for which problem (TW<sub>0</sub>) has a solution, which is also unique. This defines a function  $\tau = \tau(u_\ell)$  on  $(\alpha, \beta)$ . We then show that  $\tau(u_\ell)$  is increasing, continuous, and that

$$\tau(u) \rightarrow \infty \quad \text{if} \quad u \rightarrow \beta.$$

Finally, we write

$$\tau_* \stackrel{\text{def}}{=} \lim_{u \rightarrow \alpha^+} \tau(u).$$

*Step 2.* We show that for any  $\tau \in (0, \tau_*]$ , problem (TW<sub>0</sub>) has a solution with  $u_\ell = \alpha$ .

The proof is concluded by defining the function  $\bar{u}_\ell(\tau)$  on  $(\tau_*, \infty)$  as the inverse of the function  $\tau(u_\ell)$  on the interval  $(\alpha, \beta)$ . The resulting function  $\bar{u}(\tau)$ , defined by (1.17) on  $\mathbf{R}^+$ , then has all the properties required in Theorem 1.1.

**4.1. The function  $\tau(u)$ .** As a first result we prove that  $\tau(u)$  is well defined on the interval  $(\alpha, \beta)$ .

**LEMMA 4.1.** *For each  $u_\ell \in (\alpha, \beta)$  there exists a unique value of  $\tau$  such that there exists a solution of problem (TW<sub>0</sub>). This solution is unique and decreasing.*

*Proof.* It is convenient to write (4.1a) in a more conventional form, and introduce the variables

$$\xi = -\eta/\sqrt{s\tau} \quad \text{and} \quad \tilde{u}(\xi) = u(\eta).$$

In terms of these variables, problem (TW<sub>0</sub>) becomes

$$\begin{aligned} (4.3a) \quad & u'' + cu' - g(u) = 0 \quad \text{in} \quad -\infty < \xi < \infty, \\ (4.3b) \quad & \begin{cases} u(-\infty) = 0, & u(+\infty) = u_\ell, \end{cases} \end{aligned}$$

where

$$(4.4) \quad c = \frac{1}{\sqrt{s\tau}} \quad \text{and} \quad g(u) = su - f(u),$$

and the tildes have been omitted. Graphs of  $g(u)$  for  $M = 2$  and different values of  $s$  are shown in Figure 13.

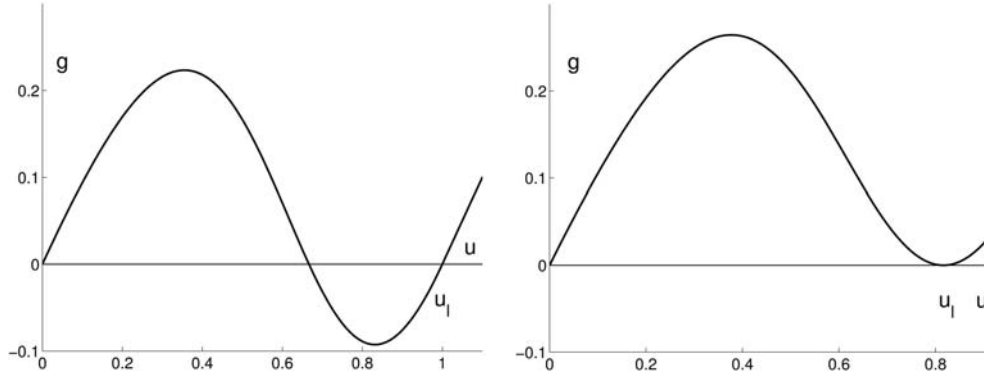


FIG. 13. The function for  $g(u)$  for  $M = 2$ , and  $s = 0.95$  (left) and  $s = s(\alpha) = 1.113$  (right).

We study problem (4.3) in the phase plane and write (4.3a) as the first order system

$$(4.5a) \quad \mathcal{P}(c, s) \begin{cases} u' = v, \\ (4.5b) \quad v' = -cv + g(u). \end{cases}$$

For  $u_\ell \in (\alpha, \beta)$  the function  $g(u)$  has three distinct zeros, which we denote by  $u_i$ ,  $i = 0, 1$ , and  $2$ , where

$$u_0 = 0 \quad \text{and} \quad u_1 < \alpha < u_2 = u_\ell.$$

Plainly the points  $(u, v) = (u_i, 0)$ ,  $i = 0, 1, 2$ , are the equilibrium points of (4.5) with associated eigenvalues

$$(4.6) \quad \lambda_\pm = -\frac{c}{2} \pm \frac{1}{2} \sqrt{c^2 + 4g'(u_i)}.$$

Since

$$g'(u_0) > 0, \quad g'(u_1) < 0, \quad \text{and} \quad g'(u_2) > 0,$$

the outer points,  $(u_0, 0)$  and  $(u_2, 0)$ , are saddles and  $(u_1, 0)$  is either a stable node or a stable spiral.

Since we are interested in a traveling wave with  $u(-\infty) = 0$  and  $u(+\infty) = u_\ell$ , we need to investigate orbits which connect the points  $(0, 0)$  and  $(u_\ell, 0)$ . The existence of a unique wave speed  $c$  for which there exists such a solution of the system  $\mathcal{P}(c, s)$ , which is unique and decreasing, has been established in [25]; see also [17]. This allows us to define the function  $c = c(u_\ell)$  for  $\alpha < u_\ell < \beta$ .

By definition,  $c(u_\ell)$  only takes on positive values. This is consistent with the identity, obtained by multiplying (4.3a) by  $u'$  and integrating the result over  $\mathbf{R}$ :

$$(4.7) \quad c \int_{\mathbf{R}} \{u'(\xi)\}^2 d\xi = \int_{\mathbf{R}} g(u(\xi)) u'(\xi) d\xi = \int_0^{u_\ell} g(t) dt \stackrel{\text{def}}{=} G(u_\ell),$$

because  $G(u_\ell) > 0$  when  $0 < u_\ell < \beta$ .

Finally, by (4.2) and (4.4), we find that  $\tau$  is uniquely determined by  $u_\ell$  through the relation

$$(4.8) \quad \tau(u_\ell) = \frac{1}{s(u_\ell)c^2(u_\ell)}.$$

This completes the proof of Lemma 4.1.  $\square$

Lemma 4.1 allows us to define a function  $\tau(u)$  on  $(\alpha, \beta)$ , such that if  $u_\ell \in (\alpha, \beta)$ , then problem  $(TW_0)$  has a unique solution  $u(\eta)$  if and only if  $\tau = \tau(u_\ell)$ . In the next lemma we show that the function  $\tau(u)$  is strictly increasing on  $(\alpha, \beta)$ .

LEMMA 4.2. *Let  $u_{\ell,i} = \gamma_i$  for  $i = 1, 2$ , where  $\gamma_1 \in (\alpha, \beta)$ , and let  $\tau(\gamma_i) = \tau_i$ . Then*

$$\gamma_1 < \gamma_2 \quad \implies \quad \tau_1 < \tau_2.$$

*Proof.* For  $i = 1, 2$  we write

$$s_i = \frac{f(\gamma_i)}{\gamma_i} \quad \text{and} \quad g_i(u) = s_i u - f(u).$$

Since

$$\frac{d}{du} \left( \frac{f(u)}{u} \right) = \frac{1}{u} \left( f'(u) - \frac{f(u)}{u} \right) < 0 \quad \text{for} \quad \alpha \leq u < \beta,$$

it follows that

$$(4.9) \quad \gamma_1 < \gamma_2 \quad \implies \quad s_1 > s_2 \quad \text{and} \quad g_1(u) > g_2(u) \quad \text{for} \quad u > 0.$$

To prove Lemma 4.2 we return to the formulation used in the proof of Lemma 4.1. Traveling waves correspond to heteroclinic orbits in the  $(u, v)$ -plane. Those associated with  $\gamma_1$  and  $\gamma_2$  we denote by  $\Gamma_1$  and  $\Gamma_2$ . They connect the origin to  $(\gamma_1, 0)$  and  $(\gamma_2, 0)$ , respectively.

We shall show that

$$(4.10) \quad \gamma_1 < \gamma_2 \quad \implies \quad c_1 = c(\gamma_1) > c(\gamma_2) = c_2.$$

We can then conclude from (4.4) that

$$\tau_2 s_2 > \tau_1 s_1 \quad \implies \quad \tau_2 > \frac{s_1}{s_2} \tau_1 > \tau_1,$$

as asserted.

Thus, suppose to the contrary that  $c_1 \leq c_2$ . We claim that this implies that near the origin the orbit  $\Gamma_1$  lies below  $\Gamma_2$ . Orbits of the system  $\mathcal{P}(c, s)$  leave the origin along the unstable manifold under the angle  $\theta$  given by

$$(4.11) \quad \theta = \theta(c, s) \stackrel{\text{def}}{=} \frac{1}{2} \left\{ \sqrt{c^2 + 4s - c} \right\}.$$

An elementary computation shows that

$$(4.12) \quad \frac{\partial \theta}{\partial c} < 0 \quad \text{and} \quad \frac{\partial \theta}{\partial s} > 0.$$

Hence, since  $s_1 > s_2$  and we assume that  $c_1 \leq c_2$ , it follows that

$$\theta_1 = \theta(c_1, s_1) > \theta(c_2, s_2) = \theta_2,$$

and hence that the orbit  $\Gamma_1$  starts out *above*  $\Gamma_2$ .

Since  $(\gamma_2, 0)$  lies to the right of the point  $(\gamma_1, 0)$  we conclude that  $\Gamma_1$  and  $\Gamma_2$  must intersect. Let us denote the first point of intersection by  $P = (u_0, v_0)$ . Then at  $P$  the slope of  $\Gamma_1$  cannot exceed the slope of  $\Gamma_2$ . The slopes at  $P$  are given by

$$\left. \frac{dv}{du} \right|_{\Gamma_i} = -c_i + \frac{g_i(u_0)}{v_0}, \quad i = 1, 2.$$

Because  $g_1(u) > g_2(u)$  for  $u > 0$  by (4.9), it follows that

$$\left. \frac{dv}{du} \right|_{\Gamma_1} > \left. \frac{dv}{du} \right|_{\Gamma_2} \quad \text{at } P,$$

so that, at  $P$ , the slope of  $\Gamma_1$  exceeds the slope of  $\Gamma_2$ , a contradiction. Therefore we find that  $c_1 > c_2$ , as asserted.  $\square$

In the next lemma we show that the function  $\tau(u)$  is continuous.

LEMMA 4.3. *The function  $\tau : (\alpha, \beta) \rightarrow \mathbf{R}^+$  is continuous.*

*Proof.* Because the function  $s(\gamma) = \gamma^{-1}f(\gamma)$  is continuous, it suffices to show that the function  $c(\gamma)$  is continuous. Since we have shown in the proof of Lemma 4.1 that  $c(\gamma)$  is decreasing (cf. (4.10)), we only need to show that it cannot have any jumps.

Suppose to the contrary that it has a jump at  $\gamma_0$ , and let us write

$$\liminf_{\gamma \searrow \gamma_0^+} c(\gamma) = c^+ \quad \text{and} \quad \limsup_{\gamma \nearrow \gamma_0^-} c(\gamma) = c^-.$$

Then, since  $c(\gamma)$  is decreasing, we may assume that  $c^- > c^+$ .

Thus, there exist sequences  $\{\gamma_n^-\}$  and  $\{\gamma_n^+\}$  with corresponding heteroclinic orbits  $(u_n^\pm, v_n^\pm)$  and wave speeds  $c_n^\pm$ , such that

$$c_n^+ \searrow c^+ \quad \text{and} \quad c_n^- \nearrow c^- \quad \text{as } n \rightarrow \infty.$$

Since the unstable manifold at  $(0, 0)$  and the stable manifold at  $(\gamma, 0)$  depend continuously on  $c$ , it follows that the corresponding orbits also converge, i.e., that there exist orbits  $(u^+, v^+)$  and  $(u^-, v^-)$  such that

$$(u_n^\pm, v_n^\pm)(\xi) \rightarrow (u^\pm, v^\pm)(\xi) \quad \text{as } n \rightarrow \infty,$$

uniformly on  $\mathbf{R}$ . This argument yields two heteroclinic orbits, one with speed  $c^+$  and one with speed  $c^-$ , which both connect the origin to the point  $(\gamma_0, 0)$ . Since by Lemma 4.1 there exists only one such orbit, we have a contradiction.

It follows that  $c^- = c^+$ , and continuity of the function  $c(\gamma)$ , and hence of  $\tau(\gamma)$ , has been established.  $\square$

In the following lemma we prove the final assertion made in Step 1, which involves the behavior of  $\tau(u)$  as  $u \rightarrow \beta$ .

LEMMA 4.4. *We have*

$$\tau(\gamma) \rightarrow \infty \quad \text{as} \quad \gamma \rightarrow \beta^-.$$

*Proof.* In view of the definition (4.7) of  $\tau$ , it suffices to show that  $c(\gamma) \rightarrow 0$  as  $\gamma \rightarrow \beta$ . Proceeding as in the proof of Lemma 4.3, we find that  $c(\gamma)$  and the orbit  $\Gamma(c(\gamma))$  converge to  $c_0$  and  $\Gamma(c_0) = \{(u_0, v_0)(t) : t \in \mathbf{R}\}$  as  $\gamma \rightarrow \beta$ . Note that

$$c(\gamma) \int_{\mathbf{R}} v^2(\xi; \gamma) d\xi = \int_0^\gamma g(t; \gamma) dt,$$

where  $g(t; \gamma) = s(\gamma)t - f(t)$ . If we let  $\gamma \rightarrow \beta$  in this identity, we obtain

$$(4.13) \quad c_0 \int_{\mathbf{R}} v_0^2(\xi) d\xi = \int_0^\beta g(t; \beta) dt = 0.$$

Because at the origin the unstable manifold points into the first quadrant when  $\gamma = \beta$  (cf. (4.9)), it follows that  $v_0 > 0$  on  $\mathbf{R}$ . Therefore, (4.13) implies that  $c_0 = 0$ , as asserted.  $\square$

**4.2. Traveling waves with  $u_\ell = \alpha$ .** In Lemmas 4.1 and 4.2 we have shown that  $\tau(u)$  is an increasing function on  $(\alpha, \beta)$ . Since  $\tau(u) > 0$  for all  $u \in (\alpha, \beta)$ , the limit

$$\tau_* \stackrel{\text{def}}{=} \lim_{u \rightarrow \alpha^+} \tau(u)$$

exists. In the following lemmas we show that  $\tau_* > 0$  and that for all  $\tau \in (0, \tau_*]$ , problem  $(\text{TW}_0)$  has a unique solution with  $u_\ell = \alpha$ .

Let  $\mathcal{S} \in \mathbf{R}^+$  denote the set of values of  $\tau$  for which problem  $(\text{TW}_0)$  has a unique solution with  $u_\ell = \alpha$ .

LEMMA 4.5. *There exists a constant  $\tau_0 > 0$  such that  $(0, \tau_0) \subset \mathcal{S}$ .*

*Proof.* We shall show that there exists a wave speed  $c_0 > 0$  such that if  $c > c_0$ , then problem (4.5) has a heteroclinic orbit connecting the origin to the point  $(\alpha, 0)$ . This then yields Lemma 4.5 when we put

$$\tau_0 = \frac{1}{c_0^2 s(\alpha)}.$$

In (4.6) we saw that the origin is a saddle and that the slope of the unstable manifold is given by

$$\theta(c) = \frac{1}{2} \left\{ \sqrt{c^2 + 4s} - c \right\}.$$

Note that

$$\theta(c) < \frac{1}{c} g'(0) = \frac{s}{c}.$$

Hence, near the origin the orbit lies below the isocline  $\mathcal{I}_v = \{(u, v) : v = c^{-1}g(u), u \in \mathbf{R}\}$ .

Since  $u' > 0$  and  $v' > 0$  in the lens shaped region

$$\mathcal{L} = \{(u, v) : 0 < u < \alpha, 0 < v < c^{-1}g(u), u \in \mathbf{R}\},$$

the orbit will leave  $\mathcal{L}$  again. To see what happens next, we consider the triangular region  $\Omega_m$  bounded by the positive  $u$ - and  $v$ -axis and the line

$$\ell_m \stackrel{\text{def}}{=} \{(u, v) : v = m(\alpha - u)\}, \quad m > 0.$$

On the axes the vector field points into  $\Omega_m$ , and on the line  $\ell_m$  it points inwards if

$$(4.14) \quad \left. \frac{dv}{du} \right|_{\ell_m} = -c + \frac{g(u)}{m(\alpha - u)} < -m.$$

Let

$$m_0 = \inf \{m > 0 : g(u) < m(\alpha - u) \text{ on } (0, \alpha)\}.$$

Then

$$-c + \frac{g(u)}{m(\alpha - u)} < -c + \frac{m_0}{m},$$

and (4.14) will hold for values of  $c$  and  $m$  which satisfy the inequality

$$-c + \frac{m_0}{m} < -m$$

or

$$c > m + \frac{m_0}{m}.$$

To obtain the largest range of values of  $c$  for which the vector field points into  $\Omega_m$  we choose  $m$  so that the right-hand side of this inequality becomes smallest; i.e., we put  $m = \sqrt{m_0}$ . We thus find that for

$$c > c_0 \stackrel{\text{def}}{=} 2\sqrt{m_0}$$

the region  $\Omega_{\sqrt{m_0}}$  is invariant, and hence, that the orbit must tend to the point  $(\alpha, 0)$ . This completes the proof of Lemma 4.5.  $\square$

The next lemma gives the structure of the set  $\mathcal{S}$ .

LEMMA 4.6. *If  $\tau_0 \in \mathcal{S}$ , then  $(0, \tau_0] \subset \mathcal{S}$ .*

*Proof.* As in earlier lemmas we prove a related result for problem (4.5). Let  $\mathcal{S}^*$  be the set of values of  $c$  for which there exists a heteroclinic orbit of problem (4.5) from  $(0, 0)$  to  $(\alpha, 0)$ . We show that if  $c_0 \in \mathcal{S}^*$ , then  $[c_0, \infty) \subset \mathcal{S}^*$ . Plainly this implies Lemma 4.6 with  $\tau_0 = 1/(c_0 s^2)$ .

As before, we denote the orbit emanating from the origin by  $\Gamma(c)$ . Suppose that  $c > c_0$ . Then, since  $\theta'(c) < 0$  it follows that  $\theta(c_0) > \theta(c)$ , so that near the origin  $\Gamma(c_0)$  lies above  $\Gamma(c)$ . We claim that  $\Gamma(c_0)$  and  $\Gamma(c)$  will not intersect for  $u \in (0, \alpha)$ . Accepting this claim for the moment, we conclude that since  $\Gamma(c_0)$  tends to  $(\alpha, 0)$ , the orbit  $\Gamma(c)$  must converge to  $(\alpha, 0)$  as well.

It remains to prove the claim. Suppose that  $\Gamma(c_0)$  and  $\Gamma(c)$  do intersect at some  $u \in (0, \alpha)$ , and let  $(u_0, v_0)$  be the first point of intersection. Then

$$(4.15) \quad \left. \frac{dv}{du} \right|_{\Gamma(c)} \geq \left. \frac{dv}{du} \right|_{\Gamma(c_0)} \quad \text{at} \quad (u_0, v_0).$$

But, from the differential equations we deduce that

$$\left. \frac{dv}{du} \right|_{\Gamma(c)} = -c + \frac{g(u_0)}{v_0} < -c_0 + \frac{g(u_0)}{v_0} = \left. \frac{dv}{du} \right|_{\Gamma(c_0)} \quad \text{at} \quad (u_0, v_0),$$

which contradicts (4.15). This proves the claim and so completes the proof of Lemma 4.6.  $\square$

We conclude this section by showing that  $\tau_* \in \mathcal{S}$ , and hence that  $\mathcal{S} = (0, \tau_*]$ .

LEMMA 4.7. *We have  $\mathcal{S} = (0, \tau_*]$ .*

*Proof.* It follows from Lemmas 4.1 and 4.2 that for every  $\varepsilon \in (0, \beta - \alpha)$ , there exists a  $\tau_\varepsilon = \tau(\alpha + \varepsilon) > 0$  such that problem (TW<sub>0</sub>) has a unique traveling wave  $u_\varepsilon(\eta)$  with speed  $s_\varepsilon = s(\alpha + \varepsilon)$ , such that

$$u_\varepsilon(-\infty) = \alpha + \varepsilon \quad \text{and} \quad u_\varepsilon(\infty) = 0.$$

This wave corresponds to a heteroclinic orbit  $\Gamma_\varepsilon = \{(u_\varepsilon(\xi), v_\varepsilon(\xi)) : \xi \in \mathbf{R}\}$  of the system  $\mathcal{P}(c_\varepsilon, s_\varepsilon)$ , where  $c_\varepsilon = 1/\sqrt{s_\varepsilon \tau_\varepsilon}$ , which connects the points  $(0, 0)$  and  $(\alpha + \varepsilon, 0)$ . It leaves the origin along the stable manifold under an angle  $\theta_\varepsilon = \theta(c_\varepsilon, s_\varepsilon)$  and enters the point  $(\alpha + \varepsilon, 0)$  along the stable manifold under the angle

$$\psi_\varepsilon = \psi(c_\varepsilon, s_\varepsilon) = \frac{1}{2} \left\{ -c_\varepsilon - \sqrt{c_\varepsilon^2 + 4g'(\alpha + \varepsilon)} \right\} \rightarrow -c_0 = -\frac{1}{\sqrt{s(\alpha)\tau_*}} \quad \text{as } \varepsilon \rightarrow 0.$$

Reversing time, i.e., replacing  $\xi$  by  $-\xi$ , we can view  $\Gamma_\varepsilon$  as the unique orbit emanating from the point  $(\alpha + \varepsilon, 0)$  into the first quadrant and entering the origin as  $\xi \rightarrow \infty$ . In the limit, as  $\varepsilon \rightarrow 0$ , we find that

$$u_\varepsilon(\xi) \rightarrow u_0(\xi) \quad \text{and} \quad v_\varepsilon(\xi) \rightarrow v_0(\xi) \quad \text{as } \varepsilon \rightarrow 0 \quad \text{for } -\infty < \xi \leq \xi_0,$$

where  $\xi_0$  is any finite number. We claim that

$$u_0(\xi) \rightarrow 0 \quad \text{and} \quad v_0(\xi) \rightarrow 0 \quad \text{as } \xi \rightarrow \infty;$$

i.e.,  $\Gamma_0 \stackrel{\text{def}}{=} \{(u_0(\xi), v_0(\xi)) : \xi \in \mathbf{R}\}$  is a heteroclinic orbit, which connects  $(\alpha, 0)$  and the origin  $(0, 0)$ .

Suppose to the contrary that  $\Gamma_0$  does not enter the origin as  $\xi \rightarrow \infty$  and possibly does not even exist for all  $\xi \in \mathbf{R}$ . Then, since

$$\frac{dv}{du} = -c_0 + \frac{g(u)}{v} > -c_0 \quad \text{if } 0 < u < \alpha, \quad v > 0,$$

$\Gamma_0$  must leave the first quadrant in finite time, either through the  $u$ -axis or through the  $v$ -axis. This means by continuity that for  $\varepsilon$  small enough  $\Gamma_\varepsilon$  must also leave the first quadrant in finite time. Since  $\Gamma_\varepsilon$  is known to enter the origin for every  $\varepsilon > 0$ , and hence never to leave the first quadrant, we have a contradiction. This proves the claim that  $\Gamma_0$  is a heteroclinic orbit, which connects  $(\alpha, 0)$  and  $(0, 0)$ .  $\square$

*Remark 4.1.* It is evident from Lemmas 4.6 and 4.7 that

$$(4.16) \quad \tau_* \geq \tau_0 = \frac{1}{4m_0s(\alpha)},$$

where  $m_0$  was defined in (4.15). For  $M = 2$ , we find that  $s(\alpha) \approx 1.11$ ,  $m_0 \approx 0.70$  and hence  $\tau_0 \approx 0.32$ . Numerically, we find that  $\tau_* \approx 0.61$ .

**5. Proof of Theorems 1.2 and 1.3.** For the proofs of Theorems 1.2 and 1.3 we turn to the system  $\mathcal{P}(c, s)$  defined in section 4. For convenience we restate it here,

$$(5.1a) \quad \mathcal{P}(c, s) \begin{cases} u' = v, \\ v' = -cv + g_s(u), \end{cases}$$

where

$$c = \frac{1}{\sqrt{s\tau}} \quad \text{and} \quad g_s(u) = su - f(u).$$

Part (a) of Theorem 1.2 is readily seen to be a consequence of the following lemma.

LEMMA 5.1. *Let  $\tau > \tau_*$  be given. Then for every  $u_\ell \in (0, \underline{u})$ , there exists a unique heteroclinic orbit of the system  $\mathcal{P}(c, s)$  in which*

$$s = s_\ell = \frac{f(u_\ell)}{u_\ell} \quad \text{and} \quad c = c_\ell = \frac{1}{\sqrt{s_\ell \tau}},$$

which connects  $(0, 0)$  and  $(u_\ell, 0)$ .

*Proof.* Let  $\Gamma_\ell$  and  $\bar{\Gamma}$  denote the orbits of  $\mathcal{P}(c_\ell, s_\ell)$  and  $\mathcal{P}(\bar{c}, \bar{s})$ , where  $\bar{c} = c(\bar{u})$  and  $\bar{s} = s(\bar{u})$ , which enter the first quadrant from the origin. They do this under the angles  $\theta(c_\ell, s_\ell)$  and  $\theta(\bar{c}, \bar{s})$ , respectively. Since  $c_\ell > \bar{c}$  and  $s_\ell < \bar{s}$ , it follows from (4.12) that

$$\theta(c_\ell, s_\ell) < \theta(\bar{c}, \bar{s}).$$

Hence, near the origin,  $\Gamma_\ell$  lies below  $\bar{\Gamma}$ . Thus,  $\Gamma_\ell$  enters the region  $\Omega$  enclosed between  $\bar{\Gamma}$  and the  $u$ -axis. Since

$$\left. \frac{dv}{du} \right|_{\Gamma_\ell} = -c_\ell + \frac{s_\ell u - f(u)}{v} < -\bar{c} + \frac{\bar{s}u - f(u)}{v} = \left. \frac{dv}{du} \right|_{\bar{\Gamma}},$$

it follows that  $\Gamma_\ell$  cannot leave  $\Omega$  though its “top”  $\bar{\Gamma}$ . We define the following subsets of the bottom of  $\Omega$ :

$$\begin{aligned} S_1 &= \{(u, v) : 0 < u < u_\ell, v = 0\}, \\ S_2 &= \{(u, v) : u = u_\ell, v = 0\}, \\ S_3 &= \{(u, v) : u_\ell < u < \bar{u}, v = 0\}. \end{aligned}$$

Inspection of the vector field show that orbits can only leave  $\Omega$  through  $S_3$ . Note that the set  $S_2$  consists of an equilibrium point.

There are two possibilities: *either*  $\Gamma_\ell$  never leaves  $\Omega$ , *or*  $\Gamma_\ell$  leaves  $\Omega$ , necessarily through the set  $S_3$ . In the first case  $\Gamma_\ell$  is a heteroclinic orbit from  $(0, 0)$  to  $(u_\ell, 0)$ , and the proof is complete.

Thus, let us assume that  $\Gamma_\ell$  leaves  $\Omega$  at some point  $(u, v) = (u_0, 0)$ . Consider the energy function

$$\mathcal{H}(u, v) = \frac{1}{2}v^2 - G_{s_\ell}(u),$$

where  $G_{s_\ell}$  is the primitive of  $g_{s_\ell}$  as defined in (4.7), and write  $H(\xi) = \mathcal{H}(u(\xi), v(\xi))$ , when  $(u(\xi), v(\xi))$  is an orbit. Then differentiation shows that

$$H'(\xi) = -c_\ell v^2(\xi) < 0.$$

Since  $\mathcal{H}(0, 0) = 0$ , it follows that

$$\mathcal{H}(u_0, 0) = -G_{s_\ell}(u_0) < 0$$



and that

$$\mathcal{H}(u(\xi), v(\xi)) = \frac{1}{2}v^2 - G_{s_\ell}(u) < -G_{s_\ell}(u_0) \quad \text{for} \quad \xi > \xi_0.$$

This means that

$$G_{s_\ell}(u) > G_{s_\ell}(u_0) > 0 \quad \text{for} \quad \xi > \xi_0.$$

Let

$$u_1 = \inf\{s \in \mathbf{R} : G_{s_\ell}(s) > G_{s_\ell}(u_0) \text{ on } (s, u_0)\}.$$

Since  $G_{s_\ell}(u_0) > 0$  it follows that  $u_1 \in (0, u_\ell)$ . Therefore

$$\left. \begin{aligned} 0 < u_1 < u(x) < u_0 \\ v^2(x) < 2\{G_{s_\ell}(u_\ell) - G_{s_\ell}(u_0)\} \end{aligned} \right\} \quad \text{for} \quad x > x_0.$$

From a simple energy argument we conclude that  $(u(x), v(x)) \rightarrow (u_\ell, 0)$  as  $x \rightarrow \infty$ . This completes the proof of Lemma 5.1.  $\square$

Part (b) follows from the following result.

LEMMA 5.2. *Let  $\tau > \tau_*$  be given. For any  $u_\ell \in (\underline{u}(\tau), \bar{u}(\tau))$  there exists no solution of the system  $\mathcal{P}(c_\ell, s_\ell)$ , with*

$$s_\ell = \frac{f(u_\ell)}{u_\ell} \quad \text{and} \quad c_\ell = \frac{1}{\sqrt{s_\ell \tau}},$$

which connects  $(0, 0)$  and  $(u_\ell, 0)$ .

*Proof.* Let  $\bar{\Gamma}$  denote the orbit corresponding to  $\bar{c}$  and  $\bar{s}$ , which connects  $(0, 0)$  and the point  $(\bar{u}, 0)$ , and let  $\Gamma_\ell$  denote the orbit which corresponds to  $c_\ell$  and  $s_\ell$ . Observe that

$$s_\ell > \bar{s} \quad \text{and} \quad c_\ell < \bar{c},$$

and hence

$$\theta(c_\ell, s_\ell) > \theta(\bar{c}, \bar{s}).$$

Therefore, near the origin,  $\Gamma_\ell$  lies above  $\bar{\Gamma}$ . Hence, to reach the point  $(u_\ell, 0)$ , the orbit  $\bar{\Gamma}_\ell$  has to cross  $\bar{\Gamma}$  somewhere, and at the first point of crossing we must have

$$\frac{dv}{du} \Big|_{\bar{\Gamma}} \geq \frac{dv}{du} \Big|_{\Gamma_\ell}.$$

However, by the equations, we have

$$\frac{dv}{du} \Big|_{\bar{\Gamma}} = -\bar{c} + \frac{g_{\bar{s}}(u)}{u} < -c_\ell + \frac{g_{s_\ell}(u)}{u} = \frac{dv}{du} \Big|_{\Gamma_\ell},$$

so that we have a contradiction.  $\square$

This completes the proof of Theorem 1.2.

The proof of Theorem 1.3 is entirely analogous to that of Theorem 1.2, and we omit it.

**6. Entropy dissipation.** In this section we study the Cauchy problem

$$(6.1a) \quad (CP) \begin{cases} u_t + (f(u))_x = \mathcal{A}_\varepsilon(u) & \text{in } S = \mathbf{R} \times \mathbf{R}^+, \\ u(\cdot, 0) = u_0(\cdot) & \text{on } \mathbf{R}, \end{cases}$$

where

$$(6.2) \quad \mathcal{A}_\varepsilon(u) = \varepsilon u_{xx} + \varepsilon^2 \tau u_{xxt} \quad (\varepsilon > 0).$$

With this choice (6.1a) becomes the regularized BL equation (1.17) for which we obtained traveling wave solutions in the previous sections. In (6.1a) and (6.2) we introduce subscripts to denote partial derivatives. Without further justification we assume that problem (CP) has a smooth, nonnegative, and bounded solution  $u^\varepsilon$  for each  $\varepsilon > 0$ , and that there exists a limit function  $u : S \rightarrow [0, \infty)$  such that for each  $(x, t) \in S$ ,

$$u^\varepsilon(x, t) \rightarrow u(x, t) \quad \text{as } \varepsilon \rightarrow 0.$$

In addition we assume the following structural properties:

(i)  $\|u^\varepsilon\|_\infty < C$  for some constant  $C > 0$ , and for each fixed  $t > 0$ ,

$$\begin{aligned} u^\varepsilon(x, t) &\rightarrow u_\ell \in \mathbf{R}^+ & \text{as } x &\rightarrow -\infty, \\ u^\varepsilon(x, t) &\rightarrow u_r \in \mathbf{R}^+ & \text{as } x &\rightarrow +\infty. \end{aligned}$$

(ii) The partial derivatives of  $u^\varepsilon$  vanish as  $|x| \rightarrow \infty$ .

(iii) Let  $U(s) = \frac{1}{2}s^2$  for  $s \geq 0$ ,  $U_\ell = U(u_\ell)$ , and  $U_r = U(u_r)$ . Then there exists a smooth function  $\lambda_\varepsilon : [0, \infty) \rightarrow \mathbf{R}$  which is uniformly bounded with respect to  $\varepsilon > 0$  in any bounded interval  $(0, T)$ , such that

$$\int_{\mathbf{R}} \{U(u^\varepsilon(x, t)) - G_\varepsilon(x, t)\} dx = 0 \quad \text{for all } t > 0,$$

where  $G_\varepsilon$  is the step function

$$G_\varepsilon(x, t) = U_\ell + (U_r - U_\ell)H(x - \lambda_\varepsilon(t)), \quad (x, t) \in S$$

in which  $H$  denotes the Heaviside function.

Note that the traveling waves constructed in this paper all have these properties.

*Remark 6.1.* The question as to which conditions on  $u_0$  would generate such a solution is left open in this paper. Clearly we need that  $u_0 : \mathbf{R} \rightarrow \mathbf{R}$  satisfies (i) and (ii), and  $U(u_0) - G \in L^1(\mathbf{R})$ . Further we require that  $u'_0 \in L^2(\mathbf{R})$ .

The main purpose of this section is to show that  $U(u^\varepsilon)$  is an entropy for (6.1a). In doing so we borrow arguments and ideas of LeFloch [26]. For completeness we recall some definitions. We say that the term  $\mathcal{A}_\varepsilon(u)$  is *conservative* if

$$(6.3) \quad \lim_{\varepsilon \rightarrow 0^+} \int_S \mathcal{A}_\varepsilon(u^\varepsilon) \varphi = 0 \quad \text{for all } \varphi \in C_0^\infty(S)$$

and we say that  $\mathcal{A}_\varepsilon(u)$  is *entropy dissipative* (for an entropy  $U$ ) if

$$(6.4) \quad \limsup_{\varepsilon \rightarrow 0^+} \int_S \mathcal{A}_\varepsilon(u^\varepsilon) U'(u^\varepsilon) \varphi \leq 0 \quad \text{for all } \varphi \in C_0^\infty(S), \varphi \geq 0.$$

We establish the following theorem.

**THEOREM 6.1.** *Let  $u^\varepsilon$  be the solution of problem (CP), and let  $u^\varepsilon$  satisfy (i), (ii), and (iii). Then, the regularization  $\mathcal{A}_\varepsilon(u)$  defined in (6.2) has the following properties:*

- (a)  $\mathcal{A}_\varepsilon(u)$  is conservative.
- (b)  $\mathcal{A}_\varepsilon(u)$  is entropy dissipative for the entropy  $U(u) = \frac{1}{2}u^2$ .

*Proof.* Part (a). For any  $\varphi \in C_0^\infty(S)$  we obtain after partial integration with respect to  $x$  and  $t$ ,

$$\int_S \mathcal{A}_\varepsilon(u^\varepsilon)\varphi = \varepsilon \int_S u^\varepsilon \varphi_{xx} - \varepsilon^2 \tau \int_S u^\varepsilon \varphi_{xxt} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Part (b). To simplify notation, we drop the superscript  $\varepsilon$  from  $u^\varepsilon$ . When we multiply (6.1a) by  $u$  we obtain

$$(6.5) \quad \partial_t U(u) + \partial_x F(u) = u\mathcal{A}_\varepsilon(u) = \varepsilon uu_{xx} + \varepsilon^2 \tau uu_{xxt},$$

where

$$(6.6) \quad F(u) = \int_0^u U'(s)f'(s) ds = \int_0^u sf'(s) ds = uf(u) - \int_0^u f(s) ds.$$

An elementary computation shows that

$$\begin{aligned} \varepsilon uu_{xx} &= \varepsilon U_{xx} - \varepsilon u_x^2, \\ \varepsilon^2 \tau uu_{xxt} &= \varepsilon^2 \tau \left( U_{xxt} - \frac{1}{2}(u_x^2)_t - (u_x u_t)_x \right). \end{aligned}$$

Hence

$$(6.7) \quad \begin{aligned} \int_S \mathcal{A}_\varepsilon(u) u \varphi &= \varepsilon \int_S U \varphi_{xx} - \varepsilon \int_S u_x^2 \varphi \\ &\quad - \varepsilon^2 \tau \int_S U \varphi_{xxt} + \frac{1}{2} \varepsilon^2 \tau \int_S u_x^2 \varphi_t + \varepsilon^2 \tau \int_S u_t u_x \varphi_x. \end{aligned}$$

Plainly

$$\varepsilon \int_S U \varphi_{xx} \rightarrow 0 \quad \text{and} \quad \varepsilon^2 \tau \int_S U \varphi_{xxt} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Since  $\varphi \geq 0$ , it remains to estimate the last two terms on the right-hand side of (6.7).

For this purpose we establish the following two estimates.

LEMMA 6.1. *Let  $T > 0$ , and let  $S_T = \mathbf{R} \times (0, T]$ . Then there exists a constant  $C > 0$  such that for all  $\varepsilon > 0$ ,*

$$(6.8) \quad \varepsilon \int_{S_T} u_x^2 \leq C$$

and

$$(6.9) \quad \varepsilon \int_{S_T} u_t^2 \leq C.$$

*Proof of (6.8).* We write (6.5) as

$$(6.10) \quad \partial_t U(u) + \partial_x F(u) = \varepsilon U_{xx} - \varepsilon u_x^2 + \varepsilon^2 \tau \left\{ U_{xxt} - \frac{1}{2}(u_x^2)_t - (u_t u_x)_x \right\}.$$

Using properties (i)–(iii) and writing  $F_\ell = F(u_\ell)$ ,  $F_r = F(u_r)$ , we find that

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbf{R}} \{U(x, t) - G_\varepsilon(x, t)\} dx - \frac{d\lambda_\varepsilon}{dt} (U_r - U_\ell) \\ & + (F_r - F_\ell) + \varepsilon \int_{\mathbf{R}} u_x^2 + \frac{1}{2} \varepsilon^2 \tau \frac{d}{dt} \int_{\mathbf{R}} u_x^2 \leq 0, \end{aligned}$$

or, when we integrate over  $(0, T)$

$$-\{\lambda_\varepsilon(t) - \lambda_\varepsilon(0)\}(U_r - U_\ell) + (F_r - F_\ell)t + \varepsilon \int_{S_T} u_x^2 + \frac{1}{2} \varepsilon^2 \tau \int_{\mathbf{R}} u_x^2(t) \leq \frac{1}{2} \varepsilon^2 \tau \int_{\mathbf{R}} (u'_0)^2,$$

from which (6.8) immediately follows.  $\square$

*Proof of (6.9).* We multiply (6.1) by  $u_t$ . This yields

$$(6.11) \quad u_t^2 + (f(u))_x u_t = u_t \mathcal{A}_\varepsilon(u) = \varepsilon u_t u_{xx} + \varepsilon^2 \tau u_t u_{xxt}.$$

Using the identities

$$u_t u_{xx} = (u_x u_t)_x - \frac{1}{2} (u_x^2)_t \quad \text{and} \quad u_t u_{xxt} = (u_{xt} u_t)_x - (u_{xt})^2,$$

we find that

$$u_t^2 + \frac{\varepsilon}{2} (u_x^2)_t \leq -f'(u) u_t u_x + \varepsilon (u_x u_t)_x + \varepsilon^2 \tau (u_{xt} u_t)_x.$$

When we integrate over  $\mathbf{R}$  and use Schwarz's inequality and properties (i) and (ii), we obtain

$$\int_{\mathbf{R}} u_t^2 + \frac{\varepsilon}{2} \frac{d}{dt} \int_{\mathbf{R}} u_x^2 \leq \frac{1}{2} \int_{\mathbf{R}} u_t^2 + \frac{K^2}{2} \int_{\mathbf{R}} u_x^2,$$

where  $K = \max\{|f'(s)| : s > 0\}$ . Hence, when we integrate over  $(0, t)$ ,

$$\int_{S_t} u_t^2 \leq \varepsilon \int_{\mathbf{R}} (u'_0)^2 + K^2 \int_{S_t} u_x^2.$$

In view of the first estimate this establishes (6.9) and completes the proof of Lemma 6.1.  $\square$

We now return to the proof of Theorem 6.1(b). For each  $\varphi \in C_0^\infty(S)$  we choose  $T > 0$  so that  $\text{supp } \varphi \subset S_T$ . Then (6.8) implies that

$$(6.12) \quad \varepsilon^2 \int_{S_T} u_x^2 \varphi_t \leq \varepsilon^2 K_1 \int_{S_T} u_x^2 \leq \varepsilon K_1 C \quad \text{with} \quad K_1 = \sup |\varphi_t|,$$

and (6.8) and (6.9) together imply that

$$(6.13) \quad \varepsilon^2 \int_{S_T} u_t u_x \varphi_x \leq \varepsilon^2 K_2 \int_{S_T} |u_t| |u_x| \leq \varepsilon K_2 C \quad \text{with} \quad K_2 = \sup |\varphi_x|.$$

Using (6.12) and (6.13) in (6.7) we conclude that, writing  $u = u^\varepsilon$  again,

$$\limsup_{\varepsilon \searrow 0} \int_S \mathcal{A}_\varepsilon(u^\varepsilon) u^\varepsilon \varphi \leq 0,$$

which is what was claimed in Theorem 6.1.  $\square$

It now follows from (6.5) that in the limit as  $\varepsilon \rightarrow 0$ ,

$$(6.14) \quad \partial_t U(u) + \partial_x F(u) \leq 0$$

holds in a weak or distributional sense. This shows that  $(U, F)$  is an entropy pair for (1.1).

The inequality in (6.14) indicates entropy dissipation. Across shocks  $\{u_\ell, u_r\}$  it can be computed explicitly. Let

$$u(x, t) = \begin{cases} u_\ell & \text{for } x < st, \\ u_r & \text{for } x > st. \end{cases}$$

Then (6.14) implies that

$$-s(U_r - U_\ell) + (F_r - F_\ell) \leq 0.$$

Hence the entropy dissipation is given by

$$(6.15) \quad E(u_\ell, u_r) \stackrel{\text{def}}{=} -s(U_r - U_\ell) + (F_r - F_\ell).$$

We conclude by observing that if  $u = u(\eta)$  is a traveling wave satisfying problem (TW), then (6.15) can be written as

$$(6.16) \quad E(u_\ell, u_r) = \int_{\mathbf{R}} \{-s(U(u))' + (F(u))'\} d\eta.$$

Applying (6.6) and the definition of  $U$  gives

$$(6.17) \quad E(u_\ell, u_r) = \int_{\mathbf{R}} u \left( -s + \frac{df}{du} \right) u' d\eta = \int_{u_\ell}^{u_r} u \left( -s + \frac{df}{du} \right) du.$$

Rewriting further

$$-s + \frac{df}{du} = \frac{d}{du} (-s(u - u_\ell) + f(u) - f(u_\ell)),$$

integrating (6.17) by parts, and using the Rankine–Hugoniot condition yields

$$E(u_\ell, u_r) = \int_{u_r}^{u_\ell} \{f(u) - f(u_\ell) - s(u - u_\ell)\} du.$$

In the special case when  $u_r = 0$  we have  $s = f(u_\ell)/u_\ell$  and thus

$$E(u_\ell, 0) = \int_0^{u_\ell} \{f(u) - su\} du.$$

Returning to the proof of Proposition 1.1 we observe that the integral is negative provided  $u_\ell < \beta$ . Thus this condition acts as an entropy condition in the sense that  $E(u_\ell, 0) < 0$  only if  $u_\ell < \beta$ .

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