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A NEW CLASS OF g-MODES IN NEUTRON STARS

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ABSTRACT

Because a neutron star is born hot, its internal composition is close to chemical equilibrium. In the fluid core, this implies that the ratio of the number densities of charged particles (protons and electrons) to neutrons, $x \equiv n_c/n_n$, is an increasing function of the mass density. This composition gradient stably stratifies the matter giving rise to a Brunt-Väisälä frequency $N \sim (xg/2H)^{1/2} \sim 500 \text{ s}^{-1}$, where g is the gravitational acceleration, and H is the density scale height. Consequently, a neutron star core provides a cavity that supports gravity modes (g-modes). These g-modes are distinct from those previously identified with the thermal stratification of the surface layers and the chemical stratification of the crust. We compute the lowest-order, quadrupolar, g-modes for cold, Newtonian, neutron star models with $M/M_\odot = 0.581$ and $M/M_\odot = 1.405$ and show that the crustal and core g-modes have similar periods. We also discuss damping mechanisms and estimate damping rates for the core g-modes. Particular attention is paid to damping due to the emission of gravitational radiation.

Subject headings: stars: neutron — stars: oscillations

1. INTRODUCTION

The nonradial oscillations of neutron stars have been analyzed extensively in a series of papers by Thorne and collaborators (Thorne & Campolattaro 1967; Price & Thorne 1969; Thorne 1969a, b; Campolattaro & Thorne 1970; Ipser & Thorne 1973). These authors assumed that chemical equilibrium rendered cold neutron stars neutrally stable and therefore concluded that their g-modes were degenerate at zero frequency (e.g., Thorne 1969a; Van Horn 1980; & McDermott, Van Horn, & Scholl 1983). McDermott et al. (1983) calculated g-modes associated with *thermal stratification* and found the quadrupole modes of completely fluid models with temperatures $T \sim 10^8$ K to be concentrated within 10 m of the stellar surface and to have periods $P > 50$ ms.[1]

Later, Finn (1987) pointed out that the *nonuniform chemical composition of the neutron star crust* would perturb the g-modes away from zero frequency. He studied modes associated with discrete changes in composition that occur at densities in the range $8 \times 10^6 < \rho < 4 \times 10^{11}$ g cm^{-3}. These are located in the outer kilometer of the star. Finn's calculations show that the lowest order, crustal g-modes have periods of a few milliseconds, shorter than those of thermal g-modes of the same multipole order.

In addition to discrete changes of chemical composition in the crust, there is also a *smooth change of chemical composition in the core* of a neutron star. More specifically, the equilibrium concentration of charged particles (protons and electrons) depends on density, and increases toward the center of the star. This concentration gradient stably stratifies the core, thus giving rise to an additional series of g-modes. The purpose of our paper is to provide a rough description of these core g-modes.

In § 2, we derive equations that govern the linear oscillations of a fluid star. Then we obtain the WKB dispersion relation which approximates the g-mode frequencies in terms of the Brunt-Väisälä frequency, N.

The stable stratification of the material in the core of a neutron star is investigated in § 3. We evaluate the Brunt-Väisala frequency, thereby obtaining a numerical estimate for the periods of the core g-modes. These turn out to be of the same order of magnitude as those of the discontinuity modes computed by Finn (1987). We also provide crude estimates for the damping timescales of these modes.

Section 4 outlines the numerical computation of the g-mode eigenfunctions, periods, and damping times due to emission of gravitational waves. We explain the differential equations, boundary conditions, and stellar equilibrium models used, emphasizing the models for the Brunt-Väisälä frequency due to the composition gradients in the core and crust.

In § 5, we present and discuss the results of the numerical calculations of quadrupole f- and g-modes for two model neutron stars, comparing them to earlier work.

Finally, in § 6, we give a short summary of the conclusions of the present work and its implications for other aspects of neutron star physics.

2. NONRADIAL OSCILLATIONS OF A NEWTONIAN, FLUID STAR

In this section we derive differential equations that govern the linear oscillations of a nonrotating, unmagnetized, inviscid, fluid star. We apply *Newtonian* mechanics and make the *Cowling approximation* (Cowling 1941), that is, we neglect the Eulerian perturbations of the local gravitational potential. Similar derivations can be found elsewhere (e.g., Cox 1980; Unno et al. 1970); we offer ours for completeness and to clarify the notation and approach of the present paper.

Under our assumptions, the equilibrium configuration of the star is spherically symmetric, and the equilibrium density and pressure, ρ_0 and p_0, depend only on the radial coordinate, r. Oscillations are characterized by a displacement vector field, $\xi(r, t)$, together with the *Eulerian* (or "local") *perturbations* of the density and pressure, $\delta\rho$ and δp. In addition, it is conve-

[1] McDermott et al. (1985) and McDermott, Van Horn, & Hansen (1988) studied neutron star models with a solid crust and found an additional set of thermal g-modes in the fluid material just below the crust, with even longer periods.

nient to use the *Lagrangian* (or "convective") *perturbations*, $\Delta\rho$ and Δp. Eulerian and Lagrangian perturbations are formally related by

$$\Delta = \delta + \boldsymbol{\xi} \cdot \nabla . \quad (1)$$

Written in terms of these variables, the continuity equation reads

$$\Delta\rho = -\rho_0 \nabla \cdot \boldsymbol{\xi} , \quad (2)$$

and the equation of motion takes the form

$$\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = -\frac{1}{\rho_0 + \delta\rho} \nabla(p_0 + \delta p) + \boldsymbol{g} , \quad (3)$$

where the local gravitational acceleration, $\boldsymbol{g}$, is taken to be constant at a fixed location in space (Cowling approximation). The equation of motion decomposes into an equilibrium equation,

$$\frac{1}{\rho_0} \nabla p_0 = \boldsymbol{g} , \quad (4)$$

in which both sides are time-independent radial vectors, and a linearized equation for the perturbations,

$$-\omega^2 \boldsymbol{\xi} = -\frac{1}{\rho_0} \nabla(\delta p) + \frac{\delta\rho}{\rho_0} \boldsymbol{g} , \quad (5)$$

where all perturbation variables depend on time through the factor $e^{-i\omega t}$.

Since the last term in equation (5) is purely radial, the component of the displacement vector perpendicular to the radial direction satisfies

$$\xi_\perp = \frac{1}{\rho_0 \omega^2} \nabla_\perp(\delta p) , \quad (6)$$

where $\nabla_\perp$ is the analogous component of the gradient operator. Substituting this expression into the continuity equation (2), and taking the Eulerian pressure perturbation in a given mode to be the product of a spherical harmonic $Y_l^m(\theta, \phi)$ and an arbitrary function of radius, we obtain

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \xi_r) - \frac{l(l+1)}{r^2\omega^2} \frac{\delta p}{\rho_0} + \frac{\Delta\rho}{\rho_0} = 0 . \quad (7)$$

To relate the density perturbations to the pressure perturbations, it is useful to introduce the variables

$$c_s^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_{\text{adiab}} = \frac{\Delta p}{\Delta \rho} , \quad (8)$$

where c_s is the adiabatic sound speed, and

$$c_{\text{eq}}^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_{\text{equilib}} = \frac{dp_0/dr}{d\rho_0/dr} . \quad (9)$$

Using these definitions, it is straightforward to rewrite equation (7) in terms of the radial displacement and the Eulerian pressure perturbation as

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \xi_r) - \frac{g}{c_s^2} \xi_r + \left[\frac{1}{c_s^2} - \frac{l(l+1)}{r^2\omega^2}\right] \frac{\delta p}{\rho_0} = 0 , \quad (10)$$

where $g \equiv |\boldsymbol{g}|$. Similarly, the radial component of the equation of motion (eq. [5]) takes the form

$$\frac{1}{\rho_0} \frac{\partial \delta p}{\partial r} + \frac{g}{c_s^2} \frac{\delta p}{\rho_0} + (N^2 - \omega^2)\xi_r = 0 , \quad (11)$$

where the *Brunt-Väisälä frequency*

$$N \equiv g\left(\frac{1}{c_{\text{eq}}^2} - \frac{1}{c_s^2}\right)^{1/2} . \quad (12)$$

From equations (10) and (11), it is clear that ξ_r is a function of radius times the spherical harmonic $Y_l^m(\theta, \phi)$, and that the same is true for Δp, $\delta\rho$, and $\Delta\rho$.

To obtain the WKB dispersion relation, we assume that the radial wavelength is much smaller than both r and the density scale height

$$H \equiv (d \ln \rho_0/dr)^{-1} = c_{\text{eq}}^2/g \sim c_s^2/g .$$

Under these conditions, the perturbations are proportional to

$$\exp\left[i \int^r dr' k(r')\right] ,$$

where $|d \ln k/dr| \ll k$. In this short wavelength limit, equations (10) and (11) reduce to

$$ik\xi_r \approx -\left[\frac{1}{c_s^2} - \frac{l(l+1)}{r^2\omega^2}\right] \frac{\delta p}{\rho_0} , \quad (13)$$

$$ik \frac{\delta p}{\rho_0} \approx -(N^2 - \omega^2)\xi_r . \quad (14)$$

When combined, equations (13) and (14) yield the WKB dispersion relation,

$$(N^2 - \omega^2)\left[\frac{l(l+1)}{r^2\omega^2} - \frac{1}{c_s^2}\right] \approx k^2 . \quad (15)$$

In the fluid core of a neutron star, $0 < c_s - c_{\text{eq}} \ll c_{\text{eq}} \sim c_s$, so $N \ll g/c_s \ll c_s k$. This inequality implies that the dispersion relation has two well-separated branches. On the higher frequency (pressure) p-mode branch

$$\omega^2 \approx [(kr)^2 + l(l+1)] \left(\frac{c_s}{r}\right)^2 , \quad (16)$$

whereas on the lower frequency (gravity) g-mode branch

$$\omega^2 \approx \frac{l(l+1)}{(kr)^2 + l(l+1)} N^2 . \quad (17)$$

Stable g-modes exist whenever $N^2 > 0$ [i.e., $(\partial p/\partial\rho)_{\text{adiab}} > (\partial p/\partial\rho)_{\text{equilib}}$] in some region of the star.

3. g-MODES IN NEUTRON STAR CORES

3.1. *Derivation of the Brunt-Väisälä Frequency*

Neutron star cores are believed to contain a fluid mixture of several species of particles. At nuclear density ($\rho_{\text{nuc}} = 2.8 \times 10^{14}$ g cm^{-3}), the only particles present are neutrons, protons, and electrons. For simplicity, we will assume that no other particles exist in the core, although additional particle species appear at only slightly higher densities (Pandharipande 1971; see Lattimer et al. 1991 for more recent references).

Because of the high density, small fractional differences between the number densities of protons and electrons create huge electric fields that quickly restore equilibrium. Thus, we can safely use a single variable, n_c, to denote the number densities of both charged particle species. In this subsection, we follow Shapiro & Teukolsky (1983) in deriving a simplified expression for the ratio $x \equiv n_c/n_n$, where n_n is the number density of neutrons, by neglecting all interactions among particles. Then, we show that its density dependence gives rise to a nonzero Brunt-Väisälä frequency and hence to stable stratification.

The equilibrium value of x is determined by the condition of chemical potential equilibrium for neutron beta decay, $n \rightarrow p + e^- + \bar{\nu}_e$, and its inverse reaction, $p + e^- \rightarrow n + \nu_e$.[2] Since neutrinos and antineutrinos escape from the star their chemical potential vanishes, and the equilibrium condition takes the form

$$\mu_n = \mu_p + \mu_e , \tag{18}$$

where μ_n, μ_p, and μ_e are the internal chemical potentials of the three species of massive particles. Under typical conditions, neutrons contribute most of the density, and each species of massive particle is highly degenerate.

The neutrons and protons are (approximately) nonrelativistic, so their Fermi energies (not including the rest-mass energy $m_N c^2$) are given by

$$E_{Fi} \approx \frac{\hbar^2}{2m_N} (3\pi^2 n_i)^{2/3} , \tag{19}$$

where m_N is the nucleon mass (approximately equal for protons and neutrons) and $i = n, p$ labels the particle species. Taking $n_n \approx \rho/m_N$, we find $E_{Fn} \approx 10^{-4}(\rho/\rho_{nuc})^{2/3}$ ergs $\approx 60(\rho/\rho_{nuc})^{2/3}$ MeV. The electrons are extremely relativistic, so their Fermi energy is

$$E_{Fe} = \hbar c(3\pi^2 n_c)^{1/3} . \tag{20}$$

If interactions among particles and finite-temperature effects are neglected, the chemical potentials can be written as

$$\mu_n \approx m_N c^2 + E_{Fn} ,$$
$$\mu_p \approx m_N c^2 + E_{Fp} , \quad \mu_e \approx E_{Fe} . \tag{21}$$

Since $n_c \ll n_n$, and therefore $E_{Fp} \ll E_{Fn}$, the equilibrium condition (eq. [18]) reduces to

$$E_{Fe} \approx E_{Fn} . \tag{22}$$

Thus, the equilibrium density ratio

$$x = \frac{n_c}{n_n} \approx \left(\frac{\hbar}{2m_N c}\right)^3 3\pi^2 n_n \approx 6 \times 10^{-3} \frac{\rho}{\rho_{nuc}} . \tag{23}$$

In an adiabatic perturbation the composition remains unchanged, so

$$\left(\frac{\partial n_c}{\partial n_n}\right)_{adiab} = \frac{n_c}{n_n} , \tag{24}$$

[2] Neutron star matter is expected to be close to chemical equilibrium since the stars are born hot.

whereas, since $n_c \propto n_n^2$ in chemical equilibrium,

$$\left(\frac{\partial n_c}{\partial n_n}\right)_{equilib} = 2 \frac{n_c}{n_n} . \tag{25}$$

The pressure and mass density of the fluid, neglecting the small contributions of the protons in the first case, and of the electrons in the second, are

$$p = \tfrac{2}{5} n_n E_{Fn} + \tfrac{1}{4} n_c E_{Fe} , \tag{26}$$

$$\rho \approx m_N(n_n + n_c) . \tag{27}$$

To linear order in the density ratio x,

$$c_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_{adiab} \approx \frac{2}{3}\frac{E_{Fn}}{m_N}\left(1 - \frac{1}{2}x\right) \approx \frac{5}{3}\frac{p}{\rho}\left(1 - \frac{1}{8}x\right) , \tag{28}$$

$$c_{eq}^2 = \left(\frac{\partial p}{\partial \rho}\right)_{equilib} \approx \frac{2}{3}\frac{E_{Fn}}{m_N}(1 - x) \approx \frac{5}{3}\frac{p}{\rho}\left(1 - \frac{5}{8}x\right) . \tag{29}$$

Equations (28) and (29) show that the equilibrium composition gradient stably stratifies the fluid core of a cold neutron star. A piece of matter raised above (lowered below) its equilibrium position, slowly enough so that its pressure can adjust to that of its surroundings, but quickly enough to freeze its chemical composition, is denser (less dense) than the surrounding matter. Thus, the buoyancy force opposes the displacement.

The Brunt-Väisälä frequency (eq. [12]) becomes

$$N \approx \left(\frac{x}{2}\right)^{1/2} \frac{g}{c_{eq}} = \left(\frac{x}{2}\frac{g}{H}\right)^{1/2} \sim 500 \text{ s}^{-1} \tag{30}$$

for typical neutron star parameters ($g \sim 10^{14}$ cm s^{-2}, $H \sim 10$ km, and $\rho \sim \rho_{nuc}$). The oscillation periods of the core g-modes range upward from $P_{min} \sim 2\pi/N \sim 10$ ms. They are of the same order of magnitude as the periods of the crustal discontinuity modes found by Finn (1987), and significantly shorter than the periods of the thermal g-modes computed by McDermott et al. (1983).

We have implicitly assumed that neutrons and charged particles will move together on all time scales comparable to the oscillation periods. This is clearly true if the neutrons are "normal" (not superfluid). In this case, binary collisions between neutrons and protons (or between neutrons and electrons if the protons are superconducting) effectively bind the neutrons and charged particles together on extremely short time scales (see, e.g., Yakovlev & Shalybkov 1990 for the numbers). If the neutrons are superfluid, relative motions between neutrons and charged particles may occur (Epstein 1988; Mendell 1991a). Nevertheless, the motions of superfluid neutrons and superconducting protons are not independent of each other, so the core g-modes would still exist, and would have periods similar to those computed here.

3.2. *Damping Mechanisms*

Three damping mechanisms for g-modes come to mind. They are the *relaxation toward chemical equilibrium* of the oscillating fluid, *viscous* damping, and damping due to the emission of *gravitational waves.* These three mechanisms are evaluated, in order, below. Afterwards, we briefly discuss the mechanism of *mutual friction* between two superfluid species (neutrons and protons), recently suggested by Mendell (1991b).

As the core fluid oscillates at fixed chemical composition, the instantaneous equilibrium composition also oscillates. Thus, the oscillating fluid is out of chemical equilibrium. Under non-equilibrium conditions, the net rate of direct plus inverse beta decays tends to relax the composition back toward equilibrium. The relaxation weakens the restoring force acting on displaced fluid elements, thus damping the oscillation.[3] It is not difficult to show that the damping time scale is comparable to the characteristic relaxation time scale.

The equilibration timescale, expressed in terms of the net beta reaction rate per unit volume, $\delta\Gamma \equiv \Gamma(p + e^- \rightarrow n + \nu_e) - \Gamma(n \rightarrow p + e^- + \bar{\nu}_e)$, reads

$$\tau_{\rm chem} \sim -\frac{\delta n_n}{\delta\Gamma} \sim \frac{\delta n_c}{\delta\Gamma}\,, \quad (31)$$

where δn_n and δn_c are the amounts by which the number densities of neutrons and charged particles differ from their equilibrium values. For reactions at constant density, $\delta n_c = -\delta n_n$. Otherwise (say, at constant pressure), δn_c and $-\delta n_n$ differ by a factor of order unity.

The deviation from chemical equilibrium is conveniently characterized by the chemical potential difference, $\delta\mu \equiv \mu_p + \mu_e - \mu_n$, which can be estimated as

$$\delta\mu = \delta E_{{\rm F}p} + \delta E_{{\rm F}e} - \delta E_{{\rm F}n} \approx \delta E_{{\rm F}e} \approx \frac{1}{3}\, E_{{\rm F}n}\, \frac{\delta n_c}{n_c}\,, \quad (32)$$

since $|\delta E_{{\rm F}p}| \ll |\delta E_{{\rm F}n}| \ll |\delta E_{{\rm F}e}|$, and $E_{{\rm F}e} \approx E_{{\rm F}n}$.

If both neutrons and protons are normal (not superfluid), and $|\delta\mu| \ll kT$, the differential reaction rate is

$$\delta\Gamma = \lambda\delta\mu\,, \quad (33)$$

where λ is a temperature-dependent proportionality constant that characterizes the reaction speed.

If, as it has been believed until recently, only the modified URCA reactions can operate (Chiu & Salpeter 1964), this parameter takes the value

$$\lambda \approx 5 \times 10^{33} T_9^6 \left(\frac{\rho}{\rho_{\rm nuc}}\right)^{2/3} {\rm ergs^{-1}\, cm^{-3}\, s^{-1}} \quad (34)$$

(Sawyer 1989),[4] where T_9 is the temperature in units of 10^9 K.

Substitution of equations (32)–(34) into equation (31) yields the *relaxation time toward chemical equilibrium*

$$\tau_{\rm chem} \sim \frac{3n_c}{\lambda E_{{\rm F}n}} \sim \frac{0.2}{T_9^6}\left(\frac{\rho_{\rm nuc}}{\rho}\right)^{2/3} {\rm yr}\,, \quad (35)$$

which is the approximate time scale for damping of g-modes due to neutrino emission.[5]

Incidentally, equation (35) justifies our use of an adiabatic equation of state in the derivation of the wave equation since for stars with core temperatures $T \ll 3 \times 10^{10}$ K the damping time scale is much longer than the periods of the lowest order g-modes ($\tau_{\rm chem}/P \sim 10^9 T_9^{-6}$).

If the regular URCA reactions can operate, as has been suggested by Lattimer et al. (1991), λ is increased by a factor $\sim 5 \times 10^5 T_9^{-2}$, $\tau_{\rm chem}$ is *decreased* by the same factor, and the temperature below which the g-modes are weakly damped is reduced to $\sim 3 \times 10^9$ K.

[3] The energy lost by the oscillation mode is emitted in the form of neutrinos.

[4] For convenience, our sign convention for λ is opposite to Sawyer's.

[5] This damping time is related to the cooling time due to thermal neutrino emission by a factor of order n_c/n_n.

In all likelihood, superfluidity of one or more particle species would *reduce* the reaction rates, thus increasing the damping time.

The *viscous damping time scale* is set by the rate at which momentum diffuses across a wavelength. We write

$$\tau_{\rm visc} \sim \frac{L^2}{\nu}\,, \quad (36)$$

where L is a characteristic wavelength of the oscillation mode and ν is the kinematic viscosity. Using the formulae of Cutler & Lindblom (1987) for the viscosity, and defining $L_6 = L/(10^6\ {\rm cm})$, we obtain

$$\tau_{\rm visc} \sim 80 L_6^2\, T_9^2 \left(\frac{\rho_{\rm nuc}}{\rho}\right)^{5/4} {\rm yr} \quad (37)$$

if the material in the star is "normal," and

$$\tau_{\rm visc} \sim 20 L_6^2\, T_9^2 \left(\frac{\rho_{\rm nuc}}{\rho}\right) {\rm yr} \quad (38)$$

if both neutrons and protons are superfluid. In the former case, ν is primarily due to the neutrons, and in the latter, it is almost completely due to the electrons.

The evaluation of the *damping time scale due to emission of gravitational radiation* is more subtle. Below, we give an approximate lower bound, which is later checked by numerical evaluation of the damping time for specific modes of our model stars.

The e-folding time for the oscillation amplitude can be written as

$$t_g = \frac{2E}{P_g}\,, \quad (39)$$

where E is the total energy stored in the oscillations, and P_g is the power released by emission of gravitational waves.

In order to estimate this time scale, it is convenient to introduce the variables η_r, $\eta_\perp$, and ρ_1 defined by the relations

$$\xi_r(r, \theta, \phi, t) = \eta_r(r) Y_l^m(\theta, \phi) e^{-i\omega t}\,,$$

$$\frac{\delta p(r, \theta, \phi, t)}{\omega^2 \rho_0 r} = \eta_\perp(r) Y_l^m(\theta, \phi) e^{-i\omega t}\,,$$

$$\delta\rho(r, \theta, \phi, t) = \rho_1\, Y_l^m(\theta, \phi) e^{-i\omega t}\,. \quad (40)$$

Note that η_r and $\eta_\perp$ are related to the radial and horizontal components of the displacement, ξ_r and $\xi_\perp$, respectively. For the simple case of azimuthal symmetry ($m = 0$), $\xi_r = \eta_r(r) P_l(\cos\theta)$ and $\xi_\theta = \eta_\perp(r) dP_l(\cos\theta)/d\theta$, where $P_l(x)$ denotes the lth Legendre polynomial.

In terms of these variables, the mode energy can be written as

$$E = \frac{\omega^2}{2}\int_0^R \rho_0 r^2 [\eta_r^2 + l(l+1)\eta_\perp^2] dr\,, \quad (41)$$

and the radiated power (in the weak-gravity approximation) is

$$P_g = \frac{G}{8\pi c^{2l+1}}\, \frac{(l+1)(l+2)}{(l-1)l} \left[\frac{4\pi\omega^{l+1}}{(2l+1)!!}\int_0^R \rho_1 r^{l+2} dr\right]^2 \quad (42)$$

(Thorne 1969b), where $(2l+1)!! \equiv (2l+1) \times (2l-1) \times (2l-3) \times \cdots \times 3 \times 1$.

Using the relations in § 2, it is possible to express the Eulerian density perturbation in terms of the two components of the displacement vector

$$\frac{\rho_1}{\rho_0} = \frac{N^2}{g}\,\eta_r + \frac{\omega^2 r}{c_s^2}\,\eta_\perp = \frac{x}{2H}\,(\eta_r + \beta\eta_\perp)\,, \tag{43}$$

where $\beta \equiv (rg/c_s^2)(\omega/N)^2$. Thus,

$$P_g = \frac{G}{8\pi c^{2l+1}}\,\frac{(l+1)(l+2)}{(l-1)l} \times \left[\frac{4\pi\omega^{l+1}}{(2l+1)!!}\int_0^R \rho_0\,\frac{x}{2H}\,(\eta_r + \beta\eta_\perp) r^{l+2}\,dr\right]^2 . \tag{44}$$

For g-modes, β is of order unity or less everywhere in the core of the neutron star. Hence, an order-of-magnitude estimate of their damping time due to emission of gravitational waves (39) is

$$\tau_g^g \sim 8[(2l+1)!!]^2\,\frac{(l-1)l}{(l+1)(l+2)}\,\frac{c^{2l+1}}{GMR^{2l-2}x^2\omega^{2l}}$$
$$\sim 10^2\left(\frac{M}{M_\odot}\right)^{-1}\left(\frac{R}{10\ \mathrm{km}}\right)^{-2}\left(\frac{x}{0.01}\right)^2\left(\frac{P}{10\ \mathrm{ms}}\right)^4\ \mathrm{yr}\,, \tag{45}$$

where $l = 2$ in the numerical evaluation. For comparison, for quadrupole f- and p-modes (for which $\beta \sim x^{-1}$ rather than unity, so that $\rho_1/\rho_0 \sim \eta_\perp/H$), a similar estimate yields

$$\tau_g^f \sim 0.2\left(\frac{M}{M_\odot}\right)^{-1}\left(\frac{R}{10\ \mathrm{km}}\right)^{-2}\left(\frac{P}{0.4\ \mathrm{ms}}\right)^4\ \mathrm{s}\,. \tag{46}$$

Equation (46) provides an adequate estimate of the damping times of f-modes. However, equation (45) should be regarded as a lower bound, because cancellation in the integral in equation (44) due to nodes in the eigenfunctions is not accounted for in its derivation. In both cases, the damping times of high order (p- and g-) modes are underestimated by progressively larger factors. These expectations are confirmed by the results of numerical computations of the damping times reported in § 5.

A comparison of equations (35) and (38) indicates that viscous damping dominates over neutrino emission in stars whose core temperatures are lower than $\sim 6 \times 10^8 L_6^{-1/4}$ K if only modified URCA reactions occur, and $\sim 5 \times 10^7 L_6^{-1/3}$ K if regular URCA reactions can operate. Damping due to emission of gravitational radiation is not likely ever to be important for g-modes, in sharp contrast to f- and p-modes (see also McDermott et al. 1988, Cutler, Lindblom, & Splinter 1990).

Mendell (1991b) has pointed out that *mutual friction* might damp the oscillations of rotating neutron stars if at least two particle species (e.g., neutrons and protons) are superfluid. Using Mendell's results, we find that the damping time due to mutual friction is always shorter than $10^5(\rho_{\mathrm{nuc}}/\rho)(P_{\mathrm{rot}}/0.1\ \mathrm{s})$s (where P_{rot} is the stellar rotation period), making it the most important damping mechanism for g-modes in the interesting temperature range $10^7\ L_6^{-1}\ (P_{\mathrm{rot}}/0.1\ \mathrm{s})^{1/2} < T/K < 2 \times 10^9\,(P_{\mathrm{rot}}/0.1\ \mathrm{s})^{-1/6}$.

4. NUMERICAL CALCULATIONS AND RESULTS

In order to advance our understanding of neutron star g-modes, we compute eigenvalues and eigenfunctions for the lowest few quadrupole g-modes of two model neutron stars.

4.1. *Differential Equations and Boundary Conditions*

With the change of variables given in equations (40), equations (10) and (11) are rewritten as

$$\frac{d\eta_r}{dr} = -\left(2 - \frac{gr}{c_s^2}\right)\frac{\eta_r}{r} + \left[l(l+1) - \frac{\omega^2 r^2}{c_s^2}\right]\frac{\eta_\perp}{r}\,, \tag{47}$$

$$\frac{d\eta_\perp}{dr} = \left(1 - \frac{N^2}{\omega^2}\right)\frac{\eta_r}{r} - \left(1 - \frac{N^2 r}{g}\right)\frac{\eta_\perp}{r}\,. \tag{48}$$

The origin is a singular point of these equations; as $r \to 0$ each of the four coefficients multiplying η_r and $\eta_\perp$ on the right-hand sides of these equations diverges as r^{-1}. The physically meaningful solutions are regular at $r = 0$ and have the form

$$\eta_r \approx l\eta_\perp \sim r^{l-1} \tag{49}$$

as $r \to 0$.

The pressure (but not necessarily the density) must vanish at the surface ($r = R$) of an unperturbed neutron star. Perturbations must satisfy the boundary condition that the Lagrangian pressure perturbation vanish, $\Delta p = 0$. Written in terms of the variables η_r and $\eta_\perp$, this boundary condition becomes

$$(\omega^2 r\eta_\perp - g\eta_r)\rho_0 = 0 \quad \text{at } r = R\,. \tag{50}$$

In our numerical calculations, equations (47) and (48) are integrated, for two closely spaced trial values of ω, for a point near to $r = 0$, where $\eta_r = l\eta_\perp$ is imposed, out to $r = R$, where the left-hand side of equation (50) is evaluated. Then, a new trial value for ω is chosen by interpolation to reduce this value, and the procedure is iterated until an accurate eigenvalue is obtained.

4.2. *Stellar Equilibrium Models*

We compute g-modes for two neutron star models, both based on the Pandharipande (1971) equation of state that includes hyperonic matter (model B in the classification of Arnett & Bowers 1974), the same as used by Finn (1987). Since only a table containing the values of the density ρ and pressure p at a relatively small number of discrete points[6] is available to us, we obtain the intermediate values necessary for the construction of the stellar model by approximating the equation of state in each interval between tabulated points by a polytrope, $p = k\rho^{1+1/n}$, with the constants k and n determined by the values of ρ and p at the endpoints. This interpolation procedure allows us to calculate the important derivative

$$c_{\mathrm{eq}}^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_{\mathrm{equilib}} = \left(1 + \frac{1}{n}\right)\frac{p}{\rho} \tag{51}$$

without a numerical differentiation.

The models are constructed by integrating the *Newtonian* equations of stellar structure outward from the center of the star, in this way determining ρ, p, c_{eq}^2, and the local gravitational acceleration g as functions of the radial coordinate r. We use Newtonian equations for consistency with the equations for the modes, which are much more easily written and understood in their Newtonian form. At any rate, larger uncertainties are introduced in our calculation by the lack of knowledge about the correct equation of state and the composition of the star than by this simplification.

Model 1, with central density $\rho_c = 0.8 \times 10^{15}$ g cm^{-3}, radius $R = 10.98$ km, and total mass $M = 0.581\ M_\odot$, is similar to

[6] The ratio between the densities at consecutive points in the important range $5 \times 10^{14} < \rho < 2 \times 10^{15}$ g cm^{-3} fluctuates between 1.2 and 1.4.

Finn's "fiducial" model. *Model 2* represents a more standard neutron star, and has central density $\rho_c = 1.6 \times 10^{15}$ g cm^{-3}, radius $R = 10.94$ km, and total mass $M = 1.405 \, M_\odot$.

4.3. *Models for the Brunt-Väisälä Frequency*

Unfortunately, currently available neutron star models do not provide enough information for us to extract the speed of sound, c_s, and the Brunt-Väisälä frequency, N, directly from them. Our procedure is to estimate N taking into account the *composition discontinuities in the crust* (Finn 1987) and the *continuous composition gradient in the core* and then to evaluate c_s from equation (12).

In the upper crust (ρ less than neutron drip density), where the density discontinuities occur, relativistically degenerate electrons dominate the pressure, and

$$p = \frac{1}{4} n_e E_{\rm Fe} = \frac{1}{4} (\hbar c)^{1/3} \left(\frac{\rho}{\alpha m_N} \right)^{4/3} , \qquad (52)$$

where m_N is the mass of a nucleon, and α is the number of nucleons per electron. In an adiabatic perturbation, α is constant, so

$$\frac{1}{c_s^2} \equiv \left(\frac{\partial \rho}{\partial p} \right)_{\rm adiab} = \frac{3\rho}{4p} , \qquad (53)$$

but, since the equilibrium value of α varies with pressure,

$$\frac{1}{c_{\rm eq}^2} \equiv \left(\frac{\partial \rho}{\partial p} \right)_{\rm equilib} = \frac{3\rho}{4p} \left(1 + \frac{4}{3} \frac{d \ln \alpha}{d \ln p} \right) . \qquad (54)$$

Thus, from equation (12), we obtain

$$N_{\rm crust}^2 = \left(\frac{4}{3} \frac{d \ln \alpha}{d \ln p} \right) \frac{g^2}{c_s^2} \approx \left(\frac{4}{3} \frac{d \ln \alpha}{d \ln p} \right) \frac{g^2}{c_{\rm eq}^2} . \qquad (55)$$

In Finn's model, $\alpha(p)$ changes in $D(=11)$discrete steps:

$$\frac{d \ln \alpha}{d \ln p} = \sum_{i=1}^{D} \Lambda_i \delta \left(\ln \frac{p}{p_i} \right) . \qquad (56)$$

Here, $\delta(x)$ is the Dirac δ-function, p_i is the pressure at which the ith discontinuity occurs, and Λ_i is the difference between the values of $\ln \alpha$ on opposite sides of the discontinuity, or equivalently, the fractional jump in the density as shown in Finn's Table 2. Since our numerical integrator cannot handle δ-functions, we make a continuous approximation to equation (56):[7]

$$\frac{d \ln \alpha}{d \ln p} \approx \sum_{i=1}^{D} \frac{\Lambda_i}{(2\pi\sigma^2)^{1/2}} \exp \left[-\frac{1}{2\sigma^2} \left(\ln \frac{p}{p_i} \right)^2 \right] . \qquad (57)$$

Our choice of $\sigma = 0.2$ compromises the convenience of a smooth function against the need to accurately model the discontinuities. Also, we do not include Finn's outermost discontinuity; it is too close to the surface to be properly taken into account by our computer code, and it appears not to affect Finn's lowest order modes.

The contribution to N^2 from the continuous chemical gradient in the core follows from equations (30) and (23):

$$N_{\rm core}^2 \approx 3 \times 10^{-3} \left(\frac{\rho}{\rho_{\rm nuc}} \right) \frac{g^2}{c_{\rm eq}^2} . \qquad (58)$$

[7] This computation can be done more elegantly by imposing "jump" conditions at the discontinuities (Finn 1987; McDermott 1990), but this is not convenient for us.

To account for the stratification in both regions, we write the Brunt-Väisälä frequency as

$$N^2 = N_{\rm crust}^2 + N_{\rm core}^2 . \qquad (59)$$

4.4. *Damping Time Due to Emission of Gravitational Radiation*

We evaluate the mode damping times due to gravitational radiation reaction numerically from[8]

$$\tau_g = \frac{c^{2l+1}}{2\pi G \omega^{2l}} \frac{(l-1)[(2l+1)!!]^2}{l(l+1)(l+2)} \times \frac{\int_0^R \rho_0 r^2 [\eta_r^2 + l(l+1)\eta_\perp^2] dr}{\{\int_0^R \rho_0 r^{l+1} [\eta_r + (l+1)\eta_\perp] dr\}^2} \qquad (60)$$

(see, e.g., Balbinski & Schutz 1982). Substantial cancellation occurs in the denominator of this expression (see eq. [45]). The integral $\int_0^R \rho_0 r^{l+1} [\eta_r + (l+1)\eta_\perp] dr$ is smaller by a factor $\sim x$ than both $|\int_0^R \rho_0 r^{l+1} \eta_r \, dr|$ and $(l+1) |\int_0^R \rho_0 r^{l+1} \eta_\perp \, dr|$. Thus, a naive order-of-magnitude estimate based on equation (60) underestimates the true damping time by at least a factor $\sim x^2 \sim 10^{-4}$. Furthermore, numerical evaluation of the damping time from equation (60) requires that the functions $\eta_r(r)$ and $\eta_\perp(r)$ be known accurately enough to allow such a subtle cancellation.

5. RESULTS

5.1. "*Fiducial*" *Models*

For model 1 with $N = N_{\rm crust}$ (see § 4.3), the input physics and model parameters are similar to those used by Finn (1987).[9] As can be seen by comparing Figure 1 with Finn's Figure 5, our g-modes are qualitatively very similar to his, giving us confidence that the smoothing of the discontinuities (see eq. [57]) in our calculation of the Brunt-Väisäla frequency does not introduce substantial errors. The periods do not agree exactly, since our use of Newtonian rather than relativistic physics introduces errors of order $GM/Rc^2 \sim 8\%$. Taking this into account, the agreement is satisfactory (see Table 1). Our damping times are smaller than Finn's by factors ~ 10. The Newtonian approximation and the strong dependence of τ_g on stellar radius and mode period (see the analytical estimates by Finn 1987) may account for part of this discrepancy. Similar discrepancies between relativistic and quasi-Newtonian estimates of the damping time of quadrupole f-modes due to gravitational radiation reaction forces have been reported by Balbinski & Schutz (1982).

An additional simplification which changes our results compared to Finn's is our use of the *Cowling approximation.* The extent to which this approximation affects f-mode periods can be appreciated by considering a *Newtonian, incompressible, uniform density, fluid star.* For this simple model, analytical calculations give the ratio

$$\frac{P(\text{Cowling approximation})}{P(\text{exact})} = \left(1 - \frac{3}{2l+1} \right)^{1/2} \approx 0.63 \quad \text{for } l = 2 . \qquad (61)$$

[8] This expression can be obtained by using the continuity equation, $\delta\rho = -\nabla \cdot (\rho_0 \xi)$, to integrate eq. (42) by parts, and replacing the result and eq. (41) in (39).

[9] Finn's model star has the parameters $M = 0.522 \, M_\odot$, $\rho_c = 10^{15}$ g cm^{-3}, and $R = 9.83$ km.

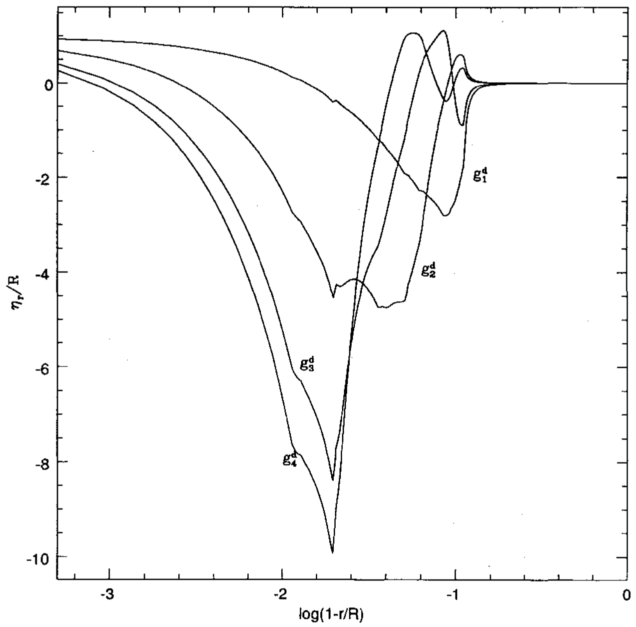

FIG. 1.—The first four (quadrupole) g-modes of our neutron star model 1 ($M = 0.581\ M_\odot$), when only the stable stratification due to the discontinuities in the stellar crust is taken into account ($N = N_{\rm crust}$). The radial displacement is plotted as a function of log $(1 - r/R)$, where R is the radius of the star, for easy comparison with Fig. 5 of Finn (1987). The modes are normalized by the condition $\eta_r(R)/R = 1$.

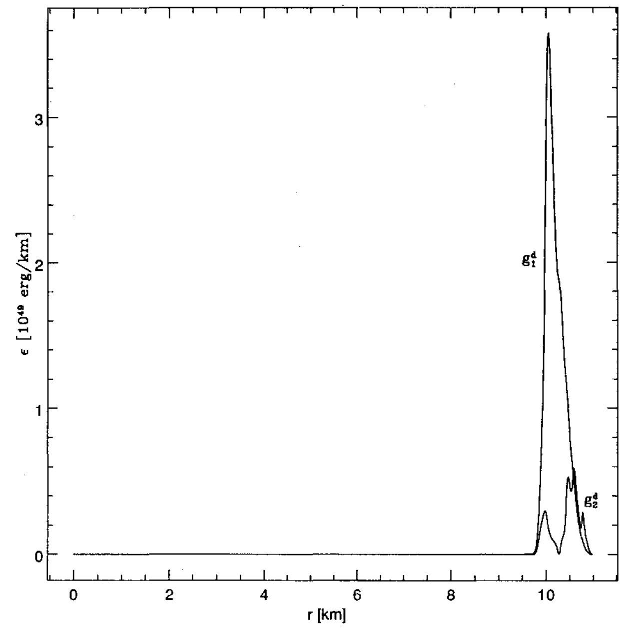

FIG. 2.—Oscillation energy per unit radial distance, $\epsilon = \frac{1}{2}\omega^2\rho_0 r^2[\eta_r^2 + l(l+1)\eta_\perp^2]$, as a function of radius for the first two g-modes of our model 1 with $N^2 = N^2_{\rm crust}$. The modes are normalized by the condition $\eta_r(R)/R = 1$.

Since neutron stars are centrally condensed, this ratio is closer to unity for them. The Cowling approximation is more accurate for the g-modes than it is for the f-mode, because radial nodes weaken the perturbations of the gravitational field.

McDermott et al. (1988) used Newtonian equations of motion and the Cowling approximation to calculate periods of oscillation modes of a relativistic neutron star model very similar to Finn's ($M = 0.503\ M_\odot$, $\rho_c = 9.44 \times 10^{14}$ g cm^{-3}, $R = 10.1$ km) and obtained $P = 0.398$ ms for the quadrupole f-mode, which is very close to our value.

The periods of the crustal discontinuity modes change very little (by a few percent upward) as the stellar mass is increased by a factor ~ 2.4 (see Table 1).

The periods of the core g-modes (see Table 2) agree well with the estimate given in § 3.1, and are similar to those of the crustal discontinuity modes. Comparing the results for models 1 and 2, one sees that these periods decrease strongly with increasing stellar mass.

The damping times of the core modes due to the emission of gravitational radiation exceed the estimate of equation (45) by a large factor ($\sim 10^2$ for g_1^c and more than 10^3 for the higher order modes in both models), showing that this is an extremely inefficient damping mechanism for g-modes.

A comparison of Figures 2 and 3 reveals that the oscillation energy of the core g-modes is distributed throughout the inner 90% in radius of the star, whereas discontinuity modes are concentrated in the outer crust, i.e., the outermost 10% in radius.

When both contributions to the stratification are taken into account, the crust and core modes retain their separate identities, as Figures 4 and 5 show. The core and crust act as a pair of weakly coupled resonant cavities, with modes in one cavity being little affected by the existence of the other cavity. Fractional period changes due to the presence of the second cavity

TABLE 1

CRUSTAL DISCONTINUITY MODES

Mode	P(ms) (Finn 1987)	τ_g(s) (Finn 1987)	P(ms) (Model 1)	τ_g(s) (Model 1)	P(ms) (Model 2)	τ_g(s) (Model 2)
f	0.528	6.49 − 01	0.397	1.03 − 01	0.298	1.10 − 02
g_1^d	5.13	1.09 + 12	5.22	1.60 + 11	5.54	2.24 + 12
g_2^d	10.5	9.56 + 15	10.8	7.84 + 14	11.3	2.61 + 15
g_3^d	14.2	1.93 + 16	14.9	2.72 + 15	15.6	4.52 + 15
g_4^d	15.3	2.46 + 17	16.5	1.84 + 16	17.1	1.14 + 16

NOTES.—Periods (in ms) and damping times due to emission of gravitational radiation (in seconds) for the quadrupole ($l = 2$) f-mode and first four g-modes of model neutron stars whose crusts are stably stratified due to composition discontinuities. Shown are the results of Finn (1987) for a relativistic 0.522 $M_\odot$ neutron star model, and our results for the Newtonian models 1 ($M = 0.581\ M_\odot$) and 2 ($M = 1.405\ M_\odot$), in which we took the Brunt-Väisälä frequency to be $N = N_{\rm crust}$ (see eq. [55]).

TABLE 2

CORE g-MODES

Mode	P(ms) (Model 1)	τ_g(s) (Model 1)	P(ms) (Model 2)	τ_g(s) (Model 2)	P(ms) (Model 1-int)	P(ms) (Model 1-rc)
f	0.397	1.03 − 01	0.298	1.10 − 02	0.397	0.247
g_1^c	10.8	3.34 + 11	4.66	7.30 + 08	4.21	12.4
g_2^c	15.0	3.47 + 13	7.25	7.15 + 10	6.39	16.0
g_3^c	18.8	2.22 + 14	10.1	1.47 + 12	7.66	21.0
g_4^c	24.3	2.04 + 15	12.4	5.33 + 12	9.22	26.8

NOTES.—Periods (in ms) and damping times due to emission of gravitational radiation (in seconds) for the quadrupole ($l = 2$) f-mode and first four g-modes of neutron star models that are stably stratified due to a smooth composition gradient in the stellar core. The first four columns of numbers show results for our Newtonian models 1 ($M = 0.581\ M_\odot$) and 2 ($M = 1.405\ M_\odot$) with Brunt-Väisälä frequency $N = N_{\rm core}$ (see eq. [58]), which are analyzed in § 5.1. The last two columns contain results for modified versions of model 1, as discussed in § 5.2. In the first case (labeled "Model 1-int"), the density ratio x is taken to be as given by eq. (62), in order to get an estimate of the effect of the strong interactions among nucleons, and in the second (labeled "Model 1-rc"), the crust is taken to be perfectly rigid, i.e., the boundary condition $\xi_r = 0$ is imposed at the crust-core boundary [$r = r_{\rm ccb} = 7.81$ km, where $\rho(r_{\rm ccb}) = 2.4 \times 10^{14}$ g cm^{-3}].

are smaller than 10^{-3} for all modes listed. The gravitational radiation damping times of the core modes are also changed very little (by a few percent) by the existence of stratification in the crust, but the damping times of some crustal modes are decreased significantly (up to factors $\sim 10^2$) by the core stratification. This latter result follows because small motions in the core (whose density is several orders of magnitude higher than that of the outer crust) substantially increase the oscillations of the stellar quadrupole moment.

5.2. *Modifications and Further Discussion*

Possibly the most important source of errors in the present determination of the core g-modes is our neglect of the *strong interactions among nucleons*, which leads us to significantly underestimate the composition ratio, x. A fit to the results obtained by R. Smith (see Table I of Sauls, Stein, & Serene 1982), who took the strong interactions into account, gives

$$x \approx 0.05(\rho/\rho_{\rm nuc})^{0.4}\ , \qquad (62)$$

instead of $x \approx 0.006\rho/\rho_{\rm nuc}$ from our equation (23). Substitution of equation (62) into equation (30) for the Brunt-Väisälä frequency yields periods for the four lowest order, quadrupolar, core g-modes of Model 1 that are shorter than our previous results by a factor ~ 2.5. These periods are listed in the column labeled "Model 1-int" in Table 2. This procedure is not completely consistent, since the strong interactions are neglected in

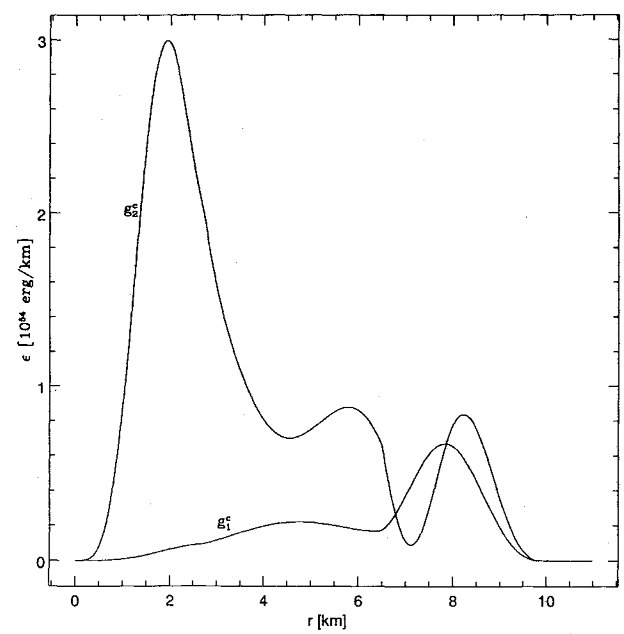

FIG. 3.—Oscillation energy per unit radial distance as a function of radius for the first two g-modes of our Model 1 with $N^2 = N^2_{\rm core}$. The normalization condition is $\eta_r(R)/R = 1$.

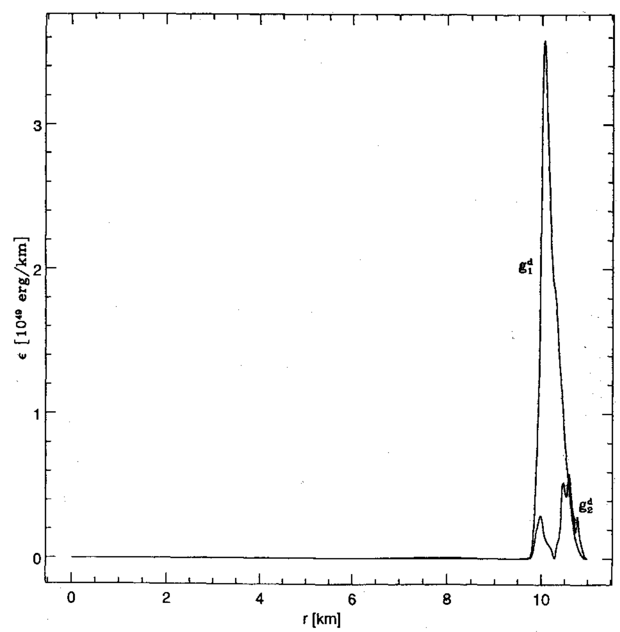

FIG. 4.—Oscillation energy per unit radial distance as a function of radius for the first two crustal g-modes of our model 1 with $N^2 = N^2_{\rm crust} + N^2_{\rm core}$. Again, the modes are normalized by the condition $\eta_r(R)/R = 1$.

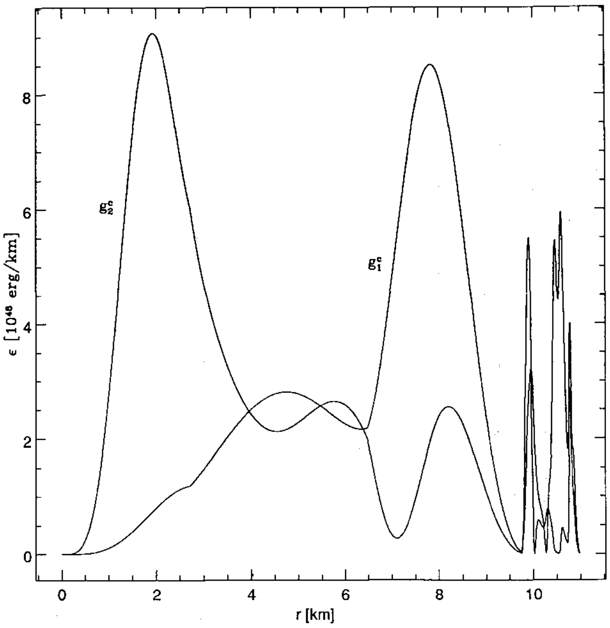

FIG. 5.—Oscillation energy per unit radial distance as a function of radius for the first two core g-modes of our model 1 with $N^2 = N^2_{\rm crust} + N^2_{\rm core}$. Again, the normalization condition is $\eta_r(R)/R = 1$.

the derivation of equation (30), but it gives an idea of the importance of the approximations made.[10]

Throughout this paper, we have taken the neutron stars to be completely fluid. This assumption was also made by Finn (1987), but was criticized by McDermott (1990), who pointed out that the *shear modulus of the crystalline stellar crust* will significantly affect the discontinuity modes. Neutron star models that take into account the finite shear modulus of the crust (but not the stable stratification associated with composition gradients) were studied by McDermott et al. (1985) and McDermott, Van Horn, & Hansen (1988). These authors find a sequence of shear (s-) modes in the crust and two sequences of interfacial (i-) modes, one trapped at the interface between the upper crust and the fluid ocean above, and the other at the boundary between the lower crust and the fluid core below.

The solid crust with its finite shear modulus modifies both the crustal discontinuity modes and the core g-modes. However, its effect on the core modes should be small. To verify this, we calculate the periods of the f-mode and the first four core g-modes of Model 1 (with $N = N_{\rm core}$) with the *modified boundary condition* $\eta_r(r_{\rm ccb}) = 0$, where $r_{\rm ccb} = 7.81$ km is the location of the crust-core boundary, taken to be where $\rho = 2.4 \times 10^{14}$ g cm^{-3}. This is equivalent to setting both the shear modulus and the mass of the crust equal to infinity. However, the g-mode periods (listed in the column labeled "Model 1-rc" in Table 1) are longer by only 7%–15% than those of the "fiducial" model 1.[11] The f-mode period is more strongly changed (and in the opposite direction). However, in a more realistic calculation with a crust of finite shear modulus, McDermott et al. (1988) find that the f-mode period is almost identical to its value for a completely fluid model.

We have been assuming that the entire neutron star (except the outer crust) is composed of neutrons, protons, and electrons. However, *additional species of particles* (muons, kaons, hyperons, and others) undoubtedly appear at densities only slightly above nuclear density (see, e.g., Lattimer et al. 1991 for references), and these would contribute to the stable stratification of the neutron star core, decreasing the periods of the g-modes. Furthermore, the lower crust is stably stratified by the *density-dependent concentration of neutrons*, which contribute an important share of the mass density without adding much to the pressure.

6. CONCLUSIONS

The present work shows that the core of a neutron star is stably stratified, and that it supports a set of core g-modes. These modes have periods ranging upward from a few milliseconds, similar to those of the discontinuity modes identified and investigated by Finn (1987), and considerably shorter than the thermal g-modes studied by McDermott et al. (1983).

It is unfortunate that, to date, no convincing detections of neutron star oscillations have been reported (see McDermott et al. 1988 for references and for a discussion of the possibilities). However, the stable stratification identified here may have other important consequences. For example, it restricts secular motions of matter inside neutron stars, because neutrons and charged particles cannot move together over large radial distances on timescales shorter than those over which weak interactions can maintain chemical equilibrium. How this might impact the evolution of neutron star magnetic fields will be discussed in a separate paper (Goldreich & Reisenegger 1992).

We are indebted to a number of colleagues for assistance with this project. The computer code we used to calculate g-modes is a modified version of a code Pawan Kumar wrote to calculate solar p-modes. Curt Cutler provided two relativistic neutron star models, on which preliminary calculations were carried out, and the table from which we constructed the equation of state. He also pointed us to Mendell's work on damping of oscillations in superfluid stars. Norman Murray offered help with the computations. Patrick N. McDermott gave valuable criticisms on a previous version of this paper, and motivated us to numerically evaluate the gravitational wave damping of the g-modes. Finally, our research was supported by NSF grant AST 89-13664, and by NASA grants NAGW 1303 and NAGW 2372.

[10] The difference between the real value and our results for noninteracting particles is probably somewhat less striking than this comparison would suggest, because the dependence of x on ρ is less strong in eq. (62) than in eq. (23).

[11] The counterintuitive effect of *increasing* the period by *decreasing* the size of the resonant cavity can be understood by a glance at the WKB dispersion relation for g-modes (eq. [17]), which shows that the frequency *decreases* with increasing radial wavenumber.

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