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A NEW CLASS OF INFINITE SPHERE PACKINGS

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The packings considered in this paper are packings of a unit sphere in N-dimensional Euclidean space by an infinite number of unequal spheres. More specifically, we are interested in complete packings, those which exhaust the volume of the packed sphere.

The osculatory or Apollonian packing in two dimensions is well known and is described for example in [13]. Recently we investigated the three dimensional osculatory packing of a sphere [4]. However, the results of that paper indicate that, for $N > 3$, N-dimensional osculatory packings are irregular and not invariant under inversion as is the case for $N = 2$ and 3. In this paper we introduce a class of packings which we call discrete packings, and produce some examples. This class is analogous to the class of lattice packings which appear in the theory of packings of equal spheres.

We shall use the systems of polyspherical coordinates developed in [4]. Section 2 contains a description of these as well as the proofs of some additional results needed here. The idea of the separation $\Delta(X, Y)$ between two spheres X and Y will again play an important role.

In § 3 we consider inversively generated configurations obviously generalizing the construction used in [4]. That is, we begin with a 'cluster' of $(N + 2)$ disjoint spheres and by successive inversions replace the spheres one at a time with new spheres in such a way that the separations between the spheres in the new cluster are the same as for the initial cluster. In terms of polyspherical coordinates the necessary inversions are represented by matrices which preserve a certain indefinite quadratic form. Repetition of the process leads to a configuration of spheres in E_N which may or may not be a packing, depending on the initial cluster.

In § 4 we give sufficient conditions under which an inversively generated configuration is a packing. The conditions force the separations between the spheres in the configuration to lie in a discrete subset of the rational numbers, hence the name 'discrete packing'. In addition to the two and three dimensional osculatory packings, we give examples of discrete packings for dimensions 2, 3, 4, 5, and 9. We do not know yet whether such packings exist in all dimensions. The examples we have found are given in § 6.

The packings described in § 4 are not in general osculatory; that is, the largest possible sphere is not generated at each step. However, they are K-osculatory, a natural generalization of osculatory, which we introduce in § 5. We prove in that section that K-osculatory packings are complete and that they have exponents strictly less than N. We then prove that all inversively generated configurations which are infinite packings are in fact K-osculatory.

In § 6, we give thirteen examples of discrete packings and discuss their exponents and cross sections. The cross sections arise when one chooses two of the spheres in the cluster to have zero curvature. The centers of the spheres in the cross section form a lattice in E_{N-1} which is invariant under a group generated by N reflections. Since Coxeter [6] has determined all such groups, it would appear that there is a possibility of classifying all discrete packings but that is not attempted here.

We gratefully acknowledge a letter from Professor J. B. Wilker in which he pointed out that the matrices B_i of [4] represent inversions. He also proposed a study of the groups generated by these inversions.

2. **Polyspherical coordinates.** We shall use the word *sphere* to mean an N-sphere or N-ball. If $\xi, \underline{a} \in E_N$ and $r \neq 0$ we write

$$\begin{aligned} S(\underline{a}, r) &= \{\underline{\xi}: |\underline{\xi} - \underline{a}| < r\} \quad \text{if} \quad r > 0 \\ &= \{\underline{\xi}: |\underline{\xi} - \underline{a}| > -r\} \quad \text{if} \quad r < 0 \,. \end{aligned}$$

Thus we specifically allow spheres with negative radius. The *curvature* $\varepsilon(X)$ of a sphere X is the reciprocal of its radius. We consider a half-space to be a sphere with zero curvature. If $b \varepsilon E_N$ and $\underline{n}$ is a unit vector in E_N we write $\prod(\underline{b}, \underline{n}) = \{\xi: (\xi - \underline{b})\cdot\underline{n} < 0\}$, and consider $\prod(\underline{b}, \underline{n})$ to be the limit as $r \to \infty$ of $S(\underline{b} - r\underline{n}, r)$. If $X = S(\underline{a}, r)$, $Y = S(\underline{b}, s)$ and $d = |\underline{a} - \underline{b}|$, then the *separation* $\Delta(X, Y)$ between X and Y is defined by $\Delta(X, Y) = (d^2 - r^2 - s^2)/2rs$, and by the limit of this expression if r or s is infinite. By *inversion* in a sphere X, we mean inversion in its boundary. The quantity $\Delta(X, Y)$ is invariant under inversions.

Given any $N + 2$ spheres $X_1, \cdots, X_{N+2}$, let Δ denote the matrix $(\Delta(X_i, X_j))$. We call Δ a *separation matrix*. If Δ is nonsingular then, as shown in [4], these spheres can be used as a basis for a system of polyspherical coordinates as follows: For any sphere Y we let $c(Y) = (\Delta(Y, X_1), \cdots, \Delta(Y, X_{N+2}))^T$. The *polyspherical coordinates* of Y with respect to $X_1, \cdots, X_{N+2}$ are defined by $a(Y) = \Delta^{-1}c(Y)$. Let ε_i be the curvature of X_i and $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_{N+2})^T$. Then the curvature of Y satisfies

$$\varepsilon(Y) = a(Y)^T\varepsilon \,. \tag{1}$$

The vector ε satisfies the generalized Descartes formula:

(2) $$\varepsilon^T \Delta^{-1} \varepsilon = 0 .$$

For any two spheres Y and Z we have

(3) $$\Delta(Y, Z) = c(Y)^T \Delta^{-1} c(Z) = a(Y)^T \Delta a(Z) = a(Y)^T c(Z) .$$

The Cartesian equations of Y are easily obtained from $a(Y)$ and the Cartesian equations for $X_1, \cdots, X_{N+2}$ as shown in [4].

The next result, needed in § 6, was mentioned in [4] but not proved. It is a generalization of a result of Mauldon [12] for the case $\Delta(X_i, X_j) = -\gamma$ for $i \neq j$, where γ is a constant.

LEMMA 2.1. *Suppose that $X_1, \cdots, X_{N+2}$ are spheres for which $\Delta = (\Delta(X_i, X_j))$ is nonsingular. Then Δ has one positive and $N+1$ negative eigenvalues.*

Conversely, let $\Delta = (\Delta_{ij})$ be a symmetric matrix with diagonal entries all equal to -1, and having one positive and $N+1$ negative eigenvalues. Then there are real nontrivial solutions $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_{N+2})$ of (2), and for any such solution there are spheres $X_1, \cdots, X_{N+2}$ with $\varepsilon(X_i) = \varepsilon_i$ and such that $\Delta(X_i, X_j) = \Delta_{ij}$.

Proof. As in the proof of Lemma 1 of [4], if $X = S(\underline{c}, r)$, let $u(X) = r^{-1}(1/2, |\underline{c}|^2 - r^2, -c_1, \cdots, -c_N)^T$ and $v(X) = r^{-1}(|\underline{c}|^2 - r^2, 1/2, c_1, \cdots, c_N)^T$. If $r = \infty$, define $u(X)$ and $v(X)$ by the limits of these expressions. Clearly $\Delta(X, Y) = u(X)^T v(Y)$ so $-\Delta = AB^T$, where A has rows $u(X_i)^T$ and B has rows $v(X_j)^T$. If the columns of A are the $(N+2)$-vectors $x_1, \cdots, x_{N+2}$ then the columns of B are $-x_2, -x_1, x_3, \cdots, x_{N+2}$. The representation $y^T \Delta y = -y^T A (By^T)^T$ shows that Δ has signature $(+, -, \cdots, -)$.

Conversely, if Δ has signature $(+, -, \cdots, -)$ then (2) has nontrivial solutions in the span of the first two eigenvectors of Δ^{-1}. We shall show that Δ has a factorization as $-AB^T$ where A has columns $(1/2)\varepsilon, x_2, \cdots, x_{N+2}$ and B has columns $-x_2, -(1/2)\varepsilon, x_3, \cdots, x_{N+2}$. The fact that the diagonal elements of Δ are all -1 then shows that the rows of A are of the form $u(X)$ for real spheres X. Note that there is a nonsingular P so that

(4) $$-\Delta = PDP^T ,$$

where D $= \operatorname{diag}(-1, 1, \cdots, 1)$. Write $y_1 = (1/2)P^{-1}\varepsilon$, and $y_2 = -\alpha^2 D y_1$, where $\alpha = 1/|y_1|$. Then one can choose $y_3, \cdots, y_{N+2}$, vectors whose first component vanishes such that $y_2, \cdots, y_{N+2}$ are mutually orthogonal. Then $y_1, y_3, \cdots, y_{N+2}$ will be orthogonal, and $D = (y_1 y_2 \cdots y_{N+2})(-y_2^T\ -y_1^T\ y_3^T \cdots y_{N+2}^T)^T$. Combining this with (4) completes the proof, taking $x_i = Py_i$.

LEMMA 2.2. *Let Δ be a nonsingular separation matrix for which $\Delta_{ij} \geqq 1$ if $i \neq j$. Then there are disjoint spheres $X_1, \cdots, X_{N+2}$ such that $\Delta(X_i, X_j) = \Delta_{ij}$ for all i and j. Furthermore, X_1 can be chosen with $\varepsilon_1 < 0$ and such that its center lies in the interior of the convex hull of the centers of $X_2, \cdots, X_{N+2}$.*

Proof. We seek a vector ε with $\varepsilon_1 < 0$ and $\varepsilon_i > 0$ for $i > 1$ which satisfies (2). We shall further require that if $\kappa = \Delta^{-1}\varepsilon$ then $\kappa_i > 0$ for all i. This ensures that the center of X_1 is in the convex hull of the centers of $X_1, \cdots, X_{N+2}$; for, if Z_j is a half-space orthogonal to all X_i except X_j and X_1 and if Z_j contains the center of X_j, then by (1),

$$(5) \qquad 0 = \varepsilon(Z_j) = \kappa_1 \Delta(Z_j, X_1) + \kappa_j \Delta(Z_j, X_j) .$$

Equation (5) shows that the signs of $\Delta(Z_j, X_1)$ and $\Delta(Z_j, X_j)$ are opposite and thus the centers of X_1 and X_j both lie in Z_j. This being true for all j proves our claim.

We thus now seek κ with all $\kappa_i > 0$ so that $\kappa^T \Delta \kappa = 0$. Since $\Delta + 2I$ has entries all exceeding 1, the Perron-Frobenius theorem [10, p. 49] shows that Δ has an eigenvalue $\rho \geqq N$, and a corresponding positive eigenvector ξ. Let $\kappa = \xi + \alpha e_1$ where e_1 is the usual unit vector and α is to be chosen. A direct computation shows that there is a positive α which makes $\kappa^T \Delta \kappa = 0$, and that for $i = 2, \cdots, N + 2$,

$$(6) \qquad 0 < -\varepsilon_1 = \alpha - \rho\xi_1 < \alpha < \alpha\Delta_{1i} + \rho\xi_i = \varepsilon_i .$$

($\varepsilon_1 < 0$ since $\kappa^T\varepsilon = 0$.) The inequalities (6) and $\Delta(X_i, X_j) \geqq 1$ for $i \neq j$ imply that X_i and X_j are disjoint for $i \neq j$.

LEMMA 2.3. *Let Δ satisfy the conditions of Lemma 2.2, and let $\Delta^{-1} = (q_{ij})$. Then $q_{ii} < 0$ for all i and $q_{ij}^2 \leqq q_{ii}q_{jj}$ for all i and j.*

There are real spheres $Y_1, \cdots, Y_{N+2}$ such that $\Delta(X_i, Y_j) = 0$ for $j \neq i$ and $\Delta(Y_i, X_i) > 0$ for all i.

Proof. Let $\Delta^{(i)}$ be the ith principal minor of Δ. If Δ has eigenvalues $\lambda_1 > 0 > \lambda_2 \geqq \cdots \geqq \lambda_{N+2}$, and $\Delta^{(i)}$ has eigenvalues $\nu_1 \geqq \nu_2 \geqq \cdots \geqq \nu_{N+1}$, then it is known [10, p. 76] that $\lambda_1 \geqq \nu_1 \geqq \lambda_2 \geqq \cdots \geqq \nu_{N+1} \geqq \nu_{N+2}$. Now $\Delta^{(i)} + 2I$ is a positive matrix with an eigenvalue exceeding $N + 1$ so $\nu_1 \geqq N - 1 > 0$. Hence $q_{ii} = \det \Delta^{(i)}/\det \Delta = \nu_1 \cdots \nu_{N+1}\lambda_1^{-1} \cdots \lambda_{N-2}^{-1} < 0$. The required sphere Y_i has coordinates $c(Y_i) = e_i/(-q_{ii})^{1/2}$.

Given any $i \neq j$ let k be different from both i and j. Then Y_i and Y_j are orthogonal to X_k and hence must intersect or at least touch each other. Thus $|\Delta(Y_i, Y_j)| \leqq 1$. But $q_{ij} = e_i^T \Delta^{-1} e_j = (-q_{ii})^{1/2}\Delta(Y_i, Y_j)(-q_{jj})^{1/2}$, hence $q_{ij}^2 \leqq q_{ii}q_{jj}$.

In terms of the biorthogonal spheres Y_i obtained in Lemma 2.3, the ith coordinate of a sphere X is given by $a_i(X) = (-q_{ii})^{1/2}\Delta(X, Y_i)$. The vector $\kappa = \Delta^{-1}\varepsilon$ used in Lemma 2.2 satisfies $\kappa_i = (-q_{ii})^{1/2}\varepsilon(Y_i)$. We will write $\Omega = (\Delta(Y_i, Y_j))$ so that $\Omega_{ij} = q_{ij}/(-q_{ii})^{1/2}(-q_{jj})^{1/2}$. According to the proof of Lemma 2.3, $\Omega_{ij} = -\cos\theta$, where θ is the angle between the outward normals to Y_i and Y_j at a point of intersection or contact. There is of course a simple relation between $\Omega^{-1} = (p_{ij})$ and Δ which is given by

$$\Delta_{ij} = p_{ij}/(-p_{ii})^{1/2}(-p_{jj})^{1/2} \,. \tag{7}$$

3. Inversively generated configurations. In this section we describe a process for generating configurations of spheres depending on a fixed separation matrix Δ. In the next section we will give conditions on under which such a configuration is a packing. The process is a generalization of the process defined in [4] when $\Delta = J - 2I$, J being the matrix with all entries equal to 1.

From now on we consider only those Δ which satisfy the conditions of Lemma 2.2. We call a disjoint collection of spheres $X_1, \cdots, X_{N+2}$ a *cluster* if $(\Delta(X_i, X_j)) = \Delta_{ij}$. Clusters exist by Lemma 2.2. A solution ε of (2) clearly corresponds to a cluster if and only if either all ε_i are nonnegative, or else one ε_i, say ε_k is negetive and $|\varepsilon_k| < \varepsilon_i$ for $i \neq k$. In fact, any two clusters are inversively equivalent. This is easily seen by inverting the cluster into a standard configuration in which X_1 and X_2 are parallel half-spaces at distance 1 apart (if $\Delta_{12} = 1$) or else concentric spheres with $\varepsilon(X_1) = -1$, and $\varepsilon(X_2)$ the smaller of the two possible values (if $\Delta_{12} > 1$). Then the curvatures and the distances between the centers of the other spheres are uniquely determined.

The sphere generating process is as follows: We begin with a cluster $X_1, \cdots, X_{N+2}$. Let $Y_1, \cdots, Y_{N+2}$ be the orthogonal spheres as in Lemma 2.3. Inversion in Y_j maps the cluster $X_1, \cdots, X_{N+2}$ into a new cluster $X_1(j), \cdots, X_{N+2}(j)$, where $X_i(j) = X_i$ if $i \neq j$ since $\Delta(X_i, Y_j) = 0$. From the $(N + 2)$ inversions in $Y_1, \cdots, Y_{N+2}$ we obtain $N + 2$ new clusters. We repeat this procedure with the new clusters obtaining $(N + 2)^2$ clusters $X_1(i, j), \cdots, X_{N+2}(i, j)$. Proceeding in this way, at the mth stage we will have $(N + 2)^m$ clusters which we can index by a parameter $\alpha = (i_1, \cdots, i_m)$ where each i_k takes values in the set $\{1, 2, \cdots, N + 2\}$ and $m = 1, 2, \cdots$. We include a single vector α with no components when $m = 0$. We denote by G the collection of all such α, and by $\mathscr{S}(\Delta)$ the collection of all spheres $X_i(\alpha)$, $\alpha \in G$, $i = 1, \cdots, N + 2$. We call $\mathscr{S}(\Delta)$ an inversively generated configuration.

LEMMA 3.1. *Let $\Delta^{-1} = (q_{ij})$, and let B_j be the matrix with ith column equal to e_i if $i \neq j$ and jth column equal to $-e_j - \sum_{i\neq j}(2q_{ji}/q_{jj})e_i$.*

For $\alpha = (i_1, \cdots, i_m) \in G$, *let* $B(\alpha) = B_{i_1}, \cdots, B_{i_m}$. *Then* $a(X_j(\alpha))$ *is the* jth *column of* $B(\alpha)$, $c(X_j(\alpha))$ *is the* jth *column of* $\Delta B(\alpha)$, *and* $\varepsilon(X_j(\alpha))$ *is the* jth *entry in* $\varepsilon^T B(\alpha)$; $\Delta(X_i(\alpha), X_j(\beta))$ *is the* (i, j)th *entry of* $B(\alpha)^T B(\beta)$.

Proof. See Lemma 2 of [4].

4. **Discrete packings.** We now turn to the question of determining conditions on Δ under which an inversively generated configuration $\mathscr{S}(\Delta)$ is a *packing*, i.e., a collection of disjoint spheres. To do this, we will impose conditions on Δ which will ensure that, for any $X, Y \in \mathscr{S}(\Delta)$, the separation $\Delta(X, Y)$ lies in a discrete subset of the reals which does not include the open interval $]-1, 1[$. Auxiliary arguments can then be used to show that, in fact, either $X = Y$ or $\Delta(X, Y) \geqq 1$ for all $X, Y \in \mathscr{S}(\Delta)$.

LEMMA 4.1. *Suppose that* $\Delta^{-1} = (q_{ij})$ *satisfies the following conditions*:

(a) $2q_{ij}/q_{ii}$ *is an integer for all* i *and* j,

(b) *there is a real number* M *and integers* c_i *such that*

$$2/(-q_{ii}) = c_i M \quad \text{for} \quad i = 1, \cdots, N+2,$$

(c) $|\Delta_{ij} + kM| \geqq 1$ *for all integers* k *and* $i, j \in \{1, \cdots, N+2\}$. *Then* $|\Delta(X, Y)| \geqq 1$ *for all* $X, Y \in \mathscr{S}(\Delta)$.

Proof. This depends on the formula

$$\Delta B_i = \Delta - (2/q_{ii})E_i \tag{8}$$

where E_i is a matrix with all entries equal to 0 except for a 1 in the (i, i)th position. For real numbers a, b, M let us write $a \equiv b (\mathrm{mod}\, M)$ if $(a-b)/M$ is an integer. Then (8) and (b) imply that $\Delta B_i \equiv \Delta (\mathrm{mod}\, M)$, and by induction using (a), that $B(\alpha)^T \Delta B(\beta) \equiv \Delta (\mathrm{mod}\, M)$. By Lemma 3.1,

$$\Delta(X_i(\alpha), X_j(\beta)) = \Delta_{ij} + k_{ij} M \tag{9}$$

for some integer k_{ij}. Thus the lemma follows by (c).

COROLLARY 4.2. *If* Δ *has odd integer entries and* Δ^{-1} *satisfies condition* (a) *of Lemma* 4.1, *then it satisfies* (b) *and* (c) *with* $M = 2$.

Proof. Writing (8) as $(-2/q_{ii})E_i = \Delta B_i - \Delta$, we see that $(-2/q_{ii})$ is an integer, and computing modulo 2 we have

$$(-2/q_{ii})E_i \equiv J B_i - J \equiv kJ (\mathrm{mod}\, 2) \tag{10}$$

for some integer k. The left member of (10) has at most one nonzero entry while the right member has all entries equal. Thus $-2/q_{ii} \equiv 0 (\mathrm{mod}\ 2)$.

LEMMA 4.3. *If Δ^{-1} satisfies the conditions of Lemma 4.1, then the entries of B_i are restricted to the set $\{0, \pm 1, \pm 2, \pm 3, \pm 4\}$. Each of the matrices $B_i B_j$ is of order 2, 3, 4, 6 or ∞ (the crystallographic restriction,* cf [8, p. 122]).

Proof. By (a), $-2q_{ij}/q_{ii}$ and $-2q_{ji}/q_{jj}$ are integer, and by Lemma 2.3, since $q_{ij} = q_{ji}$, we have $0 \leqq (2q_{ij}/-q_{ii})(2q_{ji}/-q_{jj}) \leqq 4$ for any i, j. This restricts the entries of B_i as stated. By Lemma 2.3, $B_i B_j$ represents a rotation through an angle 2θ, where

$$\cos \theta = q_{ij}/(-q_{ii})^{1/2}(-q_{jj})^{1/2} .$$

The restriction on $\cos \theta$ just proved implies $B_i B_j$ is of order 2, 3, 4, 6 or ∞.

THEOREM 4.4. *Let Δ be a separation matrix satisfying the conditions of Lemma 4.2. Suppose in addition that one of the following two conditions is satisfied for each $i = 1, \cdots, N+2$:*

(d) *there are at least two entries in row i of Δ which equal 1, or*

(e) *there is one entry in row i of Δ equal to 1 and also*

$$\Delta_{ij} < M - 1 \quad \textit{for all} \quad j = 1, \cdots, N+2 .$$

Then $\mathscr{S}(\Delta)$ is a packing.

Proof. We will assume that $\mathscr{S}(\Delta)$ is not a packing and obtain a contradiction. As in the proof of Theorem 5 of [4], by choosing a minimal counterexample, we may assume that there are two clusters $(X_i) = (X_i(\alpha))$ and $(W_i) = (X_i(\beta))$ such that $X_1 \subsetneqq W_j$ and each X_k, $k \neq 1$ is either equal to a W_m or disjoint from them all, and reciprocally for the W_k, $k \neq j$. If (d) holds then two of the X_k, say X_2 and X_3 touch X_1 in two distinct points. However, X_1 is inside W_j and has at most one point of contact with the boundary of W_j, while X_2 and X_3 are outside of W_j, a contradiction.

If (e) holds, then say $\Delta_{12} = 1$ so X_1 and X_2 are tangent. This point of tangency lies on the boundary of W_j, since X_1 is inside W_j and X_2 is outside. Thus $\Delta(W_j, X_1) = -1$ and $\Delta(W_j, X_2) = 1$. From (9), $\Delta(W_j, X_k) \equiv \Delta_{jk} (\mathrm{mod}\ M)$, so in particular $-1 = \Delta(W_j, X_1) \equiv \Delta_{j1} (\mathrm{mod}\ M)$. This implies $j = 1$, by (e). For $k \neq 1$ we thus have $1 \leqq \Delta(W_1, X_k) \equiv \Delta_{1k}$ so that $\Delta(W_1, X_k) = \Delta_{1k} + d_k M$, where the d_k are nonnegative integers with $d_1 = d_2 = 0$. We now invert in a sphere centred at the point of contact of X_1, X_2 and W_1 so that these become half-spaces. Since

X_k for $k > 1$ is disjoint from X_1 and W_1, and since $X_1 \subsetneqq W_1$ we have $\Delta(X_1, X_k) > \Delta(W_1, X_k)$. But $\Delta_{1k} = \Delta(X_1, X_k) \equiv \Delta(W_1, X_k) = \Delta_{1k} + d_k M$, which is a contradiction.

DEFINITION 4.1. We shall call a packing which satisfies the conditions of Lemma 4.1 a *discrete packing*. (Theorem 4.4 shows such packings exist.)

5. **K-osculatory packings.** In this section we introduce a class of packings which we call K-osculatory packings, and observe that these are complete and that there is an upper bound on the exponents of such packings which is strictly less that N. The proofs follow those of [2] so are not given in great detail. Next we show that if $\mathscr{G}(\Delta)$ is an infinite packing, then it is K-osculatory. The proof is similar to the proof of Theorem 9 of [4].

We recall from [1] that a complete packing of an open set U in E_N is a packing $\mathscr{C} = \{S_n\}$ of U by spheres S_n such that $U \backslash \bigcup S_n$ has measure zero. If U has finite measure, the exponent of $\mathscr{C}$ is defined by $e(\mathscr{C}, U) = \inf\{t: \sum r_n^t < \infty\}$, where r_n is the radius of S_n. We define $R_n = U \backslash (S_1^- \cup \cdots \cup S_1^-)$, and for each $\delta > 0$, write $U(\delta) = \{x \in U: \text{dist}(x, \partial U) \leqq \delta\}$. If $|U(\delta)|$ is the measure of $U(\delta)$ it is easily seen [2, p. 362] that $|U(\delta)| \to 0$ as $\delta \to 0+$. For most sets of interest, e.g., if U is a sphere, $|U(\delta)| = O(\delta)$ as $\delta \to 0+$.

DEFINITION 5.1. Let U be an open set of finite measure and $\mathscr{C} = \{S_n\}$ be a packing of U. Then $\mathscr{C}$ is said to be *K-osculatory* if there are real numbers $K' \geqq K \geqq 1$, and an integer m such that for $n \geqq m$, and $x \in R_n$, either $\text{dist}(x, S_1 \cup \cdots \cup S_n) \leqq K r_{n+1}$ or $\text{dist}(x, \partial U) \leqq K' r_{n+1}$.

THEOREM 5.1. *Let U be an open subset of E_N with finite measure. Suppose $|U(\delta)| = O(\delta^\gamma)$ as $\delta \to 0+$ for some constant $0 < \gamma \leqq 1$. If $\mathscr{C}$ is a K-osculatory packing of U, then $\mathscr{C}$ is complete and*

$$e(\mathscr{C}, U) \leqq \max(\beta, N - \gamma)\,, \tag{11}$$

where β is the unique root of the following, with $N - 1 < \beta < N$:

$$\sum_{j=0}^{N-1} \binom{N}{j} \frac{K^j}{x - j} + \frac{1}{x - N} = 0\,.$$

If $1/s = (K+1)^N - K^N + 1$, then

$$N - Ns < \beta < N - s\,. \tag{12}$$

Proof. As in Lemma 1 of [2] we see that

(13) $\omega_N \sum_{k=n+1}^{\infty} r_k^N \leqq |R_n| \leqq \omega_N \sum_{k=1}^{n} ((r_k + Kr_{n+1})^N - r_k^N) + |U(K'r_{n+1})|$.

Also, if $r(R_n)$ is the inradius of R_n [1], we have $r_{n+1} \leqq r(R_n) \leqq K'r_{n+1}$. Using this fact together with (13), the arguments of [2] suffice to prove the completeness of $\mathscr{C}$ and (11). The estimates (12) are proved as in [2, p. 361].

The proof of Theorem 5.4 will involve a subdivision of E_N into certain polyhedral sets:

DEFINITION 5.2. Let $X_i = S(\underline{a}_i, r_i)$, $i = 1, \cdots, N+1$ be disjoint spheres with finite radii. The *cell* $P = P(X_1, \cdots, X_{N+1})$ is defined in the following way: If all $r_i > 0$, then P is the convex hull of $a_1, \cdots, a_{N+1}$; if one $r_i < 0$ (so $r_j > 0$ for $j \neq i$), then P is the closure of the set difference $K \backslash H$, where K is the polyhedral cone with vertex at a_i generated by $a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_{N+1}$, and H is the convex hull of $a_1, \cdots, a_{N+1}$. We call $a_1, \cdots, a_{N+1}$ the *vertices* of P and $X_1, \cdots, X_{N+1}$ the *corners* of P.

LEMMA 5.2. *Let $X_i = S(\underline{a}_i, r_i), i = 1, \cdots, N+1$ be disjoint spheres, and let X be a sphere with radius $r > 0$ which intersects (or touches) each of $X_1, \cdots, X_{N+1}$. Then $P(X_1, \cdots, X_{N+1}) \subset \bigcup \{S^-(\underline{a}_i, r_i + r): i = 1, \cdots, N+1\} = T$.*

Proof. We treat only the case in which all radii are positive. The proof is by induction on N. We first note that T is starlike with respect to the center of X. We next show that T contains the boundary of P and this will complete the proof since a starlike set containing the boundary of a bounded convex set must contain the whole set. Let Z be a plane face of P, say the face through $a_1, \cdots, a_N$. Let X' be the perpendicular projection of X onto Z, and let $X_i' = X_i \cap Z$ for $i = 1, \cdots, N$. Then X' intersects X_i' since X_i' is also the projection of X_i onto Z, and thus by induction, $P(X_i', \cdots, X_N') \subset \bigcup \{S^-(\underline{a}_i, r_i + r): i = 1, \cdots, N) \subset T$. Thus all faces of P are in T so that $P \subset T$. The proof when one $r_i < 0$ uses similar ideas and we refer the reader to Lemma 8 of [4].

LEMMA 5.3. *Let $X = S(\underline{a}, r)$, $Y = S(\underline{b}, s)$ be disjoint spheres with $r > 0$ and with $\Delta(X, Y) = c \geqq 1$.*

(a) *If $s > 0$, then $S(\underline{a}, cr)$ intersects or touches Y.*

(b) *If $s < 0$, then $S(\underline{a}, (c + (c^2 - 1)^{1/2})r)$ intersects or touches Y.*

Proof. A simple exercise.

THEOREM 5.4. *Let Δ be a separation matrix for which $\mathscr{S}(\Delta)$ is* *an infinite packing. Let K be the largest entry in Δ. Let U be the unit sphere $S(\underline{a}_1, 1)$ and let $X_1 = S(\underline{a}_1, -1)$ be its exterior. Suppose $X_i = S(\underline{a}_i, r_i)$, $i = 1, \cdots, N+2$ have separation matrix Δ, and $\underline{a}_1$ lies in the interior of the convex hull of $\underline{a}_2, \cdots, \underline{a}_{N+2}$ possible by Lemma 2.2). Then $\mathscr{S}(\Delta)\backslash\{X_1\}$ is a K-osculatory packing of U.*

Proof. Let $S_i = X_{i+1}$ for $i = 1, \cdots, N+1$. We shall show that there is a sequence of spheres $\{S_n\}$, $S_n \in \mathscr{S}(\Delta)$ so that $\{S_n\}$ is K-osculatory with $K' = K + (K^2 - 1)^{1/2}$. Then S_n is complete hence is all of $\mathscr{S}(\Delta)$. We select S_n inductively for $n \geqq N+2$ and at the same time subdivide U by cells so that we can verify the conditions of Definition 5.1 using Lemmas 5.3 and 5.4.

To begin with, let P_i be the cell $P(X_1, \cdots, \hat{X}_i, \cdots, X_{N+2})$, where the symbol $\hat{}$ means omit X_i. Then $E_N \subset P_1 \cup \cdots \cup P_{N+2}$ since $\underline{a}_1$ is in the interior of P_1 by assumption. With each P_i is associated a unique next sphere $\mathscr{N}(P_i) = X_i(i) \in \mathscr{S}(\Delta)$. Let S_{N+2} be the $\mathscr{N}(P_i)$ with largest radius, say r_{N+2}. Using Lemma 5.4, for each i there is a sphere of radius Kr_{N+2}, or $K'r_{N+2}$ concentric with $\mathscr{N}(P_i)$ which intersects the corners of P_i. Thus Lemma 5.3 guarantees the conditions of Definition 5.1 for $x \in R_n P_i$, and hence for $x \in R_n$ since $R_n \subset \bigcup P_i$, (where here $n = N+2$). Now, if $S_{N+2} = \mathscr{N}(P_i)$, we replace P_i by $N+1$ cells, each having one corner S_{N+2} and the remaining corners being N of the corners of P_i. This is done for each i for which $S_{N+2} = \mathscr{N}(P_i)$. We renumber the new cells $P_1, \cdots, P_m$. These may overlap but they still cover E_N. Furthermore, the corners of each P_k are of the form $X_1(i), \cdots, \hat{X}_j(i), \cdots, X_{N+2}(i)$ for some j and i. Thus there is a unique $\mathscr{N}(P_k) \in \mathscr{S}(\Delta)$ defined for each k.

The induction now proceeds in an obvious way except at some point we might find that each $\mathscr{N}(P_i)$ is among the S_n already chosen. However, this would imply that $\mathscr{S}(\Delta)$ is finite contrary to our assumption.

6. Examples. In this section we exhibit and investigate thirteen Δ for which $\mathscr{S}(\Delta)$ is a packing. These are discrete packings satisfying the conditions of Lemma 4.1 and Theorem 4.4. Briefly, we seek symmetric matrices Δ which satisfy the following conditions, where $\Delta^{-1} = (q_{ij})$:

(i) $\Delta_{ii} = -1$ for $i = 1, \cdots, N+2$,

(ii) $\Delta_{ij} \geqq 1$ if $i \neq j$,

(iii) Δ has one positive and $N+1$ negative eigenvalues,

(iv) $2q_{ij}/q_{ii}$ is an integer (of absolute value $\leqq 4$) for all i, j,

(v) there is a real $M\ (\geqq 2)$ and integers c_i such that $2/(-q_{ii}) = c_i M$ for all i.

(vi) $|\Delta_{ij} + kM| \geqq 1$ for all integers k, and $i, j \in \{1, \cdots, N+2\}$.

It should be noted that there are at most a finite number of such Δ for each N. For if Ω is the matrix with entries $q_{ij}/(-q_{ii})^{1/2}(-q_{jj})^{1/2}$, then we know from the results of § 4 that $4\Omega_{ij}^2 \in \{0, 1, 2, 3, 4\}$. But Δ is uniquely constructible from Ω by (7).

Possibly the simplest Δ to search for are circulants. We shall write $A = \operatorname{circ}(a_0, \cdots, a_{m-1})$ for the $m \times m$ circulant with first row $a_0, \cdots, a_{m-1}$. We recall that A has the eigenvectors $x_0, \cdots, x_{m-1}$, where x_k is the vector $(1, \omega^k, \cdots, \omega^{(m-1)k})^T$, and $\omega = \exp(2\pi i/m)$, which correspond to the eigenvalues $\lambda_k = a_0 + a_1\omega^k + \cdots + a_{m-1}\omega^{(m-1)k}$. If A is symmetric we must have $a_i = a_{m-i}$ for all i. If Δ is a circulant then the diagonal entries q_{ii} of Δ^{-1} are all equal to $-c$ say, so $\Omega = c^{-1}\Delta^{-1}$. Thus the entries of Ω are restricted to the set $\{0, \pm 1/2, \pm 1\}$ by (iv).

In addition to circulants, we use matrices of the following block form:

$$\Delta = \begin{bmatrix} A & aJ_{m\times n} \\ aJ_{m\times n} & B \end{bmatrix} \tag{14}$$

where A and B are circulants and $J_{m\times n}$ has all entries equal to 1. For these, we need the following:

LEMMA 6.1. *Let A be an $m \times m$ circulant with eigenvalues $\lambda_0, \cdots, \lambda_{m-1}$ and eigenvectors $x_0, \cdots, x_{m-1}$. Let B be an $n \times n$ circulant with eigenvalues $\mu_0, \cdots, \mu_{n-1}$ and eigenvectors $y_0, \cdots, y_{n-1}$. Let Δ be as in (14). Then Δ has eigenvalues $\rho, \sigma, \lambda_1, \cdots, \lambda_{m-1}, \mu_1, \cdots, \mu_{n-1}$ where $\rho > \sigma$ are given by*

$$\rho, \sigma = \frac{1}{2}(\lambda_0 + \mu_0) \pm \frac{1}{2}((\lambda_0 - \mu_0)^2 + 4a^2mn)^{1/2} . \tag{15}$$

Proof. Since $J_{m\times n}y_k = \underline{0}$ for $k = 1, \cdots, n-1$ and $J_{n\times m}x_k = \underline{0}$ for $k = 1, \cdots, m-1$, the vectors $(x_k^T, \underline{0})^T$ and $(\underline{0}, y_k^T)^T$ are eigenvectors for corresponding to $\lambda_1, \cdots, \lambda_{m-1}$ and $\mu_1, \cdots, \mu_{n-1}$. Furthermore, $(\alpha x_0^T, y_0^T)^T$ is an eigenvector of Δ if α satisfies $ma\alpha^2 + \alpha(\lambda_0 - \mu_0) - na = 0$, which gives rise to the two eigenvalues given in (15).

List of examples. We begin by listing the examples we have found so far. The reader should check that the conditions (i) to (vi) and the conditions of Theorem 4.4 are satisfied. We will discuss the examples in more detail later. Both Δ^{-1} and Ω are given in case one is not a multiple of the other. We abbreviate $\alpha = 1/2^{1/2}$ and $\beta = 3^{1/2}/2$.

$N = 2$.

(2.1)
$$\Delta = \operatorname{circ}(-1, 1, 1, 1) , \qquad \Delta^{-1} = \frac{1}{4}\Omega ,$$
$$\Omega = \operatorname{circ}(-1, 1, 1, 1) , \qquad M = 8 .$$

$$(2.2)\qquad \begin{aligned} &\Delta = \mathrm{circ}\left(-1, 1, \frac{3}{2}, 1\right), \qquad \Delta^{-1} = \frac{4}{15}\Omega\,, \\ &\Omega = \mathrm{circ}\left(-1, 1, \frac{1}{2}, 1\right), \qquad M = \frac{15}{2}\,. \end{aligned}$$

$$(2.3)\qquad \begin{aligned} &\Delta = \mathrm{circ}\,(-1, 1, 2, 1)\,, \qquad \Delta^{-1} = \frac{1}{3}\Omega\,, \\ &\Omega = \mathrm{circ}\,(-1, 1, 0, 1)\,, \qquad M = 6\,. \end{aligned}$$

$$(2.4)\qquad \begin{aligned} &\Delta = \mathrm{circ}\left(-1, 1, \frac{5}{2}, 1\right), \qquad \Delta^{-1} = \frac{4}{7}\Omega\,, \\ &\Omega = \mathrm{circ}\left(-1, 1, -\frac{1}{2}, 1\right), \qquad M = \frac{7}{2}\,. \end{aligned}$$

$$(2.5)\qquad \begin{aligned} &\Delta = \mathrm{circ}\,(-1, 2, 1, 2)\,, \qquad \Delta^{-1} = \frac{1}{4}\Omega\,, \\ &\Omega = \mathrm{circ}\left(-1, \frac{1}{2}, 1, \frac{1}{2}\right), \qquad M = 8\,. \end{aligned}$$

$$(2.6)\qquad \begin{aligned} &\Delta = \begin{bmatrix} \mathrm{circ}\,(-1, 1) & J_{2\times2} \\ J_{2\times2} & \mathrm{circ}\,(-1, 3) \end{bmatrix} \quad \Delta^{-1} = \frac{1}{8}\begin{bmatrix} \mathrm{circ}\,(-4, 0) & J_{2\times2} \\ J_{2\times2} & \mathrm{circ}\,(-1, 1) \end{bmatrix} \\ &\Omega = \begin{bmatrix} \mathrm{circ}\,(-1, 0) & J_{2\times2} \\ J_{2\times2} & \mathrm{circ}\,(-1, 1) \end{bmatrix} \quad M = 4\,. \end{aligned}$$

$N = 3$.

$$(3.1)\qquad \begin{aligned} &\Delta = \mathrm{circ} = (-1, 1, 1, 1, 1)\,, \qquad \Delta^{-1} = \frac{1}{3}\Omega\,, \\ &\Omega = \mathrm{circ}\left(-1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \qquad M = 6\,. \end{aligned}$$

$$(3.2)\qquad \begin{aligned} &\Delta = \begin{bmatrix} \mathrm{circ}\,(-1,1,1) & J_{3\times2} \\ J_{2\times3} & \mathrm{circ}\,(-1, 3) \end{bmatrix} \quad \Delta^{-1} = \frac{1}{4}\begin{bmatrix} \mathrm{circ}\,(-2,0,0) & J_{3\times2} \\ J_{2\times3} & \mathrm{circ}\,(-1,0) \end{bmatrix} \\ &\Omega = \begin{bmatrix} \mathrm{circ}\,(-1,0,0) & \alpha J_{3\times2} \\ \alpha J_{2\times3} & \mathrm{circ}\,(-1, 0) \end{bmatrix} \quad M = 4\,. \end{aligned}$$

$$(3.3)\qquad \begin{aligned} &\Delta = \begin{bmatrix} \mathrm{circ}\,(-1, 1, 1) & J_{3\times2} \\ J_{2\times3} & \mathrm{circ}\,(-1, 5) \end{bmatrix} \\ &\Delta^{-1} = \frac{1}{6}\begin{bmatrix} 3J_{2\times3} & 3J_{3\times2} \\ \mathrm{circ}\,(-6, -3, -3) & \mathrm{circ}\,(-2, -1) \end{bmatrix} \\ &\Omega = \begin{bmatrix} \mathrm{circ}\,(-1, -\frac{1}{2}, -\frac{1}{2}) & \beta J_{3\times2} \\ \beta J_{2\times3} & \mathrm{circ}\,(-1, -\frac{1}{2}) \end{bmatrix} \qquad M = 2\,. \end{aligned}$$

$N = 4$.

$$(4.1)\qquad \begin{aligned} &\Delta = \operatorname{circ}\left(-1, 1, \frac{3}{2}, 1, \frac{3}{2}, 1\right), \qquad \Delta^{-1} = \frac{1}{10}\Omega, \\ &\Omega = \operatorname{circ}\left(-1, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}\right), \qquad M = 5 . \end{aligned}$$

$N = 5$.

$$(5.1)\qquad \begin{aligned} \Delta &= \begin{bmatrix} \operatorname{circ}(-1, 1, 1, 1) & J_{4\times 3} \\ J_{3\times 4} & \operatorname{circ}(-1, 3, 3) \end{bmatrix} \\ \Delta^{-1} &= \frac{1}{4}\begin{bmatrix} \operatorname{circ}(-4, -2, -2, -2) & 2J_{4\times 3} \\ 2J_{3\times 4} & \operatorname{circ}(-2, -1, -1) \end{bmatrix} \\ \Omega &= \begin{bmatrix} \operatorname{circ}(-1, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) & \alpha J_{4\times 3} \\ \alpha J_{3\times 4} & \operatorname{circ}(-1, -\frac{1}{2}, -\frac{1}{2}) \end{bmatrix}, \qquad M = 2 . \end{aligned}$$

$$(5.2)\qquad \begin{aligned} \Delta &= \begin{bmatrix} \operatorname{circ}(-1, 1, 1, 1, 1) & J_{5\times 2} \\ J_{2\times 5} & \operatorname{circ}(-1, 3) \end{bmatrix} \\ \Omega &= \begin{bmatrix} \operatorname{circ}(-1, 0, 0, 0, 0) & \frac{1}{2}J_{5\times 2} \\ \frac{1}{2}J_{2\times 5} & \operatorname{circ}(-1, -\frac{1}{2}) \end{bmatrix} \\ \Delta^{-1} &= \frac{1}{2}\Omega, \qquad M = 4 . \end{aligned}$$

$N = 9$.

$$(9.1)\qquad \begin{aligned} \Delta &= \begin{bmatrix} \operatorname{circ}(-1, 3, 1, 1, 3) & J_{5\times 6} \\ J_{6\times 5} & \operatorname{circ}(-1, 1, 1, 1, 1, 1) \end{bmatrix} \\ \Delta^{-1} = \Omega &= \begin{bmatrix} \operatorname{circ}(-1, -\frac{1}{2}, 0, 0, -\frac{1}{2}) & \frac{1}{2}J_{5\times 6} \\ \frac{1}{2}J_{6\times 5} & \operatorname{circ}(-1, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) \end{bmatrix} \\ M &= 2 . \end{aligned}$$

Discussion of the examples. The examples (2.1) to (2.5) are all possible 4×4 circulants satisfying conditions (i) to (vi), constructed by starting with Ω. The examples (2.6), (3.2), (3.3), (5.1), and (5.2) are all possible matrices of the form (14) in which A and B have the form $bJ - (b + 1)I$, with b an integer. We have eliminated those trivially equivalent to a circulant (by renumbering rows and columns). All of (2.1) to (2.5) can be put into form (14) by interchange of rows 2 and 3 and columns 2 and 3. The example (9.1) is the only example which is of the block form (14) where $A = \operatorname{circ}(-1, 3, 1, 1, 3)$ and $B = J - 2I$. The reason for this choice of A is that it is the only circulant of the form $\operatorname{circ}(-1, b, 1, \cdots, 1, b)$ with $b \geq 3$ which has all eigenvalues negative except for λ_0.

Note that it is possible to have $\Delta \neq \Delta'$ and yet $\mathscr{G}(\Delta) = \mathscr{G}(\Delta')$. In this case we say that Δ and Δ' are *equivalent* and write $\Delta \simeq \Delta'$. We can often settle the question of equivalence by looking at the set of real numbers $\{\Delta(X, Y): X, Y \in \mathscr{S}(\Delta)\}$ and using (9). In this way we find that no two of (2.1) through (2.6) are equivalent, and that $\Delta_{3.1} \not\simeq \Delta_{3.2}$. It fact $\Delta_{3.1} \simeq \Delta_{3.3}$, and we believe also that $\Delta_{5.1} \simeq \Delta_{5.2}$.

Cross sections: If we have a packing in which two spheres, say X_1, X_2, are tangent, then we may invert so these become half-spaces and the spheres in tne packing which touch both X_1 and X_2 become equal spheres all orthogonal to a plane half way between X_1 and X_2. The cross section of these spheres with this plane produces a packing (possibly empty) of E_{N-1} by equal spheres. This is effective for studying $\mathscr{S}(\Delta)$ if two rows of all off-diagonal entries equal to 1. More generally, if we only have $\Delta_{ij} = 1$ for a single (i, j), say $(i, j) = (N + 1, N + 2)$, then choosing X_{N+1} and X_{N+2} as half-spaces, we see that $X_1, \cdots, X_N$ become spheres whose curvatures can be chosen to be $\varepsilon_i = \Delta(X_i, X_{N+1}) + \Delta(X_i, X_{N+2})$. The biorthogonal spheres $Y_1, \cdots, Y_N$ become half-spaces orthogonal to X_{N+1} and X_{N+2} and thus the matrices $B_1, \cdots, B_N$ represent reflections in hyperplanes. The matrix Ω tells us the dihedral angles between those planes. Taking any cross section parallel to X_{N+1} and X_{N+2} we obtain N spheres X_i' from the X_i, and N planes Y_i' from the Y_i, $i = 1, \cdots, N$. The images of the X_i' under the group generated by the reflections in the Y_i' form a packing of E_{N-1} by spheres of at most N different sizes. These are of course cross sections of the spheres $X_k(\alpha)$, where k and the components of α are in $\{1, 2, \cdots, N\}$.

By using a similar device we can often check that $\mathscr{S}(\Delta)$ is infinite. For $\mathscr{S}(\Delta)$ is infinite if and only if the group $\Gamma(\Delta)$ generated by $B_1, \cdots, B_{N+2}$ is infinite, since the columns of an element of $\Gamma(\Delta)$ consist of coordinates of spheres in $\mathscr{S}(\Delta)$. An examination of Ω will often reveal infinite subgroups of $\Gamma(\Delta)$. For example, if $\Omega_{ij} = 1$ for some i, j, then B_iB_j is of infinite order. This is the case for our examples with $N = 2$. In the other examples Ω contains a 3×3 submatrix with off-diagonal entries all $\pm 1/2$ hence $\Gamma(\Delta)$ contains an infinite subgroup generated by the three corresponding B_i, (see Table 11 of [12, p. 142]).

In the examples with $N = 3$, making $\varepsilon_1 = \varepsilon_2 = 0$, the cross section of the packings (3.1) and (3.3) is the closest packing of circles. The cross section of (3.2) is the packing with circles centered at the points of a square lattice. In example (4.1), taking $\varepsilon_5 = \varepsilon_6 = 0$, the cross section half way between X_5 and X_6 is a packing of E_3 by equal spheres centered at the points of the body-centered cubic lattice. This is not the densest packing which has centers at the points of the face-centered cubic lattice. For (5.1) and (5.2), take $\varepsilon_1 = \varepsilon_2 = 0$. It is easily seen from Δ that the lattice of centers of the cross section is such that the points closest to (0, 0, 0, 0) are the 24 vertices of the 24-cell, which is known to give the densest packing. For example (9.1) the cross section is the densest lattice packing of E_8.

Exponents. Since the exponent of $\mathscr{S}(\Delta)$ is unchanged by an inversion which leaves one of the spheres with negative curvature, it is reasonable to speak of the exponent of $\mathscr{S}(\Delta)$, which we denote by $e(\Delta)$. Example (2.1) is the ordinary two dimensional osculatory packing discussed in [3], [9], [13], for example. The exponent S of this packing is known to satisfy [3]:

$$1.300197 < S < 1.314534 .$$

In this packing, if one takes X_1, X_2, X_3 to have positive curvature then they enclose a curvilinear triangle T. If one lets X_4 be the smaller disk touching these three, then the removal of X_4 from T leaves four curvilinear triangles, suggesting an iterative procedure. The packing (2.6) is quite similar to this. Again one can take X_1, X_2, and X_3 to enclose T but now X_4 touches only X_1 and X_2. However, the three disks X_4, $X_1(1)$, $X_2(2)$ are mutually tangent and the removal of these from T leaves seven new triangles. The methods of [3] should now be applicable to give rigorous bounds on the exponent of this packing.

The packing (3.1) is the three dimensional osculatory packing to which [4] was devoted. We described in [5] an algorithm for generating the coordinates of the spheres in this packing without duplication. By counting the number of spheres $W(K)$ of curvature K for each $K \leqq 300$ in the 'Soddy packing' of the unit sphere, which begins with spheres of curvatures $(-1, 2, 2, 3, 3)$, all mutually tangent, we obtained the heuristic result:

$$e(\Delta_{3.1}) \simeq 2.42009 . \tag{15}$$

The packing (3.2) is quite similar. We again can start with curvatures $(-1, 2, 2, 3, 3)$, but now the four spheres X_4, X_5, $X_4(4)$, $X_5(5)$ all have curvature 3 and their centers are at the vertices of a square. Using a similar algorithm to that of [5] we generated the 667062 spheres of curvature $\leqq 400$ for this packing, obtaining the estimate:

$$e(\Delta_{3.2}) \simeq 2.44445 . \tag{16}$$

This is not very different from (15) but we believe in fact that $e(\Delta_{3.2})$ is strictly greater than $e(\Delta_{3.1})$. Of course, the estimates (15) and (16) are not rigorous. The only rigorous estimates available for exponents other than S are those given by Theorem 5.1 combined with Theorem 5.4, which gives upper bounds, and the lower bound $(N-1)+.03$

of Larman [11]. The value of K given in Theorem 5.4 can often be improved by simple arguments. For example, we can show that (3.2) and (5.2) are 2-osculatory, but for reasons of space we will not expand on this here.

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