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REGULAR PAPER

A new class of irreducible pentanomials for polynomial-based multipliers in binary fields

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Abstract
We introduce a new class of irreducible pentanomials over $\mathbb{F}_2$ of the form $f(x) = x^{2b+c} + x^{b+c} + x^b + x^c + 1$. Let $m = 2b + c$ and use f to define the finite field extension of degree m. We give the exact number of operations required for computing the reduction modulo f. We also provide a multiplier based on Karatsuba algorithm in $\mathbb{F}_2[x]$ combined with our reduction process. We give the total cost of the multiplier and found that the bit-parallel multiplier defined by this new class of polynomials has improved XOR and AND complexity. Our multiplier has comparable time delay when compared to other multipliers based on Karatsuba algorithm.

Keywords Irreducible pentanomials · Polynomial multiplication · Modular reduction · Finite fields

1 Introduction

Finite field extensions $\mathbb{F}_{2^m}$ of the binary field $\mathbb{F}_2$ play a central role in many engineering applications and areas such as cryptography. Elements in these extensions are commonly represented using polynomial or normal bases. We center in this paper on polynomial bases for bit-parallel multipliers.

When using polynomial bases, since $\mathbb{F}_{2^m} \cong \mathbb{F}_2[x]/(f)$ for an irreducible polynomial f over $\mathbb{F}_2$ of degree m, we write elements in $\mathbb{F}_{2^m}$ as polynomials over $\mathbb{F}_2$ of degree smaller than m. When multiplying with elements in $\mathbb{F}_{2^m}$, a polynomial of degree up to $2m - 2$ may arise. In this case, a modular reduction is necessary to bring the resulting element back to $\mathbb{F}_{2^m}$. Mathematically, any irreducible polynomial can be used to define the extension. In practice, however, the choice of the irreducible f is crucial for fast and efficient field multiplication.

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There are two types of multipliers in $\mathbb{F}_{2^m}$: one-step algorithms and two-step algorithms. Algorithms of the first type perform modular reduction while the elements are being multiplied. In this paper, we are interested in two-step algorithms, that is, in the first step the multiplication of the elements is performed, and in the second step the modular reduction is executed. Many algorithms have been proposed for both types. An interesting application of two-step algorithms is in several cryptographic implementations that use the lazy reduction method [2,23]. For example, in [15] it is shown the impact of lazy reduction in operations for binary elliptic curves. An important application of the second part of our algorithm, the reduction process, is to side-channel attacks. Indeed, we prove that our modular reduction requires a constant number of arithmetic operations, and as a consequence, it prevents side-channel attacks.

The complexity of hardware circuits for finite field arithmetic in $\mathbb{F}_{2^m}$ is related to the amount of space and the time delay needed to perform the operations. Normally, the number of exclusive-or (XOR) and AND gates is a good estimation of the space complexity. The time complexity is the delay due to the use of these gates.

Several special types of irreducible polynomials have been considered before, including polynomials with few nonzero terms like trinomials and pentanomials (three and five nonzero terms, respectively), equally spaced polynomials, all-one polynomials [7,12,19], and other special families of polynomials [27]. In general, trinomials are preferred, but for degrees where there are no irreducible trinomials, pentanomials are considered.

The analysis of the complexity using trinomials is known [26]. However, there is no general complexity analysis of a generic pentanomial in the literature. Previous results (see [5] for details) have focus on special classes of pentanomials, including:

- $x^m + x^{b+1} + x^b + x^{b-1} + 1$, where $2 \le b \le m/2 - 1$ [9,11,18,20,28];
- $x^m + x^{b+1} + x^b + x + 1$, where $1 < b < m - 1$ [9,10,18–20,28];
- $x^m + x^{m-c} + x^b + x^c + 1$, where $1 \le c < b < m - c$ [3];
- $x^m + x^a + x^b + x^c + 1$, where $1 \le c < b < a \le m/2$ [19];
- $x^m + x^{m-s} + x^{m-2s} + x^{m-3s} + 1$, where $(m-1)/8 \le s \le (m-1)/3$ [19];
- $x^{4c} + x^{3c} + x^{2c} + x^c + 1$, where $c = 5^i$ and $i \ge 0$ [7,8].

Like our family, these previous families focus on bit operations, i.e., operations that use only AND and XOR gates. In the literature, it is possible to find studies that use computer words to perform the operations [17,21], but this is not the focus of our work.

1.1 Contributions of this paper

In this paper, we introduce a new class of irreducible pentanomials with the following format:

$$f(x) = x^{2b+c} + x^{b+c} + x^b + x^c + 1, b > c > 0. \quad (1)$$

We compare our pentanomial with the first two families from the list above. The reason to choose these two family is that [18] presents a multiplier considering these families with complexity 25% smaller than the other existing works in the literature using quadratic algorithms. Since our multiplier is based on Karatsuba's algorithm, we also compare our method with Karatsuba type algorithms.

An important reference for previously used polynomials and their complexities is the recent survey on bit-parallel multipliers by Fan and Hasan [5]. Moreover, we observe that all finite fields results used in this paper can be found in the classical textbook by Lidl and Niederreiter [13]; see [14] for recent research in finite fields.

We prove that the complexity of the reduction depends on the exponents b and c of the pentanomial. A consequence of our result is that for a given degree $m = 2b + c$, for any positive integers $b > c > 0$, all irreducible polynomials in our family have the same space and time complexity. We provide the exact number of XORs and gate delay required for the reduction of a polynomial of degree $2m - 2$ by our pentanomials. The number of XORs needed is $3m - 2 = 6b + 3c - 2$ when $b \neq 2c$; for $b = 2c$ this number is $\frac{12}{5}m - 1 = 12c - 1$. We also show that AND gates are not required in the reduction process. It is easy to verify that our reduction algorithm is "constant-time" since it runs the same amount of operations independent of the inputs and it avoids timing side-channel attacks [6].

For comparison purposes with other pentanomials proposed in the literature, since the operation considered in those papers is the product of elements in $\mathbb{F}_{2^m}$, we also consider the number of ANDs and XORs used in the multiplication prior to the reduction. In the literature, one can find works that use the standard product or use some more efficient method of multiplication, such as Karatsuba, and then add the complexity of the reduction.

In this paper, we use a Karatsuba multiplier combined with our fast reduction method. The total cost is then $Cm^{\log_2 3} + 3m - 2$ or $Cm^{\log_2 3} + \frac{12}{5}m - 1$, depending on $b \neq 2c$ or $b = 2c$, respectively. The constant C of the Karatsuba multiplier depends on the implementation. In our experiments, C is strictly less than 6 for all practical degrees, up to degrees 1024. For the reduction, we give algorithms that achieve the above number of operations using any irreducible pentanomial in our family. We compare the complexity of the Karatsuba multiplier with our reduction with the method proposed by Park et al. [18], as well as, with Karatsuba variants given in [5].

1.2 Structure of the paper

The structure of this paper is as follows. In Sect. 2, we give the number of required reduction steps when using a pentanomial f from our family. We show that for our pentanomials this number is 2 or 3. This fact is crucial since such a low number of required reduction steps of our family allows for not only an exact count of the XOR operations but also for a reduced time delay. Our strategy for that consists in describing the reduction process throughout equations, cleaning the redundant operations and presenting the final optimized algorithm. Section 3 provides the first component of our strategy. In this section, we simply reduce a polynomial of degree at most or exactly $2m - 2$ to a polynomial of degree smaller than m. The second component of our strategy is more delicate, and it allows us to derive the exact number of operations involved when our pentanomial f is used to define $\mathbb{F}_{2^m}$. Sections 4 and 5 provide those analyses for the cases when two and three steps of reduction are needed, that is, when $c = 1$ and $c > 1$, respectively. We give algorithms and exact estimates for the space and time complexities in those cases. Also, we describe a Karatsuba multiplier implementation combined with our reduction. In Sect. 6, based on our implementation, we show that the number of XOR and AND gates is better than the known space complexity in the literature. On the other hand, the time complexity (delay) in our implementation is worse than quadratic methods but comparable with Karatsuba implementations. Hence, our multiplier would be preferable in situations where space complexity and saving energy are more relevant than time complexity. We demonstrate that our family contains many polynomials, including degrees where pentanomials are suggested by NIST. Conclusions are given in Sect. 7.

2 The number of required reductions

When operating with two elements in $\mathbb{F}_{2^m}$, represented by polynomials, we obtain a polynomial of degree at most $2m-2$. In order to obtain the corresponding element in $\mathbb{F}_{2^m}$, a further division with remainder by an irreducible polynomial f of degree m is required. We can see this reduction as a process to bring the coefficient in interval $[2m-2, m]$ to a position less than m. This is done in steps. In each step, the coefficients in interval $[2m-2, m]$ of the polynomial is substituted by the equivalent bits following the congruence $x^m \equiv x^a + x^b + x^c + 1$. Once the coefficient in position $2m-2$ is brought to a position less than m, the reduction is completed.

In this section, we carefully look into the number of steps needed to reduce the polynomial by our polynomial f given in Eq. (1). The most important result of this section is that we need at most 3 steps of this reduction process using our polynomials. This information is used in the next sections to give the exact number of operations when the irreducible pentanomial given in Equation (1) is employed. This computation was possible because our family has a small number of required reduction steps.

Let $D_0(x) = \sum_{i=0}^{2m-2} d_i x^i$ be a polynomial over $\mathbb{F}_2$. We want to compute D_{red}, the remainder of the division of D_0 by f, where f has the form $f(x) = x^{2b+c} + x^{b+c} + x^b + x^c + 1$ with $2b + c = m$ and $b > c > 0$. The maximum number k_a of reduction steps for a pentanomial $x^m + x^a + x^b + x^c + 1$ in terms of the exponent a is given by Sunar and Koç [22]

$$k_a = \left\lfloor \frac{m-2}{m-a} \right\rfloor + 1.$$

In our case $m = 2b + c$ and $a = b + c$, thus

$$\begin{aligned} k_{b+c} &= \left\lfloor \frac{2b+c-2}{2b+c-b-c} \right\rfloor + 1 = \left\lfloor \frac{c-2}{b} \right\rfloor + 3 \\ &= \begin{cases} 2 & \text{if } c = 1, \\ 3 & \text{if } c > 1. \end{cases} \end{aligned} \tag{2}$$

Using the same method as in [22], we can derive the number of steps required associated to the exponents b and c. These numbers are needed in Sect. 3. We get

$$k_b = \left\lfloor \frac{2b+c-2}{2b+c-b} \right\rfloor + 1 = \left\lfloor \frac{b-2}{b+c} \right\rfloor + 2 = 2, \tag{3}$$

and

$$\begin{aligned} k_c &= \left\lfloor \frac{2b+c-2}{2b+c-c} \right\rfloor + 1 = \left\lfloor \frac{c-2}{2b} \right\rfloor + 2 \\ &= \begin{cases} 1 & \text{if } c = 1, \\ 2 & \text{if } c > 1. \end{cases} \end{aligned} \tag{4}$$

Thus, the reduction process for our family of pentanomials involves at most three steps. This is a special property that our family enjoys.

The general process for the reduction proposed in this paper is given in the next section. The special case $c = 1$, that is when our polynomials have the form $f(x) = x^{2b+1} + x^{b+1} + x^b + x + 1$, requires two steps. This family is treated in detail in Sect. 4. The general case of our family $f(x) = x^{2b+c} + x^{b+c} + x^b + x^c + 1$ for $c > 1$ involves three steps and is treated in Sect. 5.

3 The general reduction process

The general process that we follow to get the original polynomial D_0 reduced to a polynomial of degree smaller than m is depicted in Fig. 1. Without loss of generality, we consider the polynomial to be reduced as always having degree $2m-2$. Indeed, the cost to determine the degree of the polynomial to be reduced is equivalent to checking if the leading coefficient is zero.

The polynomial D_0 to be reduced is split into two parts: A_0 is the piece of the original polynomial with degree at least m and hence that requires extra work, while B_0 is formed by the terms of D_0 with exponents smaller than m and so that it does not require to be reduced. Dividing the leading term of A_0 by f with remainder we obtain D_1. In the same way

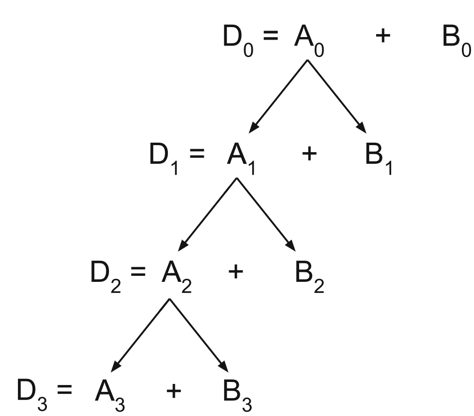


Fig. 1 Tree representing the general reduction strategy

as before, we split D_1 in two parts A_1 and B_1 and repeat the process obtaining the tree of Fig. 1.

3.1 Determining A_0 and B_0

We trivially have

$$D_0(x) = A_0(x) + B_0(x) = \sum_{i=m}^{2m-2} d_i x^i + \sum_{i=0}^{m-1} d_i x^i,$$

and hence

$$A_0 = \sum_{i=m}^{2m-2} d_i x^i \quad \text{and} \quad B_0 = \sum_{i=0}^{m-1} d_i x^i. \tag{5}$$

3.2 Determining A_1 and B_1

Using for clarity the generic form of a pentanomial over $\mathbb{F}_2$, $f(x) = x^m + x^a + x^b + x^c + 1$, dividing the leading term of A_0 by f and taking the remainder, we get

$$D_1 = \sum_{i=0}^{m-2} d_{i+m} x^{i+a} + \sum_{i=0}^{m-2} d_{i+m} x^{i+b} + \sum_{i=0}^{m-2} d_{i+m} x^{i+c} + \sum_{i=0}^{m-2} d_{i+m} x^i.$$

Separating the already reduced part of D_1 from the piece of D_1 that still requires more work, we obtain

$$A_1 = \sum_{i=m}^{m+a-2} d_{i+(m-a)} x^i + \sum_{i=m}^{m+b-2} d_{i+(m-b)} x^i + \sum_{i=m}^{m+c-2} d_{i+(m-c)} x^i, \tag{6}$$

and

$$B_1 = \sum_{i=a}^{m-1} d_{i+(m-a)} x^i + \sum_{i=b}^{m-1} d_{i+(m-b)} x^i + \sum_{i=c}^{m-1} d_{i+(m-c)} x^i + \sum_{i=0}^{m-2} d_{i+m} x^i.$$

Since $m = 2b + c$ and $a = b + c$, we have

$$A_1 = \sum_{i=2b+c}^{3b+2c-2} d_{i+b} x^i + \sum_{i=2b+c}^{3b+c-2} d_{i+b+c} x^i + \sum_{i=2b+c}^{2b+2c-2} d_{i+2b} x^i,$$

$$B_1 = \sum_{i=b+c}^{2b+c-1} d_{i+b} x^i + \sum_{i=b}^{2b+c-1} d_{i+b+c} x^i + \sum_{i=c}^{2b+c-1} d_{i+2b} x^i + \sum_{i=0}^{2b+c-2} d_{i+2b+c} x^i. \tag{7}$$

3.3 Determining A_2 and B_2

As before, we divide the leading term of A_1 by f and we obtain the remainder D_2. We get $D_2 = D_{2_a} + D_{2_b} + D_{2_c}$, where D_{2_a}, D_{2_b} and D_{2_c} refer to the reductions of the sums in Eq. (6).

We start with D_{2_a}:

$$D_{2_a} = \sum_{i=0}^{a-2} d_{i+2m-a} x^i (x^a + x^b + x^c + 1).$$

Separating D_{2_a} in the pieces A_{2_a} and B_{2_a}, we get $A_{2_a} = \sum_{i=m}^{2a-2} d_{i+2m-2a} x^i$ since $b + a - 2 < m$, and

$$B_{2_a} = \sum_{i=a}^{m-1} d_{i+2m-2a} x^i + \sum_{i=b}^{a+b-2} d_{i+2m-a-b} x^i + \sum_{i=c}^{a+c-2} d_{i+2m-a-c} x^i + \sum_{i=0}^{a-2} d_{i+2m-a} x^i.$$

Substituting $m = 2b + c$ and $a = b + c$, we get $A_{2_a} = \sum_{i=2b+c}^{2b+2c-2} d_{i+2b} x^i$, and

$$B_{2_a} = \sum_{i=b+c}^{2b+c-1} d_{i+2b} x^i + \sum_{i=b}^{2b+c-2} d_{i+2b+c} x^i + \sum_{i=c}^{b+2c-2} d_{i+3b} x^i + \sum_{i=0}^{b+c-2} d_{i+3b+c} x^i.$$

Proceeding with the reduction now of the second sum in Eq. (6), we obtain

$$D_{2_b} = \sum_{i=a}^{a+b-2} d_{i+2m-a-b} x^i + \sum_{i=b}^{2b-2} d_{i+2m-2b} x^i + \sum_{i=c}^{b+c-2} d_{i+2m-b-c} x^i + \sum_{i=0}^{b-2} d_{i+2m-b} x^i.$$

Clearly, D_{2_b} is already reduced, and thus $A_{2_b} = 0$, and

$$B_{2_b} = \sum_{i=b+c}^{2b+c-2} d_{i+2b+c} x^i + \sum_{i=b}^{2b-2} d_{i+2b+2c} x^i + \sum_{i=c}^{b+c-2} d_{i+3b+c} x^i + \sum_{i=0}^{b-2} d_{i+3b+2c} x^i.$$

We finally reduce the third and last sum in Eq. (6):

$$D_{2_c} = \sum_{i=a}^{a+c-2} d_{i+2m-a-c}x^i + \sum_{i=b}^{b+c-2} d_{i+2m-b-c}x^i + \sum_{i=c}^{2c-2} d_{i+2m-2c}x^i + \sum_{i=0}^{c-2} d_{i+2m-c}x^i.$$

Again, we easily check that D_{2_c} is reduced and so $A_{2_c} = 0$, and

$$B_{2_c} = \sum_{i=b+c}^{b+2c-2} d_{i+3b}x^i + \sum_{i=b}^{b+c-2} d_{i+3b+c}x^i + \sum_{i=c}^{2c-2} d_{i+4b}x^i + \sum_{i=0}^{c-2} d_{i+4b+c}x^i.$$

Concluding, A_2 is given by

$$A_2 = A_{2_a} + A_{2_b} + A_{2_c} = \sum_{i=m}^{2a-2} d_{i+2m-2a}x^i, \tag{8}$$

and $B_2 = B_{2_a} + B_{2_b} + B_{2_c}$ is

$$\begin{aligned} B_2 = &\sum_{i=b+c}^{2b+c-1} d_{i+2b}x^i + \sum_{i=c}^{b+2c-2} d_{i+3b}x^i + \sum_{i=b+c}^{b+2c-2} d_{i+3b}x^i \\ &+ \sum_{i=c}^{2c-2} d_{i+4b}x^i + \sum_{i=b}^{2b+c-2} d_{i+2b+c}x^i + \sum_{i=b+c}^{2b+c-2} d_{i+2b+c}x^i \\ &+ \sum_{i=b}^{2b-2} d_{i+2b+2c}x^i + \sum_{i=0}^{b+c-2} d_{i+3b+c}x^i + \sum_{i=c}^{b+c-2} d_{i+3b+c}x^i \\ &+ \sum_{i=b}^{b+c-2} d_{i+3b+c}x^i + \sum_{i=0}^{b-2} d_{i+3b+2c}x^i + \sum_{i=0}^{c-2} d_{i+4b+c}x^i. \end{aligned} \tag{9}$$

3.4 Determining A_3 and B_3

Dividing the leading term of A_2 in Eq. (8) by f, we have

$$D_3 = \sum_{i=b+c}^{b+2c-2} d_{i+3b}x^i + \sum_{i=b}^{b+c-2} d_{i+3b+c}x^i + \sum_{i=c}^{2c-2} d_{i+4b}x^i + \sum_{i=0}^{c-2} d_{i+4b+c}x^i.$$

We have that D_3 is reduced so $A_3 = 0$ and

$$B_3 = \sum_{i=b+c}^{b+2c-2} d_{i+3b}x^i + \sum_{i=b}^{b+c-2} d_{i+3b+c}x^i + \sum_{i=c}^{2c-2} d_{i+4b}x^i + \sum_{i=0}^{c-2} d_{i+4b+c}x^i. \tag{10}$$

3.5 The number of terms in A_r and B_r

Let $G(i) = 1$ if $i > 0$ and $G(i) = 0$ if $i \leq 0$. Let r be a reduction step. It is clear now that the precise number of terms for A_r and B_r, for $r \geq 0$, can be obtained using k_{b+c}, k_b and k_c given in Eqs. (2), (3) and (4). We have:

(i) The number of terms of A_0 and B_0 is 1.
(ii) For $r > 0$, the number of terms of A_r is $G(k_{b+c} - r) + G(k_b - r) + G(k_c - r)$, while the number of terms of B_r is 4 times the number of terms of A_{r-1}.

4 The family of polynomials $f(x) = x^{2b+1} + x^{b+1} + x^b + x + 1$

In this section, we consider the case when $c = 1$, that is, when $k_{b+c} = 2$, as given in Eq. (2). The polynomials in this subfamily have the form $f(x) = x^{2b+1} + x^{b+1} + x^b + x + 1$. For the subfamily treated in this section, since $k_{b+c} = 2$, we immediately get $A_2 = 0$ and the expressions in the previous section simplify. As a consequence, the desired reduction is given by

$$D_{red} = B_0 + B_1 + B_2.$$

Using Eqs. (5), (7) and (9), we obtain

$$\begin{aligned} D_{red} = &\sum_{i=0}^{2b} d_i x^i + \sum_{i=b+1}^{2b} d_{i+b}x^i + \sum_{i=1}^{b} d_{i+2b}x^i \\ &+ \sum_{i=1}^{b} d_{i+3b}x^i + \sum_{i=b}^{2b} d_{i+b+1}x^i \\ &+ \sum_{i=0}^{b-1} d_{i+2b+1}x^i + \sum_{i=b+1}^{2b-1} d_{i+2b+1}x^i \\ &+ \sum_{i=b}^{2b-2} d_{i+2b+2}x^i + \sum_{i=0}^{b-2} d_{i+3b+2}x^i + d_{3b+1}. \end{aligned} \tag{11}$$

A crucial issue that allows us to give improved results for our family of pentanomials is the fact that redundancies occur for D_{red} in Eq. (11). Let

$$T_1(j) = \sum_{i=0}^{b-2} (d_{i+2b+1} + d_{i+3b+2})x^{i+j}, \quad T_2(j) = d_{3b}x^j,$$

$$T_3(j) = d_{3b+1}x^j, \quad T_4(j) = \sum_{i=0}^{b-1} (d_{i+2b+1} + d_{i+3b+1})x^{i+j}.$$

A careful analysis of Eq. (11) reveals that T_1, T_2 and T_3 are used more than once, and hence, savings can occur. We rewrite Eq. (11) as

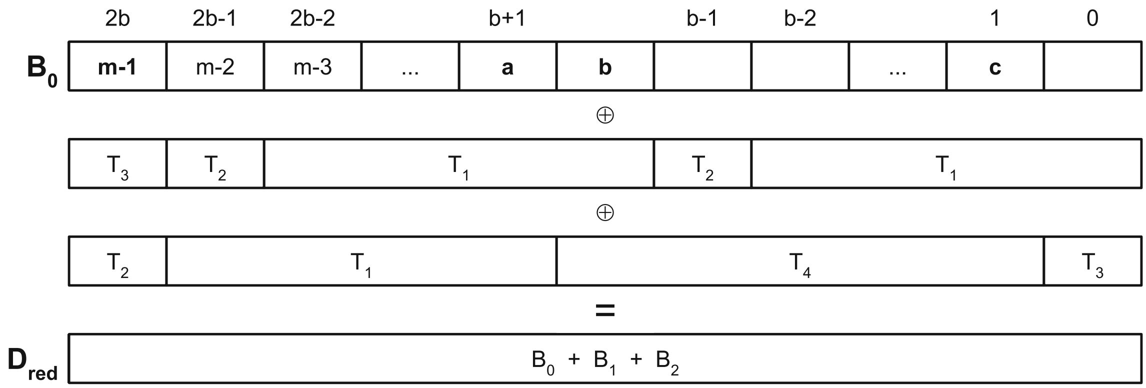


Fig. 2 Representation of the reduction by $f(x) = x^{2b+1} + x^{b+1} + x^b + x + 1$

$$D_{red} = B_0 + T_1(0) + T_1(b) + T_1(b+1) + T_2(b-1) + T_2(2b-1) + T_2(2b) + T_3(0) + T_3(2b) + T_4(1). \quad (12)$$

One can check that by plugging T_1, T_2, T_3 and T_4 in Eq. (12) we recover Eq. (11). Figure 2 shows these operations. We remark that even though the first row in this figure is B_0, the following two rows are not B_1 and B_2. Indeed, those rows are obtained from B_1 and B_2 together with the optimizations provided by T_1, T_2, T_3 and T_4.

Using Eq. (12), the number $N_\oplus$ of XOR operations is

$$N_\oplus = 6b + 1 = 3m - 2.$$

It is also easy to see from Fig. 2 that the time delay is $3T_X$, where T_X is the delay of one 2-input XOR gate.

We are now ready to provide Algorithm 1 for computing D_{red} given in Eq. (12), and as explained in Fig. 2, for the pentanomials $f(x) = x^{2b+1} + x^{b+1} + x^b + x + 1$.

Putting all pieces together, we give next the main result of this section.

Theorem 1 *Algorithm 1 correctly gives the reduction of a polynomial of degree at most* $2m-2$ *over* $\mathbb{F}_2$ *by* $f(x) = x^{2b+1} + x^{b+1} + x^b + x + 1$ *involving* $N_\oplus = 3m - 2 = 6b + 1$ *number of XORs operations and a time delay of* $3T_X$.

5 Family $f(x) = x^{2b+c} + x^{b+c} + x^b + x^c + 1$, $c > 1$

For polynomials of the form $f(x) = x^{2b+c} + x^{b+c} + x^b + x^c + 1$, $c > 1$, we have that $k_{b+c} = 3$, implying that $A_3 = 0$. The reduction is given by

$$D_{red} = B_0 + B_1 + B_2 + B_3.$$

Algorithm 1 Computing D_{red} when $f(x) = x^{2b+1} + x^{b+1} + x^b + x + 1$.

input : $D_0 = d[4b \dots 0]$ bits vector of length $4b+1$
output: D_{red}
for $i \leftarrow 0$ **to** $b-2$ **do**
  $T_1[i] \leftarrow d[i+2b+1] \oplus d[i+3b+2]$; ▹ Definition of T_1
end
for $i \leftarrow 0$ **to** $b-1$ **do**
  $T_4[i] \leftarrow d[i+2b+1] \oplus d[i+3b+1]$; ▹ Definition of T_4
end
$D_{red}[0] \leftarrow d[0] \oplus T_1[0] \oplus d[3b+1]$; ▹ Column 0 of Fig. 2
for $i \leftarrow 1$ **to** $b-2$ **do**
  $D_{red}[i] \leftarrow d[i] \oplus T_1[i] \oplus T_4[i-1]$; ▹ Columns 1 to $b-2$ of Fig. 2
end
$D_{red}[b-1] \leftarrow d[b-1] \oplus d[3b] \oplus T_4[b-2]$
$D_{red}[b] \leftarrow d[b] \oplus T_1[0] \oplus T_4[b-1]$
for $i \leftarrow b+1$ **to** $2b-2$ **do**
  $D_{red}[i] \leftarrow d[i] \oplus T_1[i-b] \oplus T_1[i-b-1]$; ▹ Columns $b+1$ to $2b-2$ of Fig. 2
end
$D_{red}[2b-1] \leftarrow d[2b-1] \oplus d[3b] \oplus T_1[b-2]$
$D_{red}[2b] \leftarrow d[2b] \oplus d[3b+1] \oplus d[3b]$
return D_{red}

Using Eqs. (5), (7), (9) and (10), we have that D_{red} satisfies

$$\begin{aligned} D_{red} = & \sum_{i=0}^{2b+c-1} d_i x^i + \sum_{i=b+c}^{2b+c-1} d_{i+b} x^i \\ & + \sum_{i=c}^{b+c-1} d_{i+2b} x^i + \sum_{i=c}^{b+2c-2} d_{i+3b} x^i \\ & + \sum_{i=b}^{2b+c-1} d_{i+b+c} x^i + \sum_{i=0}^{b-1} d_{i+2b+c} x^i \\ & + \sum_{i=b+c}^{2b+c-2} d_{i+2b+c} x^i + \sum_{i=b}^{2b-2} d_{i+2b+2c} x^i \\ & + \sum_{i=0}^{c-1} d_{i+3b+c} x^i + \sum_{i=0}^{b-2} d_{i+3b+2c} x^i. \end{aligned} \quad (13)$$

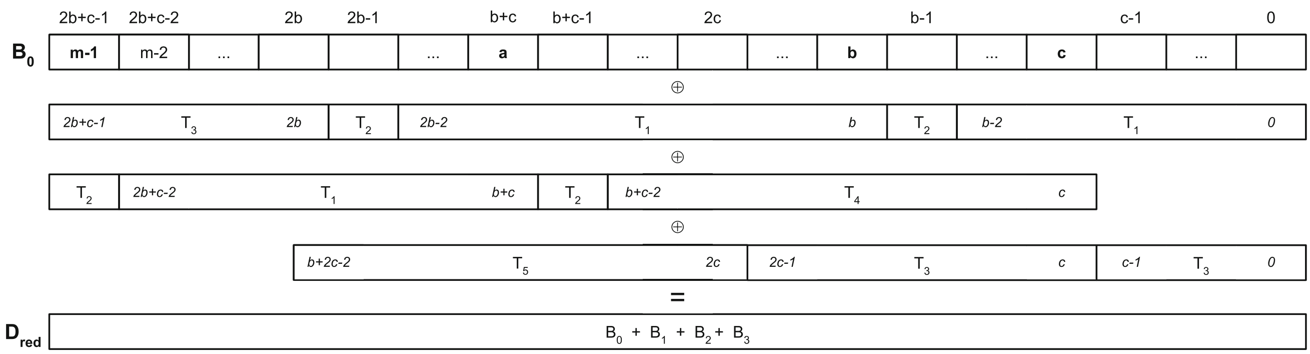


Fig. 3 Representation of the reduction by $f(x) = x^{2b+c} + x^{b+c} + x^b + x^c + 1$, $c > 1$

Let

$$T_1(j) = \sum_{i=0}^{b-2} (d_{i+2b+c} + d_{i+3b+2c}) x^{i+j}, \qquad T_2(j) = d_{3b+c-1} x^j,$$

$$T_3(j) = \sum_{i=0}^{c-1} d_{i+3b+c} x^{i+j}, \quad T_4(j) = \sum_{i=0}^{b-2} d_{i+2b+c} x^{i+j},$$

$$T_5(j) = \sum_{i=0}^{b-2} d_{i+3b+2c} x^{i+j}.$$

Again, a careful analysis of Eq. (13) shows that T_1, T_2 and T_3 are used more than once. Thus, we can rewrite Eq. (13) for D_{red} as

$$\begin{aligned} D_{red} =& B_0 + T_1(0) + T_1(b) + T_1(b+c) \\ &+ T_2(b-1) + T_2(b+c-1) \\ &+ T_2(2b-1) + T_2(2b+c-1) \\ &+ T_3(0) + T_3(c) + T_3(2b) + T_4(c) + T_5(2c). \end{aligned} \tag{14}$$

Figure 3 depicts these operations. Using Eq. (14) and Fig. 3, we have Algorithm 2. For code efficiency reasons, in contrast to Algorithm 1, in Algorithm 2 we separate the last line before the equality in Fig. 3. The additions of this last line are done in lines 17 to 20 of Algorithm 2. As a consequence, lines 3 to 16 of Algorithm 2 include only the additions per column from 0 to $2b + c - 1$ of the first three lines in Fig. 3.

The time delay is $3T_X$; after removal of redundancies and not counting repeated terms, we obtain that the number $N_\oplus$ of XORs is

$$N_\oplus = 6b + 3c - 2 = 3m - 2.$$

Theorem 2 *Algorithm 2 correctly gives the reduction of a polynomial of degree at most $2m - 2$ over $\mathbb{F}_2$ by $f(x) = x^{2b+c} + x^{b+c} + x^b + x^c + 1$ involving $N_\oplus = 3m - 2 = 6b + 3c - 2$ number of XORs operations and a time delay of $3T_X$.*

Algorithm 2 Computing D_{red} for $f(x) = x^{2b+c} + x^{b+c} + x^b + x^c + 1$.

input : $D_0 = d[2b + c - 1 \dots 0]$ bits vector of length $2b + c$
output: D_{red}
for $i \leftarrow 0$ **to** $b - 2$ **do**
  $T_1[i] \leftarrow d[i + 2b + 1] \oplus d[i + 3b + 2c]$; ▷ Definition of T_1
end
for $i \leftarrow 0$ **to** $c - 1$ **do**
  $D_{red}[i] \leftarrow d[i] \oplus T_1[i]$; ▷ Columns 0 to $c - 1$ of the first three lines of Fig. 3
end
for $i \leftarrow c$ **to** $b - 2$ **do**
  $D_{red}[i] \leftarrow d[i] \oplus T_1[i] \oplus d[i + 2b]$
end
$D_{red}[b - 1] \leftarrow d[b - 1] \oplus d[3b + c - 1] \oplus d[3b - 1]$ **for** $i \leftarrow b$ **to** $b + c - 2$ **do**
  $D_{red}[i] \leftarrow d[i] \oplus T_1[i - b] \oplus d[i + 2b]$
end
$D_{red}[b + c - 1] \leftarrow d[b + c - 1] \oplus d[3b + c - 1] \oplus T_1[c - 1]$ **for** $i \leftarrow b + c$ **to** $2b - 2$ **do**
  $D_{red}[i] \leftarrow d[i] \oplus T_1[i - b] \oplus T_1[i - b - c]$
end
$D_{red}[2b - 1] \leftarrow d[2b - 1] \oplus d[3b + c - 1] \oplus T_1[b - c - 1]$ **for** $i \leftarrow 2b$ **to** $2b + c - 2$ **do**
  $D_{red}[i] \leftarrow d[i] \oplus T_1[i - b - c] \oplus d[i + b + c]$
end
$D_{red}[2b + c - 1] \leftarrow d[2b + c - 1] \oplus d[3b + c - 1] \oplus d[3b - 1]$ **for** $i \leftarrow 0$ **to** $c - 1$ **do**
  $D_{red}[i] \leftarrow D_{red}[i] \oplus d[i + 3b + c]$; ▷ Columns 0 to $c - 1$ of the 4^{th} line of Fig. 3
end
for $i \leftarrow c$ **to** $b + 2c - 2$ **do**
  $D_{red}[i] \leftarrow D_{red}[i] \oplus d[i + 3b]$; ▷ Cols c to $b + 2c - 2$ of the 4^{th} line of Fig. 3
end
return D_{red}

5.1 Almost equally spaced pentanomials: the special case $b = 2c$

Consider the special case $b = 2c$. In this case, we obtain the almost equally spaced polynomials $f(x) = x^{5c} + x^{3c} + x^{2c} + x^c + 1$. Our previous analysis when applied to these polynomials entails

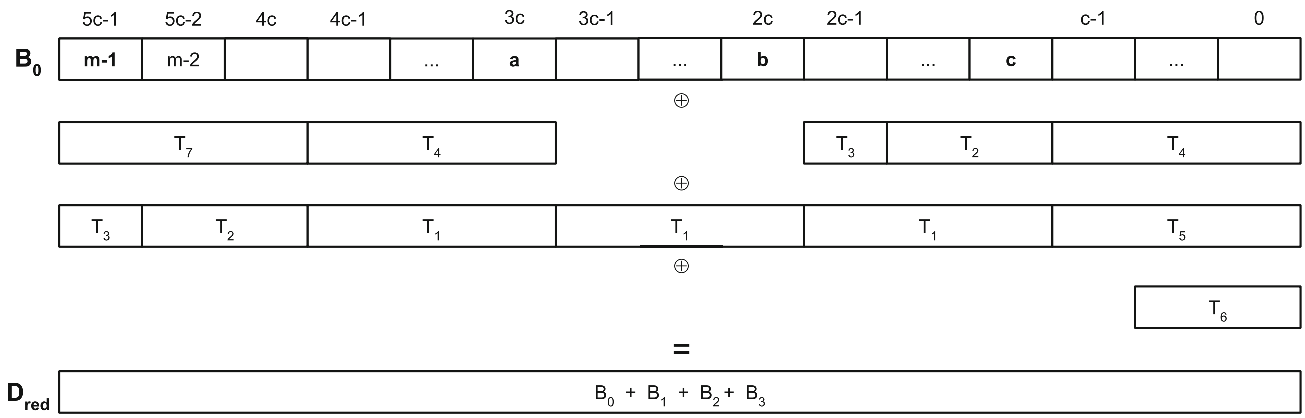


Fig. 4 Representation of the reduction by the almost equally spaced pentanomials (the special case $b = 2c$)

$$
\begin{aligned}
D_{red} = &\sum_{i=0}^{5c-1} d_i x^i + \sum_{i=3c}^{5c-1} d_{i+2c} x^i + \sum_{i=c}^{3c-1} d_{i+4c} x^i \\
&+ \sum_{i=c}^{4c-2} d_{i+6c} x^i + \sum_{i=2c}^{5c-1} d_{i+3c} x^i \\
&+ \sum_{i=0}^{2c-1} d_{i+5c} x^i + \sum_{i=3c}^{5c-2} d_{i+5c} x^i + \sum_{i=2c}^{4c-2} d_{i+6c} x^i \\
&+ \sum_{i=0}^{c-1} d_{i+7c} x^i + \sum_{i=0}^{2c-2} d_{i+8c} x^i .
\end{aligned} \tag{15}
$$

Let

$$T_1(j) = \sum_{i=c}^{2c-2} (d_{i+5c} + d_{i+4c}) x^{i+j},$$

$$T_2(j) = \sum_{i=c}^{2c-2} (d_{i+8c} + d_{i+6c}) x^{i+j},$$

$$T_3(j) = d_{8c-1} x^j, \quad T_4(j) = \sum_{i=0}^{c-1} d_{i+8c} x^{i+j},$$

$$T_5(j) = \sum_{i=0}^{c-1} d_{i+5c} x^{i+j},$$

$$T_6(j) = \sum_{i=0}^{c-2} d_{i+7c} x^{i+j}, \quad T_7(j) = \sum_{i=4c}^{5c-1} d_{i+2c} x^{i+j}.$$

In the computation of D_{red}, T_1, T_2, T_3 and T_4 are used more than once. Figure 4 shows, graphically, these operations. After removal of redundancies, the number $N_\oplus$ of XORs is $N_\oplus = 12c - 1 = \frac{12}{5} m - 1$. This number of XORs is close to $2.4m$ providing a saving of about 20% with respect to the other pentanomials in our family. Irreducible pentanomials of this form are rare but they do exist, for example, for degrees 5, 155 and 4805. We observe that the extension 155 is used in [1].

Using Eq. (15) and Fig. 4, we naturally have Algorithm 3.

Algorithm 3 Computing D_{red} for $f(x) = x^{5c} + x^{3c} + x^{2c} + x^c + 1$.

input : $D_0 = d[5c-1 \ldots 0]$ bits vector of length $5c$
output: D_{red}
for $i \leftarrow 0$ **to** $c-2$ **do**
  | $T_1[i] \leftarrow d[i+6c] \oplus d[i+5c]$ ▷ Definition of T_1
end
for $i \leftarrow 0$ **to** $c-2$ **do**
  | $T_2[i] \leftarrow d[i+9c] \oplus d[i+7c]$ ▷ Definition of T_2
end
for $i \leftarrow 0$ **to** $c-2$ **do**
  | $D_{red}[i] \leftarrow d[i] \oplus d[i+8c] \oplus d[i+5c] \oplus d[i+7c]$
end
$D_{red}[c-1] \leftarrow d[c-1] \oplus d[9c-1] \oplus d[6c-1]$
for $i \leftarrow c$ **to** $2c-2$ **do**
  | $D_{red}[i] \leftarrow d[i] \oplus T_1[i-c] \oplus T_2[i-c]$
end
$D_{red}[2c-1] \leftarrow d[2c-1] \oplus d[8c-1] \oplus T_1[c-1]$
for $i \leftarrow 2c$ **to** $3c-1$ **do**
  | $D_{red}[i] \leftarrow d[i] \oplus T_1[i-2c]$
end
for $i \leftarrow 3c$ **to** $4c-1$ **do**
  | $D_{red}[i] \leftarrow d[i] \oplus T_1[i-3c] \oplus d[i+5c]$
end
for $i \leftarrow 4c$ **to** $5c-2$ **do**
  | $D_{red}[i] \leftarrow d[i] \oplus T_2[i-4c] \oplus d[i+2c]$
end
$D_{red}[5c-1] \leftarrow d[5c-1] \oplus d[8c-1] \oplus d[7c-1]$
return D_{red}

---

6 Multiplier in $\mathbb{F}_2[x]$, complexity analysis and comparison

So far, we have focused on the second step of the algorithm, that is, on the reduction part. For the first step, the multiplication part, we simply use a standard Karatsuba recursive algorithm implementation; see Algorithm 4.

Recursivity in hardware can be an issue; see [24] and [4], for example, for efficient hardware implementations of polynomial multiplication in finite fields using Karatsuba's type algorithms.

Algorithm 4 Karatsuba Algorithm for $\mathbb{F}_{2^m}$

input : $A(x) = \sum_{i=0}^{m-1} a_i x^i$ and $B(x) = \sum_{i=0}^{m-1} b_i x^i$
output: $C(x) = A(x)B(x) = \sum_{i=0}^{2m-2} c_i x^i$

```
Function Karatsuba(A, B):
    m ← maxDegree(A, B)    ▷ compute the larger degree between polynomials A and B
    if m = 0 then
        return (A & B)    ▷ & is a bitwise AND operator
    end
    m2 = floor(m/2)    ▷ split A and B
    high_a, low_a ← split(A, m2)
    high_b, low_b ← split(B, m2)
    d0 ← Karatsuba(low_a, low_b)    ▷ recursive call of Karatsuba
    d1 ← Karatsuba((low_a ⊕ high_a), (low_b ⊕ high_b))    ▷ recursive call of Karatsuba
    d2 ← Karatsuba(high_a, high_b)    ▷ recursive call of Karatsuba
    c ← d2 x^m ⊕ (d1 ⊕ d2 ⊕ d0) x^m2 ⊕ d0
    return c
End Function
```

As can be seen our multiplier consists of two steps. The first is the multiplication itself using Karatsuba arithmetic or, if necessary, the school book method, and the second is the reduction described in the previous sections. The choice of the first step method will basically depend on whether the application requirement is to minimize area (Karatsuba), i.e., the number of ANDs and XORs gates, or to minimize the arithmetic delay (School book); see [5] for several variants of both the schoolbook and Karatsuba algorithms. Minimizing the area is interesting in applications that need to save power at the expense of a longer runtime.

We chose the Karatsuba multiplier since our goal is to minimize the area, i.e., to minimize the number of gates AND and XOR. A summary of our results compared with related works is given in Tables 1 and 2. Table 1 presents comparison costs among multipliers that perform two steps for the multiplication, that is, they execute a multiplication followed by a reduction. The table shows the multiplication algorithm used in each case. Table 2 gives a comparison among the state-of-the-art bit multipliers in the literature. The main target for us is [18] since it presents the smallest area in the literature. However, Type 3 polynomials are also considered; this is another practically relevant family of polynomials. With respect to Karatsuba variants, Table 3 of survey [5] shows asymptotic complexities of several Karatsuba multiplication algorithms without reduction.

For each entry in Table 1, we give the multiplication algorithm and the amount of gates AND, XOR as well its delay. We point that for [19] and [25], their multipliers are general for any pentanomial with $a \leq \frac{m}{2}$ instead of for a specific family such as [20]. In the case of our family, in addition to the number of XORs for the reduction, we include the cost for the multiplication due to the recursive Karatsuba implementation multiplier, that is, the XOR count is formed by the sum of the XORs of the Karatsuba multiplier and the ones of the reduction part. In our implementation, the constant of Karatsuba is strictly less than 6; see Fig. 5 for degrees up to 1024. As can be seen, for degrees powers of 2 minus 1 ($2^k - 1, k \geq 1$), the constant achieves local minimum. For the number of AND gates, we provide an interval. The actual number of AND gates depends on the value of m; it only reaches a maximum when $m = 2^k - 1$, for $k \geq 1$.

In Table 2, we provide the number of XORs and ANDs gates for Type 1 and Type 2 families in [18] and [20], Type 3 in [19] and our family of pentanomials. We point out that in [18] the authors compute multiplication and reduction as a unique block with a divide-and-conquer approach using squaring. In contrast, we separate these two parts and use Karatsuba for the multiplier followed by our reduction algorithm.

The costs for using our pentanomials for degrees proposed by NIST can be found in Table 3. The amount of XOR and AND gates are the exact value obtained from Table 1. The delay costs can be separated in T_A and T_X, delay for AND gates and XOR gates, respectively. The delay for AND gates is due to only 1 AND gate at the lowest level of the Karatsuba recursion. The delay for the XOR gates in the Karatsuba multiplier is $3\lceil \log_2 (m-1) \rceil$ since there are 3 delay XORs per level of the Karatsuba recursion. For the reduction part, we only have 3 delay XORs. Hence, the total number of XOR delays is $3\lceil \log_2 (m-1) \rceil + 3$.

Table 4 shows the number of irreducible pentanomials of degrees 163, 283 and 571 for the families considered since those are NIST degrees where pentanomials have been recommended [16]. Analyzing the table, we have that family Type 1 has the most irreducible pentanomials, but few of them have degrees recommended by NIST [16]. The first family of Type 2, proposed in [18], has restrictions in the range of c; this family presents the highest number of representatives with NIST degrees of interest. The second family of Type 2, proposed in [20], has no restrictions for c; this family presents the largest number of irreducible polynomials. Type 3 is the special case from [19]. Our family for $b \neq 2c$ has less irreducible polynomials, and it has no irreducible polynomials with degrees 163, 283 and 571. In the other side, when $b \neq 2c$ our family has 730 polynomials of degrees up to 1024 and it presents 5 pentanomials of NIST degrees.

In the following, we comment on the density of irreducible pentanomials in our family. Table 5 lists all irreducible pentanomials of our family for degrees up to 1024; $N_\oplus$ is the number of XORs required for the reduction. We leave as an

Table 1 Two steps multipliers cost comparison for different family of pentanomials

$x^m+x^a+x^b+x^c+1$ [20,25], Multiplication algorithm: Schoolbook.			
Costs	#AND	#XOR	Delay
Reduction	0	$4(m-1)$	$3T_X$
Multiplication	m^2	$(m-1)^2$	$T_A+(\lceil\log_2 m\rceil)T_X$
Multiplier	m^2	m^2+2m-3	$T_A+(3+\lceil\log_2 m\rceil)T_X$
Type I - $x^m+x^{n+1}+x^n+x+1$ [20], Multiplication algorithm: Mastrovito-like Multiplier.			
Costs	#AND	#XOR	Delay
Reduction	0	$3m+2n-1$	$3T_X$
Multiplication	m^2	m^2-2m+1	$T_A+(\lceil\log_2 m\rceil)T_X$
Multiplier	m^2	m^2+m+2n	$T_A+(3+\lceil\log_2 m\rceil)T_X$
Type I - $x^m+x^{n+1}+x^n+x+1$ [19], Multiplication algorithm: Mastrovito-like Multiplier.			
Costs	#AND	#XOR	Delay
Reduction	0	$3m-2$	$3T_X$
Multiplication	m^2	m^2-2m+1	$T_A+(\lceil\log_2(m-1)\rceil)T_X$
Multiplier	m^2	$m^2+m^\dagger$	$T_A+(3+\lceil\log_2(m-1)\rceil)T_X$
Type II - $x^m+x^{n+2}+x^{n+1}+x^n+1$ [20], Multiplication algorithm: Dual basis.			
Costs	#AND	#XOR	Delay
Reduction	0	$3m-\lceil(m-2)/2\rceil+3n-4$	$3T_X$
Multiplication	m^2	m^2-m	$T_A+(\lceil\log_2 m\rceil)T_X$
Multiplier	m^2	$m^2+2m-\lceil(m-2)/2\rceil+3n-4$	$T_A+(3+\lceil\log_2 m\rceil)T_X$
$x^m+x^a+x^b+x^c+1$, $c>1$ [19], Multiplication algorithm: Mastrovito-like Multiplier.			
Costs	#AND	#XOR	Delay
Reduction	0	$4m-4$	$4T_X$
Multiplication	m^2	m^2-2m+1	$T_A+(\lceil\log_2(m-1)\rceil)T_X$
Multiplier	m^2	m^2+2m-3	$T_A+(4+\lceil\log_2(m-1)\rceil)T_X$
Ours - $x^{2b+c}+x^{b+c}+x^b+x^c+1$, Multiplication algorithm: Karatsuba.			
Costs	#AND	#XOR	Delay
Reduction	0	$3m-2$	$3T_X$
Multiplication	$(3^{\lfloor\log_2 m\rfloor}, 3^{\lfloor\log_2 m\rfloor+1}]$	$<6m^{\log_2 3}$	$T_A+3\lceil\log_2(m-1)\rceil T_X$
Multiplier	$(3^{\lfloor\log_2 m\rfloor}, 3^{\lfloor\log_2 m\rfloor+1}]$	$<6m^{\log_2 3}+3m-2$	$T_A+3(\lceil\log_2(m-1)\rceil+1)T_X$
Ours - $x^{5c}+x^{3c}+x^{2c}+x^c+1$, Multiplication algorithm: Karatsuba.			
Costs	#AND	#XOR	Delay
Reduction	0	$(12/5)m-1$	$3T_X$
Multiplication	$(3^{\lfloor\log_2 m\rfloor}, 3^{\lfloor\log_2 m\rfloor+1}]$	$<6m^{\log_2 3}$	$T_A+3\lceil\log_2(m-1)\rceil T_X$
Multiplier	$(3^{\lfloor\log_2 m\rfloor}, 3^{\lfloor\log_2 m\rfloor+1}]$	$<6m^{\log_2 3}+(12/5)m-1$	$T_A+3(\lceil\log_2(m-1)\rceil+1)T_X$

† There is an additional XOR to reduce the time delay; see [19, p. 955]

open problem to mathematically characterize under which conditions our pentanomials are irreducible.

7 Conclusions

In this paper, we present a new class of pentanomials over $\mathbb{F}_2$, defined by $x^{2b+c}+x^{b+c}+x^b+x^c+1$. We give the exact number of XORs in the reduction process; we note that in the reduction process no ANDs are required.

It is interesting to point out that even though the cases $c=1$ and $c>1$, as shown in Figs. 2 and 3, are quite different, the final result in terms of number of XORs is the same. We also consider a special case when $b=2c$ where further reductions are possible.

There are irreducible pentanomials of this shape for several degree extensions of practical interest. We provide a detailed analysis of the space and time complexity involved in the reduction using the pentanomials in our family. For the multiplication process, we simply use the standard Karatsuba algorithm.

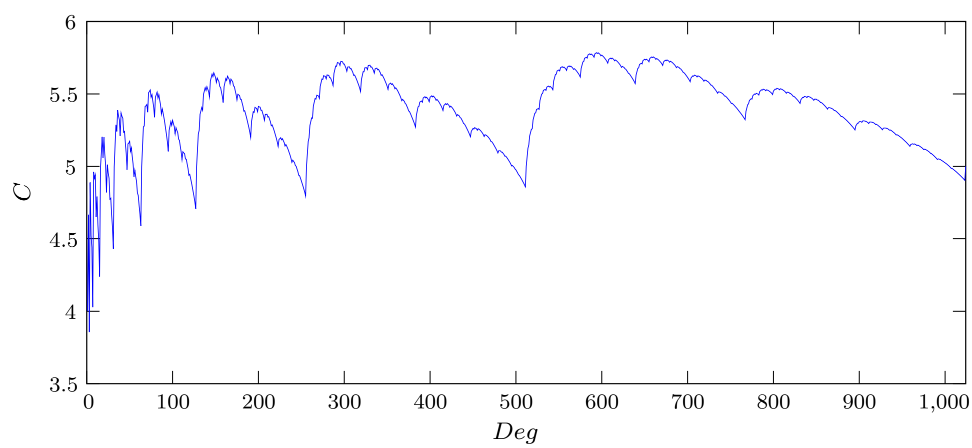
Table 2 Space and time complexities of state-of-the-art bit multipliers

Type		# XOR	# AND	Delay
Type 1	$x^m + x^{b+1} + x^b + x + 1,\ 1 < b \le \frac{m}{2} - 1$			
[18]	b is odd	$\frac{3m^2 + 24m + 8b + 21}{4}$	$\frac{3m^2 + 2m - 1}{4}$	$T_A + (3 + \lceil \log_2(m+1) \rceil) T_x$
[18]	b is even	$\frac{3m^2 + 24m + 8b + 17}{4}$	$\frac{3m^2 + 2m - 1}{4}$	$T_A + (3 + \lceil \log_2(m+1) \rceil) T_x$
Type 2	$x^m + x^{c+2} + x^{c+1} + x^c + 1$			
[18]	c is odd, $c \le \frac{3}{8}(m - 7)$	$\frac{3m^2 + 24m + 14c + 35}{4}$	$\frac{3m^2 + 2m - 1}{4}$	$T_A + (3 + \lceil \log(m+1) \rceil) T_x$
[18]	c is even, $c \le \frac{m}{2} - 1$	$\frac{3m^2 + 24m + 14c + 45}{4}$	$\frac{3m^2 + 2m - 1}{4}$	$T_A + (3 + \lceil \log(m+1) \rceil) T_x$
[20]	$c > 1$	$m^2 + 2m - \lceil (m-2)/2 \rceil + 3n - 4$	m^2	$T_A + (3 + \lceil \log(m-1) \rceil) T_x$
[20]	$c = 1$	$m^2 + m - 2$	m^2	$T_A + (3 + \lceil \log_2(m-1) \rceil) T_x$
Type 3	$x^m + x^{m-c} + x^{m-2c} + x^{m-3c} + 1$			
[19]	$\frac{m-1}{4} \le c \le \frac{m-1}{3}$	$m^2 + m - c - 1$	m^2	$T_A + (3 + \lceil \log_2(m-1) \rceil) T_x$
[19]	$\frac{m-1}{5} \le c < \frac{m-1}{4}$	$m^2 + 2m - 5c - 2$	m^2	$T_A + (3 + \lceil \log_2(m-1) \rceil) T_x$
[19]	$\frac{m-1}{8} \le c < \frac{m-1}{5}$	$m^2 + m - 2$	m^2	$T_A + (3 + \lceil \log_2(m-1) \rceil + 1) T_x$
Ours	$x^{2b+c} + x^{b+c} + x^b + x^c + 1$			
Ours	$c \ge 1, b \ne 2c$	$< 6m^{\log_2 3} + 3m - 2$	$(3^{\lfloor \log_2 m \rfloor}, 3^{\lfloor \log_2 m \rfloor + 1}]$	$T_A + 3(\lceil \log_2 (m-1) \rceil) T_x$
Ours	$c \ge 1, b = 2c$	$< 6m^{\log_2 3} + \frac{12}{5}m - 1$	$(3^{\lfloor \log_2 m \rfloor}, 3^{\lfloor \log_2 m \rfloor + 1}]$	$T_A + 3(\lceil \log_2 (m-1) \rceil + 1) T_x$

Table 3 Costs for fixed degree pentanomials proposed by NIST

Degree	XORs			ANDs	Delay
	Karatsuba	Reduction	Total		
163	17,944	487	18,431	4,419	$T_A + 27T_X$
283	43,162	847	44,009	10,305	$T_A + 30T_X$
571	132,280	1711	133,991	31,203	$T_A + 33T_X$

Fig. 5 Karatsuba constant for degrees up to 1024



The proved complexity analysis of the multiplier and reduction considering the family proposed in this paper, as well as our analysis suggests that these pentanomials are as good as or possibly better to the ones already proposed.

We leave for future work to produce a one-step algorithm using our pentanomials, that is, a multiplier that performs multiplication and reduction in a single step using our family of polynomials, as well as a detailed study of the delay obtained using this algorithm.

Table 4 Number of irreducible pentanomials for NIST degrees

Type	$\#Irred.$	163	283	571
Type 1 [18]	2025	1	2	0
Type 2 [18]	1676	3	2	2
Type 2 [20]	3430	6	4	4
Type 3 [19]	539	0	0	0
Ours, $b \neq 2c$	728	2	2	1
Ours, $b = 2c$	2	0	0	0

Table 5 Our family of irreducible pentanomials and their number of XORs $(b, c, N_{\oplus})$, $2b \neq c$

2, 1, 11	3, 2, 22	4, 1, 25	5, 1, 31	5, 2, 34
6, 1, 37	5, 3, 37	7, 2, 46	9, 5, 67	8, 7, 67
9, 6, 70	12, 1, 73	11, 3, 73	10, 7, 79	13, 3, 85
10, 9, 85	13, 4, 88	15, 6, 106	14, 9, 109	19, 2, 118
17, 6, 118	15, 10, 118	17, 11, 133	17, 12, 136	21, 5, 139
20, 7, 139	16, 15, 139	21, 6, 142	23, 5, 151	22, 7, 151
25, 2, 154	21, 11, 157	21, 13, 163	27, 5, 175	23, 13, 175
29, 2, 178	25, 10, 178	23, 14, 178	25, 12, 184	28, 7, 187
32, 1, 193	28, 9, 193	31, 4, 196	23, 20, 196	30, 7, 199
28, 15, 211	27, 18, 214	35, 3, 217	31, 11, 217	27, 22, 226
29, 20, 232	35, 10, 238	31, 19, 241	38, 7, 247	31, 21, 247
41, 3, 253	38, 9, 253	37, 12, 256	35, 19, 265	39, 12, 268
34, 25, 277	45, 4, 280	33, 29, 283	47, 2, 286	40, 17, 289
38, 23, 295	48, 7, 307	40, 23, 307	46, 15, 319	42, 23, 319
53, 2, 322	45, 18, 322	41, 26, 322	45, 19, 325	38, 33, 325
41, 28, 328	52, 7, 331	41, 29, 331	47, 20, 340	45, 26, 346
43, 30, 346	49, 19, 349	41, 35, 349	45, 28, 352	57, 6, 358
51, 18, 358	45, 30, 358	46, 31, 367	55, 14, 370	52, 25, 385
63, 4, 388	62, 7, 391	45, 44, 400	51, 34, 406	59, 19, 409
50, 41, 421	63, 18, 430	68, 9, 433	63, 19, 433	59, 27, 433
56, 33, 433	67, 12, 436	69, 11, 445	60, 31, 451	75, 2, 454
56, 41, 457	63, 29, 463	62, 31, 371	59, 37, 463	75, 6, 466
71, 14, 466	65, 26, 466	61, 36, 472	77, 5, 475	74, 15, 487
63, 37, 487	67, 30, 490	65, 34, 490	73, 19, 493	71, 30, 514
87, 2, 526	87, 6, 538	75, 30, 538	69, 42, 538	82, 17, 541
71, 46, 562	70, 49, 565	81, 28, 568	77, 36, 568	85, 21, 571
65, 61, 571	83, 28, 580	95, 10, 598	85, 30, 598	75, 50, 598
95, 12, 604	98, 9, 613	86, 33, 613	81, 43, 613	78, 49, 613
77, 51, 613	103, 3, 625	91, 28, 628	87, 37, 631	78, 55, 631
101, 11, 637	74, 65, 637	104, 7, 643	81, 54, 646	79, 60, 652
79, 61, 655	101, 18, 658	85, 53, 667	112, 1, 673	91, 44, 676
90, 47, 679	79, 69, 679	81, 66, 682	105, 19, 685	90, 49, 685
95, 43, 697	79, 75, 697	102, 31, 703	99, 37, 703	91, 53, 703
97, 42, 706	94, 49, 709	104, 31, 715	119, 2, 718	105, 30, 718
110, 23, 727	103, 37, 727	105, 34, 730	99, 46, 730	88, 73, 745
99, 52, 748	118, 15, 751	103, 45, 751	95, 61, 751	115, 23, 757
105, 43, 757	93, 67, 757	125, 4, 760	93, 68, 760	127, 2, 766
87, 83, 769	123, 14, 778	130, 1, 781	97, 67, 781	128, 7, 787
108, 47, 787	103, 59, 793	92, 81, 793	119, 30, 802	99, 70, 802
117, 36, 808	120, 31, 811	105, 61, 811	119, 34, 814	106, 63, 823

Table 5 continued

131, 14, 826	133, 13, 835	140, 1, 841	95, 91, 841	123, 37, 847
111, 61, 847	115, 54, 850	118, 49, 853	113, 59, 853	141, 6, 862
107, 76, 868	130, 31, 871	125, 42, 874	125, 43, 877	142, 15, 895
139, 22, 898	125, 50, 898	115, 70, 898	131, 43, 913	154, 1, 925
142, 25, 925	155, 3, 937	107, 102, 946	154, 9, 949	114, 89, 949
109, 99, 949	145, 34, 970	137, 50, 970	135, 54, 970	123, 78, 970
146, 33, 973	145, 36, 976	133, 60, 976	121, 85, 979	161, 6, 982
143, 44, 988	123, 84, 988	129, 74, 994	153, 29, 1.003	156, 25, 1009
115, 107, 1.009	118, 105, 1.021	169, 4, 1.024	145, 52, 1.024	137, 68, 1024
125, 92, 1.024	139, 67, 1.033	135, 78, 1.042	129, 90, 1.042	129, 91, 1045
135, 84, 1.060	174, 7, 1.063	126, 103, 1.063	157, 42, 1.066	161, 35, 1069
154, 49, 1.069	133, 93, 1.075	171, 18, 1.078	153, 54, 1.078	135, 90, 1078
179, 5, 1.087	130, 103, 1.087	169, 27, 1.093	162, 41, 1.093	142, 81, 1093
133, 99, 1.093	122, 121, 1.093	124, 121, 1.105	130, 113, 1.117	173, 29, 1123
167, 43, 1.129	144, 89, 1.129	189, 4, 1.144	177, 28, 1.144	161, 60, 1144
163, 62, 1.162	133, 123, 1.165	140, 111, 1.171	147, 101, 1.183	193, 10, 1186
185, 27, 1.189	189, 20, 1.192	197, 6, 1.198	175, 50, 1.198	160, 81, 1201
135, 132, 1.204	170, 63, 1.207	166, 71, 1.207	149, 109, 1.219	153, 102, 1222
191, 28, 1.228	189, 37, 1.243	161, 93, 1.243	159, 100, 1.252	179, 61, 1255
155, 109, 1.255	203, 14, 1.258	161, 98, 1.258	198, 25, 1.261	170, 81, 1261
150, 121, 1.261	149, 132, 1.288	205, 21, 1.291	189, 54, 1.294	163, 109, 1303
151, 134, 1.306	173, 93, 1.315	148, 143, 1.315	209, 22, 1.318	187, 66, 1318
196, 49, 1.321	190, 63, 1.327	183, 77, 1.327	194, 57, 1.333	172, 105, 1345
223, 4, 1.348	173, 108, 1.360	225, 6, 1.366	204, 49, 1.369	155, 149, 1375
162, 137, 1.381	161, 140, 1.384	204, 55, 1.387	193, 77, 1.387	199, 69, 1399
225, 18, 1.402	213, 42, 1.402	195, 78, 1.402	197, 76, 1.408	183, 108, 1420
234, 7, 1.423	203, 69, 1.423	209, 59, 1.429	161, 155, 1.429	235, 10, 1438
235, 12, 1.444	179, 124, 1.444	218, 49, 1.453	169, 147, 1.453	201, 90, 1474
225, 44, 1.480	173, 148, 1.480	220, 63, 1.507	248, 9, 1.513	247, 12, 1516
254, 1, 1.525	213, 90, 1.546	217, 83, 1.549	201, 115, 1.549	224, 71, 1555
238, 47, 1.567	261, 6, 1.582	183, 163, 1.585	227, 76, 1.588	218, 95, 1591
178, 175, 1.591	265, 4, 1.600	241, 53, 1.603	196, 143, 1.603	267, 2, 1606
269, 2, 1.618	265, 10, 1.618	261, 18, 1.618	241, 58, 1.618	225, 90, 1618
221, 98, 1.618	207, 126, 1.618	205, 130, 1.618	246, 49, 1.621	272, 1, 1633
196, 153, 1.633	192, 161, 1.633	203, 140, 1.636	254, 39, 1.639	194, 161, 1645
257, 37, 1.651	212, 127, 1.651	239, 77, 1.663	255, 46, 1.666	227, 102, 1666
245, 67, 1.669	234, 89, 1.669	197, 163, 1.669	209, 140, 1.672	244, 71, 1675
247, 68, 1.684	195, 172, 1.684	195, 173, 1.687	213, 138, 1.690	274, 17, 1693
193, 180, 1.696	280, 9, 1.705	215, 139, 1.705	243, 84, 1.708	218, 135, 1711
239, 94, 1.714	219, 134, 1.714	241, 91, 1.717	216, 145, 1.729	225, 130, 1738
223, 134, 1.738	215, 150, 1.738	249, 84, 1.744	256, 71, 1.747	208, 167, 1747
211, 163, 1.753	231, 124, 1.756	255, 77, 1.759	199, 189, 1.759	230, 129, 1765
213, 163, 1.765	249, 92, 1.768	295, 2, 1.774	265, 66, 1.786	255, 86, 1786
286, 25, 1.789	285, 30, 1.798	255, 90, 1.798	225, 150, 1.798	267, 67, 1801
263, 75, 1.801	211, 181, 1.807	293, 18, 1.810	285, 36, 1.816	247, 116, 1828
259, 94, 1.834	266, 81, 1.837	253, 107, 1.837	221, 171, 1.837	285, 44, 1840
300, 17, 1.849	252, 113, 1.849	279, 61, 1.855	265, 91, 1.861	249, 124, 1864
244, 137, 1.873	227, 172, 1.876	273, 84, 1.888	252, 127, 1.891	311, 13, 1903
271, 93, 1.903	266, 103, 1.903	259, 117, 1.903	265, 109, 1.915	255, 131, 1921

Table 5 continued

252, 137, 1.921	215, 212, 1.924	298, 47, 1.927	231, 181, 1.927	305, 36, 1936
245, 157, 1.939	323, 2, 1.942	243, 162, 1.942	259, 131, 1.945	223, 203, 1945
279, 92, 1.948	238, 175, 1.951	274, 105, 1.957	325, 6, 1.966	292, 73, 1969
322, 15, 1.975	319, 22, 1.978	303, 54, 1.978	253, 154, 1.978	310, 47, 1999
329, 14, 2.014	314, 47, 2.023	323, 30, 2.026	257, 162, 2.026	314, 49, 2029
323, 34, 2.038	289, 102, 2.038	255, 170, 2.038	307, 68, 2.044	243, 198, 2050
329, 27, 2.053	253, 179, 2.053	237, 211, 2.053	256, 175, 2.059	339, 11, 2065
308, 73, 2.065	303, 83, 2.065	243, 203, 2.065	287, 116, 2.068	243, 205, 2071
266, 161, 2.077	305, 91, 2.101	320, 63, 2.107	301, 101, 2.107	343, 19, 2113
243, 220, 2.116	293, 122, 2.122	349, 11, 2.125	285, 139, 2.125	253, 203, 2125
266, 183, 2.143	254, 207, 2.143	307, 102, 2.146	325, 69, 2.155	357, 6, 2158
315, 90, 2.158	349, 26, 2.170	329, 67, 2.173	340, 49, 2.185	347, 37, 2191
341, 50, 2.194	297, 138, 2.194	285, 164, 2.200	283, 173, 2.215	270, 199, 2215
349, 42, 2.218	301, 139, 2.221	301, 141, 2.227	261, 221, 2.227	365, 18, 2242
297, 156, 2.248	365, 21, 2.251	268, 217, 2.257	371, 13, 2.263	371, 14, 2266
287, 182, 2.266	374, 9, 2.269	361, 36, 2.272	328, 103, 2.275	375, 10, 2278
260, 241, 2.281	279, 204, 2.284	313, 139, 2.293	257, 251, 2.293	297, 173, 2299
264, 239, 2.299	381, 6, 2.302	304, 161, 2.305	260, 249, 2.305	355, 62, 2314
321, 130, 2.314	372, 31, 2.323	341, 93, 2.323	293, 189, 2.323	364, 49, 2329
287, 203, 2.329	351, 76, 2.332	377, 26, 2.338	369, 42, 2.338	325, 130, 2338
299, 182, 2.338	378, 25, 2.341	321, 140, 2.344	347, 91, 2.353	332, 121, 2353
361, 66, 2.362	303, 182, 2.362	278, 233, 2.365	305, 187, 2.389	392, 15, 2395
311, 180, 2.404	386, 31, 2.407	271, 261, 2.407	395, 14, 2.410	307, 190, 2410
297, 210, 2.410	320, 169, 2.425	351, 108, 2.428	389, 35, 2.437	361, 93, 2443
357, 102, 2.446	404, 9, 2.449	343, 133, 2.455	287, 245, 2.455	403, 14, 2458
335, 150, 2.458	325, 170, 2.458	293, 234, 2.458	397, 27, 2.461	286, 255, 2479
393, 42, 2.482	365, 101, 2.491	395, 44, 2.500	411, 14, 2.506	283, 270, 2506
381, 76, 2.512	397, 45, 2.515	285, 269, 2.515	321, 203, 2.533	407, 38, 2554
299, 254, 2.554	321, 211, 2.557	336, 185, 2.569	320, 217, 2.569	411, 38, 2578
403, 54, 2.578	355, 150, 2.578	339, 182, 2.578	322, 217, 2.581	423, 18, 2590
403, 59, 2.593	389, 91, 2.605	358, 153, 2.605	321, 228, 2.608	320, 231, 2611
379, 115, 2.617	425, 27, 2.629	389, 99, 2.629	353, 173, 2.635	435, 10, 2638
400, 81, 2.641	396, 89, 2.641	351, 181, 2.647	326, 231, 2.647	295, 294, 2650
422, 41, 2.653	382, 121, 2.653	363, 164, 2.668	319, 252, 2.668	303, 284, 2668
311, 270, 2.674	401, 91, 2.677	325, 243, 2.677	373, 148, 2.680	443, 14, 2698
417, 66, 2.698	413, 74, 2.698	375, 150, 2.698	345, 210, 2.698	301, 298, 2698
362, 177, 2.701	381, 140, 2.704	364, 175, 2.707	443, 19, 2.713	367, 173, 2719
405, 98, 2.722	448, 17, 2.737	375, 163, 2.737	407, 102, 2.746	405, 106, 2746
377, 162, 2.746	427, 67, 2.761	316, 289, 2.761	439, 45, 2.767	339, 245, 2767
318, 287, 2.767	461, 4, 2.776	393, 140, 2.776	457, 13, 2.779	445, 37, 2779
423, 83, 2.785	403, 124, 2.788	335, 262, 2.794	413, 107, 2.797	392, 151, 2803
344, 249, 2.809	387, 166, 2.818	355, 230, 2.818	389, 164, 2.824	466, 15, 2839
362, 223, 2.839	321, 306, 2.842	353, 243, 2.845	462, 31, 2.863	411, 133, 2863
394, 169, 2.869	441, 76, 2.872	436, 89, 2.881	338, 287, 2.887	443, 78, 2890
373, 218, 2.890	421, 123, 2.893	480, 7, 2.899	380, 207, 2.899	435, 102, 2914
411, 150, 2.914	405, 162, 2.914	369, 234, 2.914	376, 223, 2.923	420, 137, 2929
435, 108, 2.932	399, 180, 2.932	458, 63, 2.935	445, 89, 2.935	354, 271, 2935

Table 5 continued

437, 107, 2.941	401, 179, 2.941	425, 133, 2.947	483, 18, 2.950	350, 287, 2959
429, 132, 2.968	369, 252, 2.968	397, 197, 2.971	392, 207, 2.971	364, 265, 2977
494, 7, 2.983	387, 222, 2.986	494, 9, 2.989	429, 139, 2.989	475, 50, 2998
425, 150, 2.998	375, 250, 2.998	431, 140, 3.004	466, 71, 3.007	419, 165, 3007
337, 332, 3.016	427, 156, 3.028	407, 196, 3.028	347, 316, 3.028	487, 37, 3031
457, 98, 3.034	355, 302, 3.034	485, 43, 3.037	365, 284, 3.040	415, 187, 3049
418, 183, 3.055				

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References

1. Agnew, G.B., Mullin, R.C., Vanstone, S.A.: An implementation of elliptic curve cryptosystems over $\mathbb{F}_{2^{155}}$. IEEE J. Sel. Areas Commun. **11**(5), 804–813 (1993)
2. Bernstein, D.J., Schwabe, P.: Neon crypto. In: International Workshop on Cryptographic Hardware and Embedded Systems, p. 320339. Springer (2012)
3. Cilardo, A.: Fast parallel $GF(2^m)$ polynomial multiplication for all degrees. IEEE Trans. Comput. **62**(5), 929–943 (2013)
4. El Hadj Youssef, W., Machhout, M., Zeghid, M., Bouallegue, B., Tourki, R.: Efficient hardware architecture of recursive Karatsuba-Ofman multiplier. In: IEEE DTIS 2008 3rd International Conference on Design and Technology of Integrated Systems in Nanoscale Era, pp. 1–6. (2008)
5. Fan, H., Hasan, M.A.: A survey of some recent bit-parallel $GF(2^n)$ multipliers. Finite Fields Their Appl. **32**, 5–43 (2015)
6. Fan, J., Guo, X., De Mulder, E., Schaumont, P., Preneel, B., Verbauwhede, I.: State-of-the-art of secure ECC implementations: a survey on known side-channel attacks and countermeasures. In: 2010 IEEE International Symposium on Hardware-Oriented Security and Trust (HOST), pp. 76–87 (2010)
7. Halbutogullari, A., Koç, C.K.: Mastrovito multiplier for general irreducible polynomials. IEEE Trans. Comput. **49**, 503–518 (2000)
8. Hasan, M.A., Wang, M., Bhargava, V.K.: Modular construction of low complexity parallel multipliers for a class of finite fields $GF(2^m)$. IEEE Trans. Comput. **41**, 962–971 (1992)
9. Imaña, J.L.: High-speed polynomial basis multipliers over $GF(2^m)$ for special pentanomials. IEEE Trans. Circuits Syst. I Regul. Pap. **63**(1), 58–69 (2016)
10. Imaña, J.L., Hermida, R., Tirado, F.: Low complexity bit-parallel multipliers based on a class of irreducible pentanomials. IEEE Trans. Very Large Scale Integr. Syst. (VLSI) **14**(12), 1388–1393 (2006)
11. Imaña, J.L., Hermida, R., Tirado, F.: Low complexity bit-parallel polynomial basis multipliers over binary fields for special irreducible pentanomials. Integr. VLSI J. **46**(2), 197–210 (2013)
12. Itoh, T., Tsujii, S.: Structure of parallel multipliers for a class of fields $GF(2^m)$. Inf. Comput. **83**(1), 21–40 (1989)
13. Lidl, R., Niederreiter, H.: Finite Fields. Cambridge University Press, Cambridge (1997)
14. Mullen, G.L., Panario, D.: Handbook of Finite Fields. CRC Press, Boca Raton (2013)
15. Negre, C., Robert, J.: Impact of optimized field operations AB, AC and AB+CD in scalar multiplication over binary elliptic curve. In: International Conference on Cryptology in Africa, pp. 279–296. Springer (2013)
16. NIST. Digital signature standard. Report p. 41, National Institute of Standards and Technology (NIST) (2000)
17. Oliveira, T., López, J., Rodríguez-Henríquez, F.: Software implementation of Koblitz curves over quadratic fields. In: International Conference on Cryptographic Hardware and Embedded Systems, pp. 259–279. Springer (2016)
18. Park, S.-M., Chang, K.-Y., Hong, D., Seo, C.: New efficient bit-parallel polynomial basis multiplier for special pentanomials. Integr. VLSI J. **47**(1), 130–139 (2014)
19. Reyhani-Masoleh, A., Hasan, M.A.: Low complexity bit parallel architectures for polynomial basis multiplication over $GF(2^m)$. IEEE Trans. Comput. **53**, 945–959 (2004)
20. Rodríguez-Henríquez, F., Koç, C.K.: Parallel multipliers based on special irreducible pentanomials. IEEE Trans. Comput. **52**, 1535–1542 (2003)
21. Scott, M.: Optimal irreducible polynomials for $GF(2^m)$ arithmetic. IACR Cryptol. ePrint Arch. **2007**, 192 (2007)
22. Sunar, B., Koç, C.K.: Mastrovito multiplier for all trinomials. IEEE Trans. Comput. **48**, 522–527 (1999)
23. Unterluggauer, T., Wenger, E.: Efficient pairings and ECC for embedded systems. In: International Workshop on Cryptographic Hardware and Embedded Systems, pp. 298–315. Springer (2014)
24. von zur Gathen, J., Shokrollahi, J.: Efficient FPGA-based Karatsuba multipliers for polynomials over $\mathbb{F}_2$. In: International Workshop on Selected Areas in Cryptography, pp. 359-369. Springer (2005)
25. Wu, H.: Low Complexity Bit-Parallel Finite Field Arithmetic Using Polynomial Basis. Cryptographic Hardware and Embedded Systems, pp. 280–291. Springer, Berlin (1999)
26. Wu, H.: Bit-parallel finite field multiplier and squarer using polynomial basis. IEEE Trans. Comput. **51**(7), 750–758 (2002)
27. Wu, H.: Bit-parallel polynomial basis multiplier for new classes of finite fields. IEEE Trans. Comput. **57**, 1023–1031 (2008)
28. Zhang, T., Parhi, K.K.: Systematic design of original and modified Mastrovito multipliers for general irreducible polynomials. IEEE Trans. Comput. **50**(7), 734–749 (2001)