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CORE DISCUSSION PAPER

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A NEW CLASS OF PRODUCTION FUNCTIONS AND AN ARGUMENT AGAINST PURELY LABOR-AUGMENTING TECHNICAL CHANGE*

Jakub GROWIEC†

June 19, 2006

Abstract. This paper follows Jones (2005) in his approach to deriving the global production function from microfoundations. His framework is generalized by allowing for dependence between the Pareto distributions of labor- and capital-augmenting developments. Using the Clayton copula family to capture this dependence, we derive a "Clayton-Pareto" class of production functions that nests both the Cobb-Douglas and the CES. Embedding the resultant production function in a neoclassical growth framework, we draw conclusions for the long-run direction of technical change. Jones' result of Cobb-Douglas global production functions and purely labor-augmenting technical change hinges on the assumption of *independence* of marginal Pareto distributions. In our more general case, the shape of local production functions matters for the shape of the global production function, and technical change augments both factors in the long run. Furthermore, the elasticity of substitution between capital and labor may exceed unity and thus yield endogenous growth.

Keywords: global production function, technology frontier, CES, Pareto distribution, Clayton copula.

JEL Classification Numbers: E23, O30, O40.

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1 Introduction

Imagine you are just setting up a transportation company. Your first task is to hire drivers and to buy trucks. At this point, you realize that you have to choose not only how many, but also what type of trucks to buy: in the market there are large ones, small ones, fast and slow ones, more and less fuel-consuming ones. What bothers you is that you will have to try to optimize over all these things simultaneously. Now, assume that you have succeeded, and let us go fast forward to when your company is already well-established. It turns out that in the meantime, your company has accumulated some extra capital but did not hire new drivers. It is now relatively capital-abundant. In such case, you should use mostly large, heavy trucks: only these can assure a high enough marginal product of labor. What if it was relatively labor-abundant? Then you should clearly decide to use trucks that are small but fast: you would care more about the marginal product of capital then. Now, fast forward again. Imagine that there has been a technological breakthrough in the truck industry: the new trucks are larger, faster, *and* less fuel-consuming than the old ones. Clearly, marginal products of capital and labor have both increased. But which one has increased by more? And how will you adjust your demand for capital and labor in response to such a change? You start feeling a bit confused...

We have to cut this story here, because this paper is not going to be about trucks and drivers. It is going to be about endogenous technology choice by firms and about the implied direction of technical change, but in the economy *as a whole*.

This paper accepts Jones' (2005) view that the production function, commonly assumed by macroeconomists to be a primitive, is in fact only a reduced form which should be derived from microfoundations. It is also acknowledged that such economy-wide production function has to be viewed as an assembly of a multiplicity of production techniques, particular methods of producing the final good.

Generalizing the Jones' setup, we derive from idea-based microfoundations a new class of production functions, baptized herein "Clayton-Pareto" functions. It is indeed a large class: it nests both the Cobb-Douglas function and the CES.[1] Unfortunately, we are unable to provide closed-form formulae for all Clayton-Pareto functions. We carry out a detailed study of solvable special cases instead.

We take Jones' results as a benchmark, and show that by the means of a slight modification of his assumptions, the "Cobb-Douglas global production function and purely labor-augmenting technical change" result can be overturned.

To our knowledge, both the derivation of a class of production functions that nests the Cobb-Douglas and the CES, and the provision of an idea-based microfoundation

[1]The parameters of our microfounded global production function are interrelated. In consequence, it makes sense to indicate that "the Clayton-Pareto family nests both the Cobb-Douglas and the CES": saying that it nests the CES only (which itself nests the Cobb-Douglas) would not reflect the result properly. At the intersection of conditions guaranteeing the CES, and the Cobb-Douglas result, we would obtain a Cobb-Douglas function with exponents $(.5, .5)$ only.

for the CES production function, are novel to the literature.

We also build a formal link between Jones (2005), and Caselli and Coleman (forthcoming). We find an idea-based explanation for (a generalization of) the shape of the technology frontier Caselli and Coleman postulate, and thus in a sense, we bring their model to a common denominator with Jones'. In this respect, we acknowledge that Caselli and Coleman interpret their model differently than Jones does. Namely, in place of firms, they put countries; in place of the local production function, they put *country*'s production function; in place of the global production function, they put *world* production function. In the outcome, they obtain the *world* technology frontier and not just a technology frontier faced by a representative firm, as Jones does.[2] The models of Jones, and Caselli and Coleman have been empirically justified on diverse bases, but they are characterized by profound mathematical unity, which we would like to uncover.

As a by-product of our analytic method, we also enrich the literature by providing alternative proofs for Jones' results, and finding explicit solutions for the firms' technology choices wherever it is possible.

To discuss the long-run implications of the new class of production functions, we embed our framework in the standard neoclassical (Solow, 1956) growth model. We find that Jones' (and Acemoglu's, 2003) results of technical change being purely labor-augmenting in the long run do not hold in general, but do hold in certain important cases, among which independence of the capital- and labor-augmenting developments is probably the most prominent. In general, technical change tends to augment both capital and labor, even in the long run. One simple intuition for this result is that if productivities of production factors are correlated, then it is impossible to achieve higher and higher productivity of labor without altering the productivity of capital as well. We shall study this issue in greater detail in section 4.

We shall also discuss the possibility of endogenous growth that arises if the global production function exhibits elasticity of substitution greater than one in the long run.[3]

It is beyond all doubt that the problems discussed by this paper are important for growth theory. Indeed, we try to find answers to such fundamental questions as: "What is the true shape of the economy-wide production function?", "What direction of technical change should we expect in the long run?", and "What drives the elasticity of substitution between production factors?". We manage to get new insights, because we derive what most economists assume. We believe that if one endows her model with solid microfoundations, she is also able to draw more refined

[2]The idea that countries with different factor endowments use different production technologies has been studied earlier, among others, by Atkinson and Stiglitz (1969) as well as Basu and Weil (1998).

[3]Such possibility has already been noticed in the seminal work of Solow (1956). There exists substantial literature that deals with these issues, including de La Grandville (1989), Jones and Manuelli (1990), Yuhn (1991), Klump and de La Grandville (2000), and Palivos and Karagiannis (2004).

macro-scale conclusions; on the other hand, she may also encounter new unexpected puzzles along the way. In any case, new knowledge is attained.[4]

The remainder of the article is structured as follows. In the next section, we generalize Jones' setup by allowing for dependence between capital- and labor-augmenting developments, and we derive the class of Clayton-Pareto functions. In section 3, we discuss the solvable special cases and describe the determinants of the elasticity of substitution. In section 4, we determine the long-run direction of technical change and check the conditions for endogenous growth. Section 5 concludes.

2 The shape of production functions

We shall now lay out the basic framework of our analysis. Most of its features come from the original Jones' (2005) framework. The concept of a technology frontier has been taken from Caselli and Coleman (forthcoming).

We consider a representative firm. Each individual production technique that this firm can use, named a *local production function* (LPF) hereafter, should be intuitively associated with a "recipe", or a list of instructions to follow. In order to produce, the firm has to pick a single LPF from the range of available ones, and follow it. Thanks to this interpretation, the LPFs should be fairly rigid and not allow for much substitutability between factors of production.[5]

The second important assumption is that factors can be utilized, according to a given LPF, with certain efficiency levels only. In this paper, we are going to identify the notion of an *idea* with these efficiency levels. Since we are going to consider only two factors of production here: capital K and labor L, an idea is consequently going to be a pair (a, b), where b and a are unit productivities of capital and labor, respectively.[6] Technical change is then identified with the sequential arrival of new, better and better, ideas. The *global production function* (GPF) is the convex hull of LPFs, or to put it in different words, it is such an assembly of LPFs, that for each K and L, productivities b and a are chosen optimally by the firm whose choice is constrained within the set of available ideas. The *technology frontier* is a subset of this set that contains only the ideas which could possibly be used by the profit-maximizing firm.

[4]Earlier literature that deals with closely related issues includes Houthakker (1955-56), Uzawa (1961), Kortum (1997) and Acemoglu (2003). Moreover, endogenous technology choice has common features with the assignment problem with heterogenous workers and tasks, analyzed by Sattinger (1975); and vintage capital theory where technological improvements are embodied in consecutive vintages of machines. This strand of literature dates back to Solow (1960), as well as Solow *et al.* (1966).

[5]The extreme but most intuitive case would be the Leontief function ("fixed coefficients"), where factors have to be used up in fixed proportions; more flexible functional forms are plausible as well, as we shall see shortly.

[6]We reverse the order of a and b in order to stick to Jones' (2005) notation where possible.

Following Jones, we shall assume that each dimension of an idea, be it a or b, is randomly drawn from a Pareto distribution.[7]

We would like to make it clear at this point that when a new idea arrives, it is already a pair (a, b), chosen from some joint, bivariate distribution. The individual Pareto distributions of factor productivities serve only as marginal distributions here: ideas are inherently complex and multi-dimensional.[8] This fact is a source of a huge ambiguity, however: we do not have any empirical evidence on the pattern of dependence between the marginal idea distributions.

To resolve this ambiguity, Jones arbitrarily assumes *in*dependence (p. 528):

> Assumption 1. The parameters describing an idea are drawn from independent Pareto distributions: (...)

but does not offer any motivation for this assumption.

In this paper, this assumption is waived. Jones' setup is generalized by allowing for dependence.

We remain agnostic as to what should be the appropriate measure of dependence between the marginal idea distributions. Nevertheless, we pick the Clayton family of copulas[9] to show that marginal Pareto distributions imply neither a Cobb-Douglas global production function, nor purely labor-augmenting technical change in the long run. On the contrary, they produce a wide variety of possibilities that we analyze in detail.

2.1 The derivation procedure

Our derivation procedure of the GPF from the LPFs and the technology frontier can be decomposed into three steps.

1. The *first step* consists in deriving the technology frontier from the joint distribution of ideas. This is done in proposition 2.

2. The *second step* is to find the optimal factor efficiency levels a^* and b^* (i.e. to pick the optimal LPF), given the available technology level N and the associated technology frontier, as well as stocks of capital K and labor L. This task is accomplished in proposition 3.

[7] Empirical evidence for prevalence of Pareto distributions in scientific productivity dates back to Lotka (1926). Theoretical literature linking Pareto distributions of ideas to exponential steady-state growth includes Kortum (1997) and Gabaix (1999).

[8] See Olsson (2005) for a very interesting elaboration of this claim.

[9] See Nelsen (1999) for an introduction to the copula theory. We indicate that there exist many families of copulas, different to the one we have chosen. The Clayton family belongs to the Archimedean class, as i.e. the Frank and the Gumbel family do; another widely recognized class of copulas is the elliptic class (Nelsen, 1999).

3. The *third step* (and the last one) consists in building a convex hull of LPFs – i.e. in inserting the optimal pair of technologies (a^*, b^*) to the LPF, separately for each pair of endowments (K, L). Convex hull of the LPFs is the GPF, as stated in proposition 4.

2.2 The technology frontier

We proceed directly to the *first step* of our derivation procedure.[10] We assume that research activity brings about discoveries of new ideas, whose levels are stochastic and drawn from Pareto distributions:

Assumption 1 *Unit factor productivities $\tilde{a}$ and $\tilde{b}$ are Pareto-distributed:*

$$P(\tilde{a} > a) = \left(\frac{\gamma_a}{a}\right)^{\alpha}, \qquad a \geq \gamma_a > 0, \qquad \alpha > 0, \tag{1}$$

$$P(\tilde{b} > b) = \left(\frac{\gamma_b}{b}\right)^{\beta}, \qquad b \geq \gamma_b > 0, \qquad \beta > 0. \tag{2}$$

As opposed to Jones (2005), who assumes that $\tilde{a}$ and $\tilde{b}$ are independent, we allow them to be mutually dependent. We shall produce a multiplicity of joint (bivariate) ideas distributions, keeping (1) and (2) as marginal distributions, using the Clayton family of copulas:

Assumption 2 *Dependence between the unit factor productivities $\tilde{a}$ and $\tilde{b}$ is represented by the Clayton copula, specified in* (3)*.*

Let us now elaborate this assumption. In fact, all the members of the Clayton family of copulas are characterized by the following formula (Nelsen, 1999):

$$C(u, v) = \max\left\{0, (u^{-\delta} + v^{-\delta} - 1)^{-1/\delta}\right\}, \tag{3}$$

where u and v are random variables, uniformly distributed over the unit interval, and $\delta \geq -1$ captures the degree and sign of dependence between the marginal idea distributions. $\delta = 0$ denotes independence, and thus calls for a replacement of (3) with $C(u, v) = uv$. We consider δ to be *the* crucial parameter here, because it is exactly the Jones' (2005) assumption of $\delta = 0$ that we relax, and whose relaxation yields so interesting results.

In the next step, we shall replace u and v with suitable cumulative distribution functions (CDFs). For the ease of exposition, we shall not do it in the main text, but relegate this point to Appendix A.1, which contains the proof of the following proposition:

[10] We consider the following subsection vital for the conveyed point, nevertheless a reader who is unfamiliar with, or sceptical about the probabilistic setting may want to take Proposition 2 as an assumption and proceed directly to subsection 2.3.

Proposition 1 *Distribution of the two-dimensional random variable* $(\tilde{a}, \tilde{b})$ *is given by*

$$P(\tilde{a} > a, \tilde{b} > b) = \max\left\{0, \left(\left(\frac{\gamma_a}{a}\right)^{-\alpha\delta} + \left(\frac{\gamma_b}{b}\right)^{-\beta\delta} - 1\right)^{-\frac{1}{\delta}}\right\}, \tag{4}$$

if $\delta \in [-1, +\infty)\backslash\{0\}$*; or*

$$P(\tilde{a} > a, \tilde{b} > b) = \left(\frac{\gamma_a}{a}\right)^{\alpha}\left(\frac{\gamma_b}{b}\right)^{\beta}, \tag{5}$$

if $\delta = 0$ *(the marginal distributions are independent).*

Proof. See Appendix A.1. ■

The Clayton copula parameter δ measures the degree and sign of dependence between individual factor productivities: if $\delta < 0$, they are negatively correlated; if $\delta > 0$, they are positively correlated. The probability (4) stands a chance of being zero only if $\delta < 0$.

We note that imposing $\delta = 0$, which stands for independence, leads to (5), which is equation (20) of Jones' paper.

We shall now define formally one of the most important concepts of the paper: the technology frontier. Application of this definition will make the correspondence between the stochastic arrival of ideas and the deterministic GPF clearer.

Definition 1 *The* technology frontier *is a curve in the* (a, b) *space, such that the probability* $P(\tilde{a} > a, \tilde{b} > b)$ *is constant along this curve.*

This seemingly simple definition helps us move away from tedious probabilistic considerations back to the deterministic world. It is consistent with the approach of Caselli and Coleman (forthcoming), and section II of the Jones' article. However, its motivation turns out to be not so simple after all. In quest of such, we build a model of research, where each invention needs to be understood and connected with the existing stock of knowledge by a given percentage of researchers before it comes into industrial use. An outline of this model has been relegated to Appendix B, so that the main line of reasoning within this section is not obstructed by a digression.

Using definition 1, together with equations (4) and (5), we can write the formula for the technology frontier now. It is contained in the following

Proposition 2 *Assume* $\delta \in [-1, +\infty)\backslash\{0\}$*. Then, the technology frontier* $H(a, b)$ *is given by*

$$H(a, b) = \gamma a^{\alpha\delta} + b^{\beta\delta} = N, \tag{6}$$

where γ *and* N *are positive constants. They are functions of model parameters, defined by* $\gamma \equiv \frac{\gamma_b^{\beta\delta}}{\gamma_a^{\alpha\delta}}$*, and* $N \equiv \gamma_b^{\beta\delta}[P(\tilde{a} > a, \tilde{b} > b)^{-\delta} + 1]$*. See that* $N \in [\gamma_b^{\beta\delta}, 2\gamma_b^{\beta\delta}]$ *if* $\delta < 0$*, and* $N \geq 2\gamma_b^{\beta\delta}$ *if* $\delta > 0$*.*

If $\delta = 0$, then the technology frontier becomes

$$H(a, b) = a^{\alpha} b^{\beta} = N, \tag{7}$$

where $N \equiv \frac{\gamma_a^{\alpha}\gamma_b^{\beta}}{P(\tilde{a}>a,\tilde{b}>b)} \geq \gamma_a^{\alpha}\gamma_b^{\beta}$.

Proof. It is easy and requires only algebraic manipulations. ∎

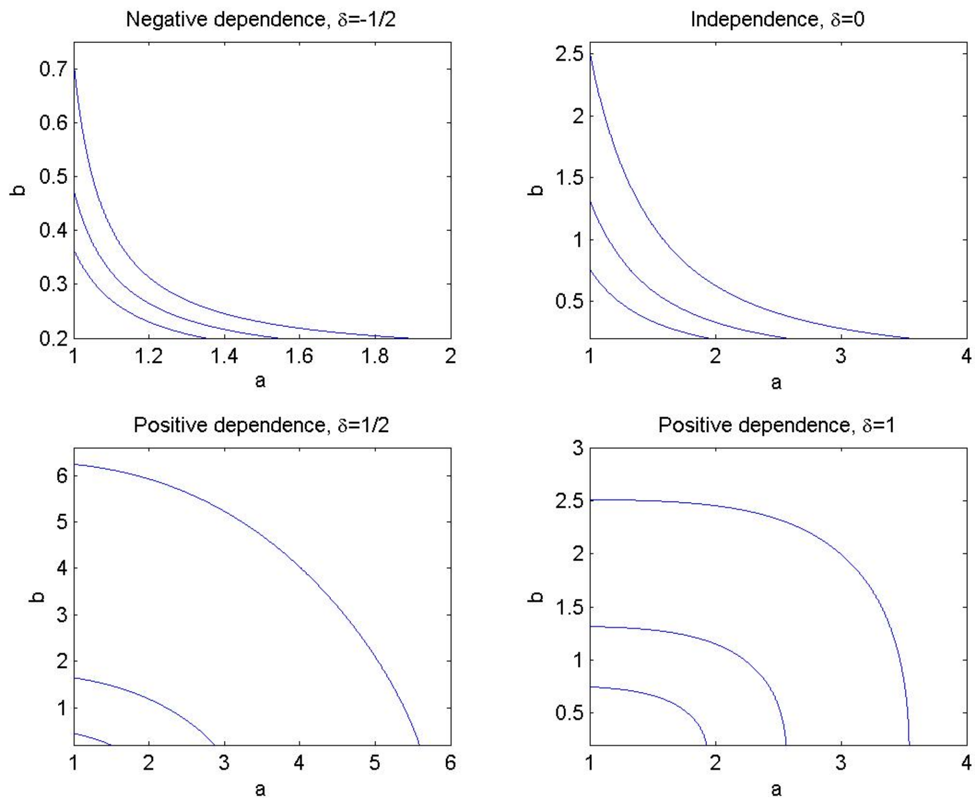

Figure 1: Examples of technology frontiers. Assumed parameter values: $\alpha = 5, \beta = 2.5, \gamma_a = 1, \gamma_b = .2; N \in \{.5, 2, 10\}$ (increasing from bottom to top) for $\delta \in \{0, .5, 1\}$, and $N \in \{9, 10, 11\}$ (increasing from top to bottom) for $\delta = -.5$.

We note that equation (7) above is Jones' equation (8).

Please note the following consequence of proposition 2: if there is negative dependence between individual factor productivities ($\delta < 0$), then perspectives for technological progress are limited: N is bounded. Thus, negative dependence between marginal idea distributions implies a strong "fishing-out" effect: with $N = \gamma_b^{\beta\delta}$, the probability $P(\tilde{a} > a, \tilde{b} > b)$ becomes zero, and further progress is impossible. Intuitively, because of the negative dependence, all further labor-(capital-)augmenting innovations would have to be unambiguously capital-(labor-)impairing. Such perspectives are clearly not desirable for a researcher.

On the other hand, if there is positive dependence between individual factor productivities ($\delta > 0$), then R&D can well go on forever. We will need these results for our considerations on the long-run direction of technical change in section 4.

A number of illustrative examples of technology frontiers, given by proposition 2, can be found in figure 1.

In the following analysis, we shall also assume that $\alpha\delta > \theta$ and $\beta\delta > \theta$, so that the curvature of the LPF is always greater than the curvature of the technology frontier. This would guarantee an interior solution to the optimization problems, which will ensue in section 2.3. These two inequality constraints have been derived from the second-order conditions in Appendix A.3.

2.3 Endogenous technology choice

Let us now proceed to the *second step* of our derivation procedure.

We shall make an important assumption that assures analytical tractability of our model. Namely, we assume the local production functions to be CES:

Assumption 3 *The local production function (LPF) $\tilde{Y}$ is given by*

$$\tilde{Y}(K,L;a,b) = \tilde{A}\left(\psi(bK)^{\theta} + (1-\psi)(aL)^{\theta}\right)^{\frac{1}{\theta}}, \tag{8}$$

where $\tilde{A} > 0$, $\theta \in (-\infty, 1]\backslash\{0\}$ and $\psi \in (0,1)$, or

$$\tilde{Y}(K,L;a,b) = \tilde{A}(bK)^{\psi}(aL)^{1-\psi}, \tag{9}$$

where $\tilde{A} > 0$ and $\psi \in (0,1)$, if $\theta = 0$.

We take this assumption from Caselli and Coleman (forthcoming). At this point, we do not have to impose $\theta < 0$ on top of it, as they do. Doing so would be natural for anyone who wants to stick to the "recipe" interpretation of the LPF, but we shall be more flexible here.[11]

The second step of our derivation procedure can be written as a competitive firm's optimization problem. Such firm faces a continuum of LPFs (production techniques) given by (8) or (9), and indexed by a and b along the available technology frontier.

[11] Assuming $\theta < 0$ is equivalent to saying that the elasticity of substitution of the LPF, $\sigma = \frac{1}{1-\theta}$, lies below unity. If we wanted to stick to the Jones' "recipe" understanding of a LPF, it would indeed seem reasonable to assume $\theta \approx -\infty$, so that each production technique (LPF) is approximately Leontief. See Jones (2005, p. 517-8) for an intuitive clarification of his understanding of a LPF. On the other hand, if the LPF denotes a country's production function (so the GPF denotes the world production function), then we would expect rather $\theta \approx 0$ than $\theta \approx -\infty$.

The firm chooses one of them in order to maximize its profit. It also optimally chooses its demand for capital and labor.

The problem of the competitive firm can be decomposed into two separate phases: first, choosing the optimal technology (a, b) for given endowments K, L and prices r, w; and then choosing the demand for both factors of production, taking their prices as given. Because we are only interested in finding the shape of the "Clayton-Pareto" global production function, and not in the general equilibrium of the economy, we shall skip the second phase.[12]

The first phase of the competitive firm's optimization problem in the typical case $\delta \neq 0$, $\theta \neq 0$ can be written as

$$\max_{a,b} \left\{ \tilde{A} \left(\psi (bK)^{\theta} + (1-\psi)(aL)^{\theta} \right)^{\frac{1}{\theta}} - rK - wL \right\} \text{ s.t. } \gamma a^{\alpha\delta} + b^{\beta\delta} = N. \tag{10}$$

First-order conditions imply that:[13]

$$\frac{b^{\beta\delta-\theta}}{a^{\alpha\delta-\theta}} = \frac{\gamma\psi}{1-\psi} \frac{\alpha}{\beta} k^{\theta}, \tag{11}$$

where we denoted $k \equiv K/L$ for convenience. Solving this equation for b yields

$$b = ck^{\frac{\theta}{\beta\delta-\theta}} a^{\frac{\alpha\delta-\theta}{\beta\delta-\theta}}, \tag{12}$$

which, plugged into (6), gives

$$\Phi(a, k; N) = \gamma a^{\alpha\delta} + c^{\beta\delta} k^{\frac{\theta\beta\delta}{\beta\delta-\theta}} a^{\frac{\beta\delta(\alpha\delta-\theta)}{\beta\delta-\theta}} - N = 0, \tag{13}$$

where $c \equiv \left(\frac{\gamma\psi}{1-\psi} \frac{\alpha}{\beta} \right)^{\frac{1}{\beta\delta-\theta}}$.

Before we go further with the calculations, we shall consider the following qualitative

Proposition 3 *Let $\alpha\delta > \theta$ and $\beta\delta > \theta$. Then, the optimization problem (10) allows a unique positive solution $(a^*(k; N), b^*(k; N))$.*

Proof. See Appendix A.2. ■

Equation (13) suffices to infer existence and uniqueness of the solution; unfortunately, it contains a sum of two arbitrary powers of a and thus, unless some particular equality constraint holds, ($\delta = 0$, $\theta = 0$ and $\alpha = \beta$ are the most interesting),[14] no

[12] Caselli and Coleman (forthcoming) close the model by assuming that the produced good is the numeraire and in the whole economy, stocks of available capital K and labor L are fixed. Jones (2005) skips the second phase.

[13] Exact derivations have been relegated to the appendix A.3.

[14] Another possibility is that one of the exponents equals 2, 3, or 4 times the other. An example of an analytically tractable Clayton-Pareto production function that is neither Cobb-Douglas nor CES, is included in subsection 3.5.

explicit formula for a as a function of capital per worker k and technology level N can be obtained from it.

Comparing our results to Jones', we can already hear the first alarm bell going off. Namely, optimal technology levels a^* and b^* depend on θ, the curvature parameter of the LPF. Hence, Jones' (2005, p. 528) claim, that

> [o]f course, the intuition regarding the global production function suggests that it is determined by the distribution of ideas, not by the shape of the local production function

does not hold in general. Equation (13) clearly suggests that in the typical "Clayton-Pareto" case, shape of the LPF *matters* for the shape of the GPF.

2.4 The global production function

We shall now pass to the *third step* of our procedure. We shall find the convex hull of LPFs, with $a^* \equiv a^*(k; N)$ and $b^* \equiv b^*(k; N)$ defined as above.

The most straightforward (and underestimated) way to do it is to insert a^* and b^* directly into the LPF formula (8). Another one is to characterize it alternatively in terms of partial elasticities, following Jones, Eq. (7):

$$\frac{\varepsilon_K}{1-\varepsilon_K} = \frac{\eta_b}{\eta_a}, \tag{14}$$

where $\varepsilon_K = \frac{\partial Y}{\partial K}\frac{K}{Y}$ is the partial elasticity of the global production function Y with respect to K; and η_a, η_b are partial elasticities of the technology frontier H with respect to a and b, respectively. Deriving equation (14) requires usage of the Envelope Theorem and the degree-one homogeneity of the GPF.

We shall point out that although both methods give the same results for the *shape* of the global production function, they are not equivalent. The first method is probably computationally more demanding, but it does not incur a loss of information due to differentiation, as the second one does. Thus, only thanks to the first method, all parameters of the global production function, including its intercept term, can be recovered.

Summing up, we obtain the following characterization of a GPF, which belongs to the Clayton-Pareto family:

Proposition 4 *Given assumptions 1, 2 and 3, propositions 1 and 2, and the firms' optimizing behavior summarized in* (10)*, the GPF is given by*

$$Y(K, L; N) \equiv \tilde{A}\left(\psi(b^*K)^\theta + (1-\psi)(a^*L)^\theta\right)^{\frac{1}{\theta}}, \tag{15}$$

where a^* and b^* satisfy (11), (12) *and* (13)*. This implies that*

$$\frac{\varepsilon_K}{1-\varepsilon_K} = \frac{\psi}{1-\psi}\left(\frac{b^*K}{a^*L}\right)^\theta = \frac{\beta}{\gamma\alpha}\frac{(b^*)^{\beta\delta}}{(a^*)^{\alpha\delta}}. \tag{16}$$

Proof. Already given in text. ■

Proposition 4 implicitly defines all GPFs that belong to the Clayton-Pareto class. A few of them have been depicted in figure 2.

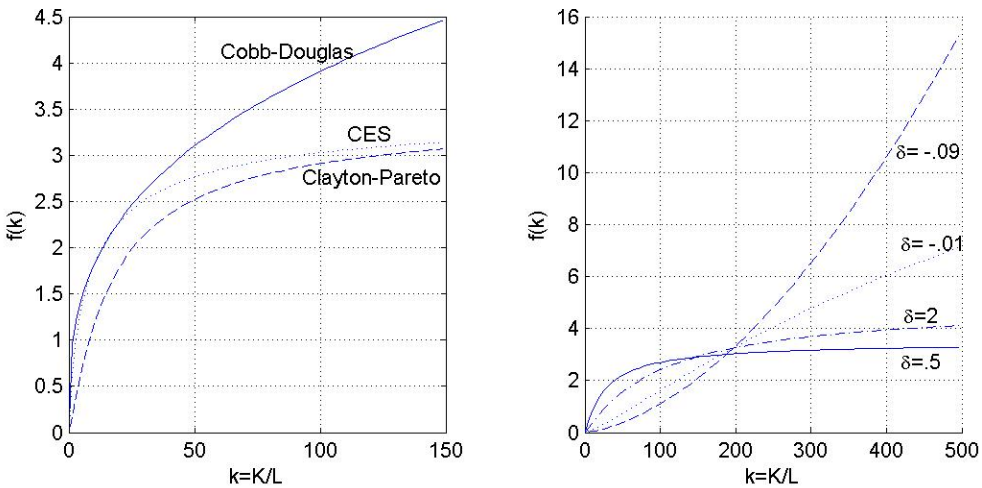

Figure 2: Left panel: a representative Clayton-Pareto production function compared to the Cobb-Douglas and the CES. Right panel: impact of the degree of dependence δ on the resultant Clayton-Pareto production functions. Assumed parameter values (unless indicated otherwise): $\tilde{A} = 1, \gamma_a = 1, \gamma_b = .2, \theta = -1, \alpha = 5, \beta = 2.5, \psi = .5, \delta = .5,$ $P(\tilde{a} > a, \tilde{b} > b) = .1$. For the CES case, we equalized $\alpha = \beta = 5$; for the Cobb-Douglas case, $\delta = 0$.

Let us now indicate the following property of Clayton-Pareto functions. Namely, not all of them are really *production* functions: some of them are not concave and thus violate the diminishing marginal utility requirement. Such situation is most likely to emerge if $\alpha\delta \approx \theta$, i.e. the second-order conditions are satisfied, but with a very narrow margin. In figure 2, we present such case for $\delta = -.09$. See also that for $\delta = -.01$, the Clayton-Pareto function is concave, albeit its implied marginal product of capital decreases *very* slowly. As we shall see in the next section, the concavity condition can be put in a simple analytical form given a very particular parameter choice. In general, however, this condition is given by a large and unintuitive formula.

3 Special cases

The drawback of our characterization of the Clayton-Pareto class of production functions is that we did not obtain a closed form. To make up for this obvious deficiency, we shall now dwell more on the solvable special cases. They all have meaningful economic interpretations. We find that the Clayton-Pareto family nests the Cobb-Douglas function and the CES function.

3.1 Independent Pareto distributions

Independence of the marginal idea distributions, i.e. $\delta = 0$, yields a particularly nice and interpretable result – the benchmark result due to Jones (2005). It is a link between *independent* Pareto idea distributions and Cobb-Douglas production functions.

With $\delta = 0$, the technology frontier is given by equation (5), which implies that the firm's optimality condition equivalent to (13) can be solved explicitly as

$$\begin{cases} a^*(k;N) & = N^{\frac{1}{\alpha+\beta}} \tilde{c}^{\frac{\beta}{\alpha+\beta}} k^{\frac{\beta}{\alpha+\beta}} \\ b^*(k;N) & = N^{\frac{1}{\alpha+\beta}} \tilde{c}^{-\frac{\alpha}{\alpha+\beta}} k^{-\frac{\alpha}{\alpha+\beta}}, \end{cases} \tag{17}$$

where $\tilde{c} = \left(\frac{\alpha}{\beta}\frac{\psi}{1-\psi}\right)^{1/\theta}$ and $\theta < 0$. As Jones finds out, the GPF comes to satisfy:

$$\frac{\varepsilon_K}{1-\varepsilon_K} = \frac{\beta}{\alpha}, \tag{18}$$

so it needs to be Cobb-Douglas with constant returns to scale, and exponents proportional to α and β, i.e. it needs to be

$$Y(K,L;N) = A(N) K^{\frac{\beta}{\alpha+\beta}} L^{\frac{\alpha}{\alpha+\beta}}. \tag{19}$$

Further calculations yield that the factor-neutral productivity term $A(N)$ is given by

$$A(N) = \tilde{A} N^{\frac{1}{\alpha+\beta}} \left[\psi^{\frac{\beta}{\alpha+\beta}} (1-\psi)^{\frac{\alpha}{\alpha+\beta}} \left(\left(\frac{\alpha}{\beta}\right)^{-\frac{\alpha}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} \right) \right]^{\frac{1}{\theta}}.$$

Moreover, if we take $\theta \to -\infty$, so that the LPFs are approximately Leontief, then we obtain $A(N) \to \tilde{A} N^{\frac{1}{\alpha+\beta}}$. This precisely corresponds to Jones' equation (28).

In this particular case of independent Pareto distributions, we could have missed the second step (finding a^* and b^*), and yet we would have arrived at almost the same result (the only difference is that now we would have failed to compute $A(N)$). This is because in (18), the GPF is required to exhibit *constant* partial elasticities with respect to its arguments, which is a defining property of Cobb-Douglas functions. So here, shapes of the LPFs have no impact on the shape of the GPF.[15]

[15] We have to maintain the assumption, that local production functions exhibit constant returns to scale, though. Moreover, to remain in the interior solution, we still have to assume that the elasticity of substitution of the LPFs is everywhere lower than unity. Finally, shapes of the LPFs – described by the parameters θ and ψ – do have an impact on $A(N)$.

3.2 Cobb-Douglas local production functions

It turns out that the "conjugate" of the Jones' "no-impact" assertion holds as well: if local production functions are Cobb-Douglas (i.e. $\theta = 0$), then the global production function is Cobb-Douglas, irrespective of the shape of the technology frontier.[16] This is fairly confusing a result, taking into account the Jones' assertion, which we cited in section 2.3.

Namely, once we assume that the LPFs are given by (9), we obtain that the shape of the technology frontier takes no part in determining the shape of the GPF. To see this implication, note that in the last step of our procedure, we would obtain

$$\frac{\varepsilon_K}{1-\varepsilon_K} = \frac{\psi}{1-\psi}, \tag{20}$$

implying a GPF of a Cobb-Douglas form:

$$Y(K, L; N) = A(N)K^{\psi}L^{1-\psi}, \tag{21}$$

irrespective of the shape of the technology frontier, provided that the two marginal distributions of unit productivities are positively dependent, which is necessary for an interior solution.

Further calculations yield $A(N) = \tilde{A}\bar{c}^{\frac{\psi}{\beta\delta}}\left(\frac{N}{\gamma+\bar{c}}\right)^{\frac{\psi}{\beta\delta}+\frac{1-\psi}{\alpha\delta}}$, where $\bar{c} = \frac{\gamma\psi}{1-\psi}\frac{\alpha}{\beta}$. This means that technology choice is irrelevant for the shape of GPF, but it is not irrelevant for the level of $A(N)$. To facilitate comparisons, we shall point out here, that in the optimum, firms choose technologies a^* and b^* according to:

$$\begin{cases} a^*(k; N) &= \left(\frac{N}{\gamma+\bar{c}}\right)^{\frac{1}{\alpha\delta}}, \\ b^*(k; N) &= \left(\frac{N\bar{c}}{\gamma+\bar{c}}\right)^{\frac{1}{\beta\delta}}. \end{cases} \tag{22}$$

Note that in this case, technology choice is independent of the endowments in capital K and labor L.

3.3 CES production function

Let us now assume that the shape parameters of both Pareto idea distributions are equal, i.e. $\alpha = \beta$. We shall see shortly, that in this particular knife-edge case, an explicit formula for the GPF can also be obtained. Moreover, this function exhibits a constant elasticity of substitution.

[16] Cobb-Douglas LPFs are not plausible, if one maintains a "recipe" understanding of a LPF, because they are characterized by unitary elasticity of substitution between factors. On the other hand, for the Caselli and Coleman's understanding of a LPF as a country-wide production function, they are perfectly plausible.

With $\alpha = \beta$, the technology frontier (6) becomes:

$$H(a,b) = \gamma a^{\alpha\delta} + b^{\alpha\delta} = N, \tag{23}$$

which is apparently Caselli and Coleman's equation (5). In their paper, it has been assumed without any explicit justification; therefore, by writing this equation down as a special case of our model, we can claim that we provide some "Clayton-Pareto" microfoundation for it.

Namely, instead of admitting that

> [t]he particular functional form of equation (5) is dictated by technical convenience, but it is rather flexible, (...)

(Caselli and Coleman, forthcoming, p. 16 of the Working Paper version), we view the equation as microfounded using Pareto distributions and the Clayton copula. On the other hand, within our framework, its flexibility is questionable since it hinges upon the knife-edge assumption $\alpha = \beta$.

The first order condition (11) of the representative firm's optimization problem (10) can be now solved as:

$$\begin{cases} a^*(k;N) &= \left(\frac{N}{\gamma + c^{\alpha\delta} k^{\frac{\alpha\delta\theta}{\alpha\delta-\theta}}}\right)^{\frac{1}{\alpha\delta}}, \\ b^*(k;N) &= \left(\frac{N c^{\alpha\delta} k^{\frac{\alpha\delta\theta}{\alpha\delta-\theta}}}{\gamma + c^{\alpha\delta} k^{\frac{\alpha\delta\theta}{\alpha\delta-\theta}}}\right)^{\frac{1}{\alpha\delta}}. \end{cases} \tag{24}$$

Thus, the GPF satisfies the equation:

$$\frac{\varepsilon_K}{1-\varepsilon_K} = \frac{c^{\alpha\delta} k^{\frac{\alpha\delta\theta}{\alpha\delta-\theta}}}{\gamma}, \tag{25}$$

so it takes the CES form (see Arrow *et al.*, 1961):[17]

$$Y(K,L;N) = A(N)(\zeta K^{\xi} + (1-\zeta)L^{\xi})^{1/\xi}, \tag{26}$$

where

$$\xi = \frac{\alpha\delta\theta}{\alpha\delta - \theta}, \tag{27}$$

$$\zeta = \frac{c^{\alpha\delta}}{c^{\alpha\delta} + \gamma}, \tag{28}$$

$$A(N) = \tilde{A} N^{\frac{1}{\alpha\delta}} \left((1-\psi)^{\frac{\alpha\delta}{\alpha\delta-\theta}} \gamma^{-\frac{\theta}{\alpha\delta-\theta}} + \psi^{\frac{\alpha\delta}{\alpha\delta-\theta}} \right)^{\frac{\alpha\delta-\theta}{\alpha\delta\theta}}, \tag{29}$$

[17] To prove this, note that $\frac{\varepsilon_K}{(1-\varepsilon_K)} = \frac{K/L}{MRS}$, where MRS is the marginal rate of substitution between capital and labor. Straightforward algebraic manipulations yield that the elasticity of substitution, $\sigma = \frac{\partial(K/L)}{\partial MRS}\frac{MRS}{K/L} = \frac{1}{1-\xi}$, where $\xi = \frac{\alpha\delta\theta}{\alpha\delta-\theta}$. Now use the theorem due to Arrow *et al.* (1961) to get (26). To obtain the coefficients A and ζ, more algebra is necessary. They have been computed by equalizing $\tilde{Y}(K,L;a^*,b^*) = Y(K,L;N)$, and then gradually simplifying the resultant expression.

and $c = \left(\frac{\gamma\psi}{1-\psi}\right)^{\frac{1}{\alpha\delta-\theta}}$. The assumption $\alpha\delta > \theta$ guarantees an interior solution to the competitive firm's optimization problem.

Let us also note that if we take $\theta \to -\infty$, so we assume that the LPFs are approximately Leontief, then our result simplifies greatly: $\xi \to -\alpha\delta$, $A(N) \to \tilde{A}N^{\frac{1}{\alpha\delta}}\left(\frac{1}{\gamma+1}\right)$, $\zeta \to \frac{1}{\gamma+1}$ and $c \to 1$.

The CES result deserves a longer discussion. It follows in the next subsection.

3.4 Elasticity of substitution: a microfoundation

The exponent ξ of the CES function, derived in (27), implies that the elasticity of substitution σ of the GPF is equal to

$$\sigma = \frac{1}{1-\xi} = \frac{\alpha\delta-\theta}{\alpha\delta-\theta-\alpha\delta\theta}. \tag{30}$$

Assuming that the LPFs be approximately Leontief, i.e. $\theta \to -\infty$, implies $\xi \to -\alpha\delta$ and accordingly, $\sigma \to \frac{1}{1+\alpha\delta}$.

One important remark is due here. To assign a serious economic interpretation to the above formulae, i.e. to assure that the elasticity of substitution σ is positive so that the marginal products of production factors are decreasing, we need to assume $\alpha\delta - \theta - \alpha\delta\theta > 0$.[18]

Having this assumption in mind, we see that the elasticity of substitution σ of the global CES function and the degree of dependence between marginal Pareto distributions δ are inversely related (which is confirmed in figure 3). Moreover, we also see that for the smallest possible values of δ, σ explodes to infinity, and for $\delta \to \infty$, it approaches the (low) elasticity of substitution of the LPF:

$$\lim_{\delta\to\left(\frac{\theta}{\alpha(1-\theta)}\right)_+} \frac{\alpha\delta-\theta}{\alpha\delta-\theta-\alpha\delta\theta} = +\infty \qquad \text{and} \qquad \lim_{\delta\to+\infty} \frac{\alpha\delta-\theta}{\alpha\delta-\theta-\alpha\delta\theta} = \frac{1}{1-\theta}. \tag{31}$$

The inverse relation between δ and σ gets the most apparent, once θ is evaluated at $-\infty$. Indeed, in this case, the relation between δ and θ is linear, elasticity of substitution σ exceeds unity if and only if $\delta < 0$, and it approaches zero as $\delta \to +\infty$.

Hence, we may conclude that what drives the elasticity of substitution of the GPF is actually the *difference* between the curvature of the LPF, described by θ, and the curvature of the technology frontier, described by $\alpha\delta$: the greater the difference, the lower the elasticity of substitution of the GPF, or to put in other words, the closer is σ to the (low) elasticity of substitution of the LPF.

[18] In the previous section, we indicated that not all Clayton-Pareto functions are concave. In the particular CES case, it turns out that the resultant GPF is in fact convex (so that the marginal product of capital is *in*creasing) if $\alpha\delta > \theta$ but $\alpha\delta - \theta - \alpha\delta\theta < 0$. We see that $\alpha\delta - \theta - \alpha\delta\theta > 0$ is a short closed-form concavity condition for the CES case, whose equivalent cannot be, unfortunately, obtained for the general Clayton-Pareto case.

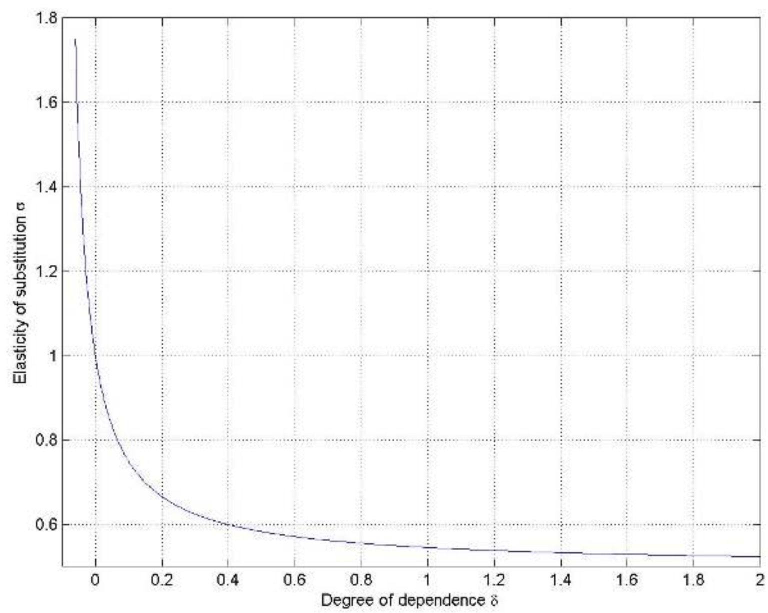

Figure 3: ELASTICITY OF SUBSTITUTION σ AS A FUNCTION OF THE DEGREE OF DEPENDENCE δ. ASSUMED PARAMETER VALUES: $\alpha = 5$, $\theta = -1$.

An intuitive argument in favor of this result goes as follows. The elasticity σ is the percentage change in the demand for capital per worker, announced by the firms which are assumed to maintain constant production, in response to a one per cent increase in the marginal rate of substitution. In our setup, there exist two channels of adjustment: first, through the demand for capital and labor, along a given LPF; and second, through the choice of technologies a and b, along the technology frontier and across LPFs. They both add to the total adjustment of the capital/labor ratio along the GPF, and neither of these two channels of adjustment should be ignored.

To get a diagramatic representation of this reasoning, one should go back to figure 1, take one particular technology frontier, and add an isoquant of the LPF to it. This isoquant should be tangent to the technology frontier and located to its north-east (which is guaranteed by second-order optimality conditions). Now, the extreme cases are the easiest to understand, and intermediate cases are also quite straightforward once one understands the extreme ones. On the one extreme, if $\delta = +\infty$, then a and b are fixed. Thus, all adjustment is necessarily made along the LPF, whose elasticity of substitution is $\frac{1}{1-\theta}$. On the other extreme, if $\alpha\delta - \theta - \alpha\delta\theta = 0$, then all adjustment happens via a and b, so capital and labor are in fact perfect substitutes and the GPF is linear ($\sigma = +\infty$). All other cases are somewhere in between these extremes, so that adjustment is made through both available channels.

Let us also point out that with $\alpha = \beta$, the elasticity of substitution σ is a constant.

It is fully determined by the deep parameters of the economy: the distribution of ideas and the curvature of the LPFs. Thus, technical change and capital accumulation do not influence its magnitude.[19]

The elasticity of substitution between capital (which is an accumulable factor), and labor (which cannot be accumulated), has sizeable effects on the long-run performance of an economy (see e.g. the two theorems by Klump and de La Grandville, 2000). In particular, $\sigma > 1$ may imply endogenous growth in absence of technical progress and explosive growth when technical progress is present (see Jones, Manuelli, 1990 and Palivos, Karagiannis, 2004), as we shall see in the section 4.

3.5 A Clayton-Pareto example that is neither Cobb-Douglas nor CES

As noted before, it is usually impossible to find a closed form for $a^*(k;N)$, implicitly defined by (13). There exist several exceptions to this rule, however, and one of them is such that $\frac{\alpha}{\beta} \cdot \frac{\beta\delta-\theta}{\alpha\delta-\theta} = \frac{1}{2}$. In such case, (13) becomes quadratic. Solving it and choosing the positive root yields the following optimal technology levels a^* and b^*:

$$\begin{cases} a^*(k;N) &= \dfrac{\left(-\gamma+\sqrt{\gamma^2+4Nc^{\beta\delta}k^{\frac{\beta\delta\theta}{\beta\delta-\theta}}}\right)^{\frac{1}{\alpha\delta}}}{2^{\frac{1}{\alpha\delta}}c^{\frac{\beta}{\alpha}}k^{\frac{\beta\theta}{\alpha(\beta\delta-\theta)}}}, \\ b^*(k;N) &= \dfrac{\left(-\gamma+\sqrt{\gamma^2+4Nc^{\beta\delta}k^{\frac{\beta\delta\theta}{\beta\delta-\theta}}}\right)^{\frac{2}{\beta\delta}}}{2^{\frac{2}{\beta\delta}}ck^{\frac{\theta}{\beta\delta-\theta}}}. \end{cases}$$

Plugging these technology choices into the LPFs yields the following Clayton-Pareto production function:

$$y = f(k) = \tilde{A}\Bigg\{\psi\left(\left(-\gamma+\sqrt{\gamma^2+4\,Nc^{\beta\,\delta}k^{\frac{\beta\,\delta\,\theta}{\beta\,\delta-\theta}}}\right)^{\frac{2}{\beta\,\delta}}2^{-\frac{2}{\beta\,\delta}}c^{-1}k^{\frac{\beta\delta-2\theta}{\beta\,\delta-\theta}}\right)^{\theta} + $$

$$+(1-\psi)\left(\left(-\gamma+\sqrt{\gamma^2+4\,Nc^{\beta\,\delta}k^{\frac{\beta\,\delta\,\theta}{\beta\,\delta-\theta}}}\right)^{\frac{1}{\alpha\,\delta}}2^{-\frac{1}{\alpha\,\delta}}c^{-\frac{\beta}{\alpha}}k^{-\frac{\beta\,\theta}{\alpha\,(\beta\,\delta-\theta)}}\right)^{\theta}\Bigg\}^{\frac{1}{\theta}}.$$

Having chosen a baseline parameter configuration, satisfying $\frac{\alpha}{\beta}\frac{\beta\delta-\theta}{\alpha\delta-\theta} = \frac{1}{2}$, we present the shape of this function in the left panel of figure 4.

The right panel of this figure confirms that this Clayton-Pareto function is characterized by a variable elasticity of substitution. It belongs neither to the CES family nor to the VES family (Revankar, 1971): its elasticity of substitution is a non-linear function of capital per worker k.

[19] Of course, for non-CES Clayton-Pareto production functions ($\alpha \neq \beta$), this property typically does not hold.

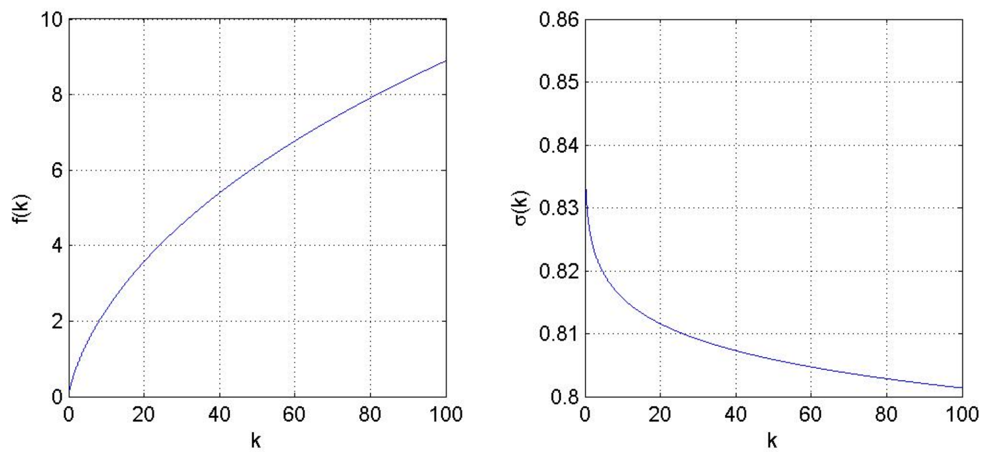

Figure 4: Example of a closed-form Clayton-Pareto production function. Assumed parameter values: $\tilde{A} = 1, \gamma_a = 1, \gamma_b = .2,\ \theta = -1, \alpha = 2, \beta = 5, \psi = .5, \delta = .1, P(\tilde{a} > a, \tilde{b} > b) = .1$. Left panel: the function. Right panel: its elasticity of substitution.

We do not present here the analytical formula for the elasticity of substitution of this Clayton-Pareto function, computed from

$$\sigma(k) = \frac{f'(k)\,(f(k) - kf'(k))}{-kf(k)f''(k)},$$

because it is very large and thus completely uninformative.[20] We hope that the reader will find our graphical representation sufficient.

4 The direction of technical change

We shall now embed our microfounded model, developed in former sections, in the standard neoclassical growth framework (Solow, 1956) to draw conclusions about the long-run direction of technical change.

From now on, we shall assume exogenous technical progress in the form of a decline in $P(\tilde{a} > a, \tilde{b} > b)$: as time passes, more and more efficient technologies are invented. With definition 1 in mind, it is clear that so defined an R&D activity is equivalent to pushing the technology frontier further and further. The choice of an optimal technology pair (a, b) from the technology frontier that is available at each instant of time, is (as in the previous sections) left to firms. This means that we shall explicitly

[20] Matlab-generated LaTeX code of this formula, typed with a 12-point Times font, is 11.5 A4-pages long. For the curious reader, it is available from the author upon request.

take advantage of the fact that in our framework, the direction of technical change is endogenous.[21]

Capital evolves according to the usual equation of motion:

$$\dot{K} = sY - \delta_K K, \tag{32}$$

where $\delta_K > 0$ is the capital depreciation rate, and $s \in (0,1)$ is the exogenous savings rate. Labor force grows at a constant rate $\hat{L} \equiv \frac{\dot{L}}{L} = n \geq 0$. Equation (32) implies that along the balanced growth path (BGP) – if there exists one, which is a very stringent condition – the product/capital ratio stays constant.

We shall distinguish here between two different interpretations of technical change: "crude" and "actually realized". Crude technical change is the change in factor productivities *given* factor endowments K and L, whereas actually realized technical change takes into account also the evolution of factor endowments over time. We are of course going to concentrate on actually realized technical change. A brief treatment of crude technical change has been relegated to Appendix A.4.

In further derivations, we shall dwell on the three particular cases, $\delta = 0$, $\theta = 0$, and $\alpha = \beta$, that offer closed-form solutions only. They already give a taste of the vast multiplicity of long-run outcomes of our model. In particular, the CES case turns out to be the most revealing: it allows us to draw preliminary conclusions about the dynamic properties of Clayton-Pareto production functions in general.

4.1 Independent Pareto distributions

We start with the case of independent Pareto distributions, studied by Jones (2005). Having assumed $\hat{N} = g$ and $y = Y/L$, and used $\hat{Y} = \hat{K}$, we find that the growth rate of the economy along the BGP is given by

$$\hat{Y} = \frac{g}{\alpha+\beta} + \frac{\beta}{\alpha+\beta}\hat{K} + \frac{\alpha}{\alpha+\beta}\hat{L} \quad \Rightarrow \quad \hat{y} = \frac{g}{\alpha}. \tag{33}$$

From (17), we also have that $\hat{a} = \frac{g}{\alpha+\beta} + \frac{\beta}{\alpha+\beta}\frac{g}{\alpha} = \frac{g}{\alpha}$ and $\hat{b} = \frac{g}{\alpha+\beta} - \frac{\alpha}{\alpha+\beta}\frac{g}{\alpha} = 0$. This is exactly the Jones' benchmark result of purely labor-augmenting technical change in the long run. Despite the fact, that for each given K and L, technical progress is factor-neutral in this case (see Appendix A.4), proportions of factors actually used in production change along the BGP in such a way that the unit productivity of capital – b – stays exactly constant.

4.2 Cobb-Douglas local production functions

We can now proceed to the case of Cobb-Douglas local production functions.

[21]Please note that our mechanism is different to the one present in Solow *et al.* (1966), or alternatively, Acemoglu (2003).

In such case, we still obtain existence of a BGP and a Cobb-Douglas GPF. However, we observe that technical change ceases to be purely labor-augmenting in the long run. Namely, along the BGP,

$$\hat{Y} = \Big(\frac{\psi}{\beta\delta} + \frac{1-\psi}{\alpha\delta}\Big)g + \psi\hat{K} + (1-\psi)\hat{L} \quad \Rightarrow \quad \hat{y} = \Big(\frac{\psi}{(1-\psi)\beta\delta} + \frac{1}{\alpha\delta}\Big)g. \tag{34}$$

Now, the growth rate of the economy $\hat{y}$ depends not only on α, as in equation (33), but also on β (as well as ψ and δ). Redoing the same exercise as in the previous subsection, we show that in this case, along the BGP, $\hat{a} = \frac{g}{\alpha\delta} > 0$ and $\hat{b} = \frac{g}{\beta\delta} > 0$. This means that technical change augments *both* factors of production in the long run.

4.3 CES production function

The case of a CES global production function turns out to be more revealing than the two Cobb-Douglas ones discussed just above, and it also gives a hint on what one can expect in the general Clayton-Pareto case.

First of all, we note that in the CES case, a BGP (with a positive growth rate of capital per worker) does not exist. This is the first difficulty we have to overcome. To get some meaningful outcome concerning the long run, we have to dwell on the asymptotics: by "in the long run" we will mean "as $t \to \infty$" rather than "along the BGP".

Second, we have to consider the two cases separately: $\xi > 0$ ($\sigma > 1$) where capital and labor are gross substitutes; and $\xi < 0$ ($\sigma < 1$) where they are gross complements.

The gross-substitutes case ($\xi > 0$) emerges if $0 > \alpha\delta > \theta$ or if $\alpha\delta > \theta > 0$. Simple calculations yield that in such case,

$$\hat{k} = sA(\zeta + (1-\zeta)k^{-\xi})^{1/\xi} - \delta_K - n \quad \Rightarrow \quad \lim_{k\to\infty} \hat{k} = sA\zeta^{1/\xi} - \delta_K - n. \tag{35}$$

Hence, there is endogenous growth via capital accumulation if $sA\zeta^{1/\xi} - \delta_K - n > 0$. Such endogenous growth is possible here thanks to the high elasticity of substitution of the analyzed global production function.[22] Since capital and labor are gross substitutes, marginal product of capital never declines to zero, and the Inada condition in infinity does not hold.[23]

Note that equation (35) makes economic sense only if $\delta < 0$ (so N is bounded) or if we exogenously fix N (assume out all R&D activity). If $A = A(N)$ were to grow exogenously on top of the elasticity-driven endogenous growth, we would have

[22] The reasoning follows Palivos and Karagiannis (2004), and Jones and Manuelli (1990), in that order.

[23] In the general Clayton-Pareto case, σ is not constant. However, one can clearly expect that endogenous growth via capital accumulation shall appear if $\lim_{k\to\infty}\sigma(k) > 1$ (see Palivos, Karagiannis, 2004).

arrived at an implausible explosive case of an economy that reaches infinite production in finite time.

Moreover, in the gross-substitutes case $\xi > 0$, with $\delta > 0$ and positive growth in $A(N)$, the condition for endogenous growth $sA\zeta^{1/\xi} - \delta_K - n > 0$ *must* (from some time on) be satisfied. Thus, such economy is bound to explode to infinity in finite time. For obvious reasons, we rule this case out.

Within the case of an endogenously, and yet non-explosively growing economy, individual factor productivities approach certain limits as $k \to \infty$ with time: $b^* \to N^{\frac{1}{\alpha\delta}}$, and $a^* \to 0$ if $\delta > 0$ or $a^* \to +\infty$ if $\delta < 0$.[24] Hence, it is clearly a^*, the productivity of labor (the unaccumulable factor) that drives the long-run endogenous growth here: in the optimum, if $\delta > 0$, so the unit productivities are positively dependent, it collapses to zero; and if $\delta < 0$, so the unit productivities are negatively dependent, it explodes to infinity.

A completely different case to consider is the one where $\xi < 0$. If capital and labor are gross complements, endogenous growth is impossible. On the other hand, since $\xi < 0$ implies $\delta > 0$ and $\theta < 0$, *exo*genous growth driven by the R&D activity (represented by perpetual growth in N) is both possible and plausible.

We find that in the gross-complements case there exists an *asymptotic BGP* – a BGP that cannot be reached in finite time, but is gradually converged to as $k \to \infty$ with time. We have that

$$\hat{Y} = \frac{g}{\alpha\delta} + \varepsilon_K \hat{K} + (1 - \varepsilon_K)\hat{L} \quad \Rightarrow \quad \hat{y} = \frac{g}{\alpha\delta} + \varepsilon_K \hat{k}. \tag{36}$$

Let us now calculate the limit of $\hat{y}$ as $t \to \infty$. Because of exogenous growth in N, we have that $A \to \infty$ and $k \to \infty$. In consequence, the partial elasticity ε_K disappears in the limit: $\varepsilon_K = A^\xi \zeta \left(\frac{k}{y}\right)^\xi \to 0$. This suffices to show that $\hat{y} \to \frac{g}{\alpha\delta}$. And by the virtue of the capital's equation of motion, we know that $\hat{k} \to \frac{g}{\alpha\delta}$ as well.

As for the growth rates of unit factor productivities, we use (24) to show that $\hat{a} \to \frac{g}{\alpha\delta}$ and $\hat{b} \to \frac{g}{\alpha\delta} + \left(\frac{\theta}{\alpha\delta - \theta}\right)\frac{g}{\alpha\delta} = \frac{g}{\alpha\delta - \theta} > 0$. Hence, we see that in the CES case, technical change augments *both* factors of production in the long run; not only labor, as it was in the Jones' case.[25]

There is a minor twist to this reasoning, however. As $\theta \to -\infty$, so local production functions become approximately Leontief, we get that $\hat{b} \to 0$. In the limit, where LPFs are truly Leontief, technical change is purely labor-augmenting, even in the CES case. We shall however insist that $\theta = -\infty$ is a very particular parameter choice: it is both knife-edge (non-typical) and extreme.

In place of a written summary of this section, we present the time paths of unit factor productivities effectively realized along the BGP (or asymptotic BGP, in the

[24] If $\delta < 0$ and there is exogenous technical progress, then the furthest attainable technology frontier is that with $N = \gamma_b^{\alpha\delta}$, and so b^* would in such case approach γ_b.

[25] And that is what one should expect in the typical Clayton-Pareto case, given that the long-run elasticity of substitution is less than unity (i.e. if $\lim_{k\to\infty} \sigma(k) < 1$).

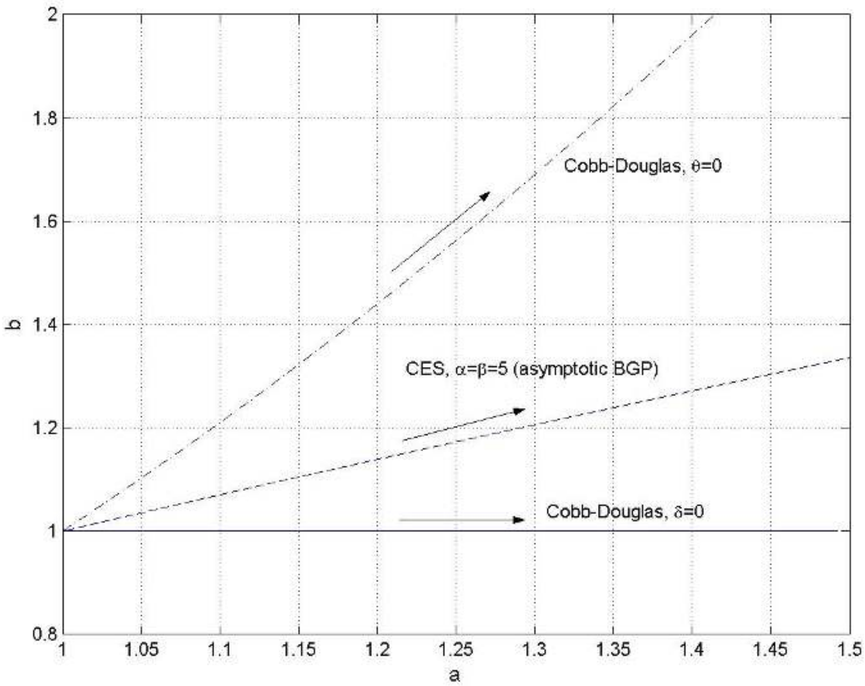

Figure 5: The direction of technical change along a balanced growth path. Unless stated otherwise, assumed parameter values are: $g = .02$, $\alpha = 5, \beta = 2.5, \theta = -1, \delta = .5$.

CES case) in figure 5. The starting point (a_0, b_0), and the long-run growth rate of the economy g have been chosen arbitrarily. See that if the LPFs are not Leontief, then technical change is purely labor-augmenting only in the Jones' case of $\delta = 0$.

In Appendix C, we employ the framework of this section to analyze the effects of ongoing shifts in the parameters of the Pareto distributions of ideas. We show that once they are allowed to change over time, our results concerning the long-run direction of technical change may be again quite different.

5 Conclusion

In this article, we have derived from microfoundations the "Clayton-Pareto" class of production functions. We have discussed the properties of some of its most interesting members. We have also analyzed the long-run implications of such production functions for the direction of technical change, i.e. we checked when it is labor-augmenting, capital-augmenting, or both.

Our "Clayton-Pareto" class of production functions has been obtained assuming that each of the unit factor productivities is Pareto-distributed, that dependence between these marginal distributions is captured by the Clayton copula, and that local production functions are CES. This class has been shown to nest both the Cobb-Douglas functions and the CES. Contrary to Jones' presumptions, its shape typically depends on the shapes of local production functions.

Embedding our microfounded model in the neoclassical (Solow, 1956) growth framework, we have proven that in general, technical progress tends to augment both factors of production in the long run. We could find only two exceptions to this rule, which would lead to purely labor-augmenting technical change: the case of independent marginal distributions, which implies a Cobb-Douglas GPF; and the case of Leontief local production functions, while the GPF is either CES or Cobb-Douglas.

We have also proven that in some cases, the neoclassical growth model with Clayton-Pareto production exhibits endogenous growth via capital accumulation. Moreover, assuming exogenous technological progress on top of it leads to explosivity.

Summing up: the goals that we achieved in this paper have been to critically re-examine the assumption of Cobb-Douglas production functions and purely labor-augmenting technical change, ubiquitous in contemporary growth theory; and to show that even a slight departure from the original Jones' (2005) framework can lead to novel results.

This article can be extended in several ways. First, our setup can be generalized to include further factors of production, such as human capital or non-renewable and renewable resources. Second, alternative copulas may be used to capture dependence between the marginal idea distributions. We doubt that the resultant production functions will follow widely recognized shapes, like e.g. the CES, but an analysis of, say, Frank-Pareto, or Gumbel-Pareto production functions, could certainly add new interesting insights to this strand of literature. Third, one may also wish to analyze the behavior of our microfounded model under R&D-based endogenous growth. Fourth, one may want to relax the assumption that the LPFs be CES. And last but not least, empirical studies in this field would be of enormous value.

A Mathematical appendix

A.1 Proof of proposition 1

Recall that the marginal idea distributions of $\tilde{a}$ and $\tilde{b}$ are Pareto, given by (1) and (2), respectively.

Because the Pareto distribution is nicely defined in terms of its survival function and not the cumulative distribution function (CDF) itself, we find it useful to substitute $X = -\tilde{a}$, $Y = -\tilde{b}$.[26]

CDFs of X and Y satisfy:

$$P(\tilde{a} > a) = P(X < -a) \equiv F_X(-a) = \left(\frac{\gamma_a}{a}\right)^{\alpha},$$

$$P(\tilde{b} > b) = P(Y < -b) \equiv F_Y(-b) = \left(\frac{\gamma_b}{b}\right)^{\beta},$$

where $a \geq \gamma_a > 0$ and $b \geq \gamma_b > 0$. Applying the Clayton copula formula given in (3) to the CDFs of X and Y yields:

$$F(a,b) = C(F_X(a), F_Y(b)) = \max\left\{0, \left(\left(\frac{\gamma_a}{-a}\right)^{-\alpha\delta} + \left(\frac{\gamma_b}{-b}\right)^{-\beta\delta} - 1\right)^{-\frac{1}{\delta}}\right\},$$

where $a \leq -\gamma_a < 0$ and $b \leq -\gamma_b < 0$.

Finally, we apply $P(\tilde{a} > a, \tilde{b} > b) = F(-a, -b)$ to get (4).

If $\delta = 0$, then

$$P(\tilde{a} > a, \tilde{b} > b) = F_X(-a)F_Y(-b) = \left(\frac{\gamma_a}{a}\right)^{\alpha}\left(\frac{\gamma_b}{b}\right)^{\beta},$$

which is directly (5). ■

A.2 Proof of proposition 3

We apply the Implicit Function Theorem to the Φ function, defined in (13). First of all, we notice that $\Phi \in C^1(\mathbb{R}^3_+)$. Second, we observe that

$$\frac{\partial \Phi}{\partial a} = \gamma\alpha\delta a^{\alpha\delta-1} + \left(\frac{\beta\delta(\alpha\delta-\theta)}{\beta\delta-\theta}\right)c^{\beta\delta}k^{\frac{\theta\beta\delta}{\beta\delta-\theta}}a^{\frac{\beta\delta(\alpha\delta-\theta)}{\beta\delta-\theta}-1}, \tag{37}$$

and so, using $\alpha\delta - \theta > 0$ and $\beta\delta - \theta > 0$, we obtain $\frac{\partial\Phi}{\partial a} > 0$ if and only if $\delta > 0$, and $\frac{\partial\Phi}{\partial a} < 0$ if and only if $\delta < 0$. Hence (since $\delta \neq 0$),[27] clearly $\frac{\partial\Phi}{\partial a} \neq 0$ for all $k > 0$ and

[26] Instead of inverting the random variables, we could also rotate the Clayton copula.

[27] The $\delta = 0$ (independence) case allows a unique, closed-form solution to the discussed problem. No sophisticated reasoning is necessary in this case. See section 3.1.

$N > 0$, so for all $k > 0$ and $N > 0$, there locally exists an implicit function a^* such that $\Phi(a^*(k;N), k; N) = 0$.

To obtain uniqueness of a^*, we shall note that for each given configuration of exogenous parameter values, and given $a, k, N > 0$, both partial derivatives of the implicit function, $\frac{\partial a}{\partial k}$ and $\frac{\partial a}{\partial N}$ have a constant sign. By the Implicit Function Theorem, we have $\frac{\partial a}{\partial k} = -\frac{\frac{\partial \Phi}{\partial k}}{\frac{\partial \Phi}{\partial a}}$ and $\frac{\partial a}{\partial N} = -\frac{\frac{\partial \Phi}{\partial N}}{\frac{\partial \Phi}{\partial a}}$. Constancy of the sign of $\frac{\partial \Phi}{\partial a}$ we have already proven above. $\frac{\partial \Phi}{\partial N} = -1$ so it is obviously always negative. Since

$$\frac{\partial \Phi}{\partial k} = \left(\frac{\beta\delta\theta}{\beta\delta - \theta}\right) c^{\beta\delta} a^{\frac{\beta\delta(\alpha\delta-\theta)}{\beta\delta-\theta}} k^{\frac{\beta\delta\theta}{\beta\delta-\theta}-1},$$

we get $\frac{\partial \Phi}{\partial k} > 0$ if and only if $\delta < 0, \theta < 0$ or $\delta > 0, \theta > 0$; and $\frac{\partial \Phi}{\partial k} < 0$ if $\delta > 0$ and $\theta < 0$. Thus, signs of $\frac{\partial a}{\partial k}$ and $\frac{\partial a}{\partial N}$ never change. Uniqueness of a^* follows from a juxtaposition of this fact with global differentiability of Φ.

From (12), we have that if a positive $a^*(k;N)$ exists and is unique, then automatically so is $b^*(k;N)$. ■

A.3 Second-order conditions

To save on algebraic manipulations, we shall simplify the competitive firm's optimization problem (10). Solving it is a task equivalent to finding extreme values of the following Lagrangian:

$$\mathcal{L}(a, b, \lambda) = \tilde{A}^\theta[\psi(bK)^\theta + (1-\psi)(aL)^\theta] - \lambda(\gamma a^{\alpha\delta} + b^{\beta\delta} - N).$$

The Lagrangian $\mathcal{L}$ should be maximized (if $\theta > 0$), or minimized (if $\theta < 0$).

We have

$$\begin{cases} \mathcal{L}_a(a,b,\lambda) &= \tilde{A}(1-\psi)\theta a^{\theta-1} L^\theta - \lambda\gamma\alpha\delta a^{\alpha\delta-1} = 0, \\ \mathcal{L}_b(a,b,\lambda) &= \tilde{A}\psi\theta b^{\theta-1} L^\theta - \lambda\beta\delta b^{\beta\delta-1} = 0, \\ \mathcal{L}_\lambda(a,b,\lambda) &= \gamma a^{\alpha\delta} + b^{\beta\delta} - N = 0. \end{cases}$$

Moving the terms with λ in the first two equations to the RHS, and dividing sidewise yields (11).

In the optimum, we also have $\lambda = \frac{\tilde{A}^\theta(1-\psi)\theta L^\theta a^{\theta-\alpha\delta}}{\gamma\alpha\delta} = \frac{\tilde{A}^\theta\psi\theta K^\theta b^{\theta-\beta\delta}}{\gamma\beta\delta}$.

The second derivatives of the Lagrangian are (after inserting appropriate expressions for λ):

$$\begin{cases} \mathcal{L}_{aa}(a,b) &= \tilde{A}^\theta(1-\psi)\theta a^{\theta-2} L^\theta(\theta - \alpha\delta), \\ \mathcal{L}_{ab}(a,b) &= \mathcal{L}_{ba}(a,b) = 0, \\ \mathcal{L}_{bb}(a,b) &= \tilde{A}^\theta\psi\theta b^{\theta-2} K^\theta(\theta - \beta\delta). \end{cases}$$

If $\mathcal{L}_{aa}$ and $\mathcal{L}_{bb}$ are both negative, then we have a maximum; and if they are both positive, we have a minimum. It is straightforward to see, that we need $\alpha\delta > \theta$ and $\beta\delta > \theta$ for our optimization criteria to be satisfied. ■

A.4 Factor-neutrality?

In this appendix, we shall confront the simple intuition that for given and *fixed* factor endowments K and L, an exogenous change in N (which is driven by a decrease in $P(\tilde{a} > a, \tilde{b} > b)$) is factor-neutral. Namely, we shall check whether a percentage change in N modifies a^* and b^* – *before* the firm is allowed to choose their preferred technology – proportionately. Contrary to this intuition, it turns out to be so only in three specific cases: $\delta = 0$ (the Jones' case), $\theta = -\infty$ (where local production functions are Leontief), and $\alpha = \beta$ (the CES case).

To prove this, we shall use (12) to obtain

$$\varepsilon_b \equiv \frac{\partial b}{\partial N}\frac{N}{b} = \frac{\partial b}{\partial a}\frac{\partial a}{\partial N}\frac{N}{b} = \left(\frac{\alpha\delta - \theta}{\beta\delta - \theta}\right)\frac{\partial a}{\partial N}\frac{N}{a} \equiv \left(\frac{\alpha\delta - \theta}{\beta\delta - \theta}\right)\varepsilon_a, \tag{38}$$

so $\varepsilon_a = \varepsilon_b$ only if $\delta = 0$, $\theta = -\infty$ or $\alpha = \beta$, which is the result announced just above.

Moreover, in the generic case $\delta \neq 0, \theta \neq -\infty$ and $\alpha \neq \beta$, we obtain that for any given K and L, $|\varepsilon_a| > |\varepsilon_b|$ if and only if $\alpha < \beta$. Intuitively, it means that "crude" technical change (not corrected for firms' technology choices yet) always favors the direction, whose distribution tail is fatter, so the prospects for further progress are greater.[28]

B A model of research

In this appendix, we shall present a model of research, which we had in mind when stating Definition 1.

We assume a continuum of researchers, located along the unit interval $I = [0, 1]$. Each researcher $i \in I$ draws one technology pair (a_i, b_i) from the joint idea distribution $(\tilde{a}, \tilde{b})$. An application of the Law of Large Numbers implies that for all values of $a \geq \gamma_a, b \geq \gamma_b$, there must exist some researcher who has drawn a technology pair arbitrarily close to (a, b) in terms of the usual Pythagorean metric on $\mathbb{R}^2$.

Let us now define the dependence between individual draws of ideas and the technology that becomes the output of the research process. We assume that a technology pair (a, b) can be made available for production only after a given fraction $x \in (0, 1)$ of researchers understands it and helps incorporate it into the existing stock of knowledge.[29] To put it formally, a technology pair (a, b) becomes available only if a given fraction x of researchers have drawn a technology pair that is not inferior to (a, b).

The reader may now wonder what we mean by *not inferior* in a two-dimensional setup. We explain this as follows. Let us consider a single ray from the origin: a

[28]We used absolute values of ε_a and ε_b, because these values are negative if $\delta < 0$. In such case, advancing to further and further technology frontiers, or decreasing $P(\tilde{a} > a, \tilde{b} > b)$, is associated with *decreasing* N.

[29]See Olsson (2005) for a detailed presentation of the foundations of this line of reasoning.

semi-straight line $\{(a,b) \in \mathbb{R}_+^2 : b = \omega a\}$, where $\omega \in (0,+\infty)$ is given. Along such a ray, all points are indexed by a single number $a \geq \gamma_a$. And on the real line, there exists a natural ordering.

Using the Law of Large Numbers again, we obtain that there exists a unique $\bar{a} > \gamma_a$ such that for all $\gamma_a < a < \bar{a}$, technology $(a, \omega a)$ is understood by at least x per cent of researchers. The "frontier" technology $(\bar{a}, \omega\bar{a})$ is characterized by $P(\tilde{a} > \bar{a}, \tilde{b} > \omega\bar{a}) = x$, and all technologies $(a, \omega a)$ for which $P(\tilde{a} > a, \tilde{b} > \omega a) > x$ are *inferior* to the technology $(\bar{a}, \omega\bar{a})$ because *both* their coordinates are lower. Thus, the "frontier" technology pair $(\bar{a}, \omega\bar{a})$ is the best one attainable within the given ray.

We carry this procedure out for all rays from the origin, letting the index ω go from 0 to $+\infty$. In the final outcome, we obtain that the best attainable ("frontier") technologies are located along the curve $\{(a,b) \in \mathbb{R}_+^2 : P(\tilde{a} > a, \tilde{b} > b) = x\}$. We refer to the x-th contour of the idea cumulative distribution function as to the *technology frontier*.

Until now there has been no technological progress (and in fact, no time dimension) in this model of research. We introduce it by assuming that as time goes on, researchers (whom we assume to be infinitely-lived for simplicity) gradually accumulate knowledge, and so it is easier and easier for them to understand and accomodate further innovations. Hence, further and further technology frontiers become attainable in the passage of time. Again for simplicity, we do not model the knowledge accumulation process explicitly. Instead, we proxy it with an exogenous decline in x which yields equivalent results.

Employing our model of research helps simplify the original Jones' framework greatly. First, we do not have to use Fréchet distributions or Poisson processes to obtain our results. Second, we are able to achieve more generality, which would not be possible if we insisted on maintaining full stochasticity until the end of analysis: one can easily imagine that if the simplest case already involves Fréchet distributions, then more complex ones would simply be intractable.

C Shifts in the distribution of ideas

In the main text, we have identified technical progress with downward shifts in the probability $P(\tilde{a} > a, \tilde{b} > b)$. When doing so, we were holding all other model parameters constant. We shall now relax this assumption and examine the effect of ongoing shifts in the distribution of ideas.

The problem may seem far-fetched at the first glance. Actually, it is the opposite due to the existence of positive feedbacks in research activity: research consists not only in discovering laws and concepts that are an inherent part of the universe, but also in creating new possibilities for further research, new areas of knowledge that could have *a priori* never appeared. Shifts in the idea distribution can be identified with the creation of such new fields of research as well as with changes in the unit efficiency of R&D itself.

In our model, the joint distribution of ideas is characterized by the quintuple of parameters $(\gamma_a, \gamma_b, \alpha, \beta, \delta)$. To keep things as simple as possible, and still come to face a few *paradoxes*, we shall look at the consequences of ongoing changes in γ_b (the cutoff point of the Pareto capital-augmenting idea distribution) only, and keep all other parameters constant. Moreover, we shall do it only in the three particular cases of interest, namely the two Cobb-Douglas cases and the CES case.

In the Jones' (2005) "independence" $\delta = 0$ case, we have

$$g = \hat{N} = \alpha\hat{\gamma}_a + \beta\hat{\gamma}_b - \hat{x},$$

where we have written $x \equiv P(\tilde{a} > a, \tilde{b} > b)$ for brevity. Instead of assuming that $\hat{x} = -g$, which we implicitly did in the main text, we could have taken some different combination of growth rates of parameters, say $\hat{\gamma}_b = \frac{g}{\beta}$ and $\hat{\gamma}_a = \hat{x} = 0$. In any case, if $\hat{\gamma}_b > 0$ in the long run, so that the marginal distribution of unit capital productivity is continuously shifted to the right, then we immediately obtain a paradox: the firm would not like to adopt any capital-augmenting improvements and yet it has to. Since $b \geq \gamma_b > 0$, we arrive at a corner solution with $\hat{b} = \hat{\gamma}_b > 0$ from some time on. Thus, the Jones' result of purely labor-augmenting technical change ceases to hold in the long run.

In the case of Cobb-Douglas LPFs ($\theta = 0$), we have

$$g = \hat{N} = \beta\delta\hat{\gamma}_b + \widehat{(x^{-\delta} + 1)},$$

and we want to assume $\gamma_b > 0$ again. For simplicity, we shall take $\hat{x} = 0$ and $\hat{\gamma}_b = \frac{g}{\beta\delta}$. This implies $\hat{\gamma} = g$, and so, $\hat{b} = \frac{g}{\beta\delta} - \frac{1}{\beta\delta}(\widehat{\gamma + \frac{\gamma\psi\alpha}{(1-\psi)\beta}}) = \frac{g}{\beta\delta} - \frac{g}{\beta\delta} = 0$. Hence, we arrive at the same paradox as above. For the firms, it would be optimal not to implement any capital-augmenting inventions, but from some time on, they will be forced to do so. The condition $b \geq \gamma_b > 0$ will ultimately be binding. Purely labor-augmenting technical change (brought about by the shifts in γ_b!) does not stand the test of rightward-shifting capital-augmenting idea distribution.

We shall now turn to the CES case $\alpha = \beta$. By taking $\xi < 0$, we shall assume away the possibility of endogenous growth. Just like above, we shall let $\hat{x} = 0$ and $\hat{\gamma}_b = \frac{g}{\alpha\delta}$. Again, it implies $\hat{\gamma} = g$. Apparently, we obtain a *different* paradox now: as technological progress shifts γ_b only, and leaves all other model parameters (including x) intact, the factor-neutral productivity level A approaches a constant. To show this, we proceed in two steps. First, we observe that $\hat{A} \to \frac{\hat{N}}{\alpha\delta} - \frac{\theta}{\alpha\delta-\theta}\frac{\alpha\delta-\theta}{\alpha\delta\theta}\hat{\gamma} = \frac{g}{\alpha\delta} - \frac{g}{\alpha\delta} = 0$. Second, we use of slightly more tedious algebra to reveal that there indeed exists a limit value of A. It equals $\bar{A} \equiv \tilde{A}N_0^{\frac{1}{\alpha\delta}}(1-\psi)^{\frac{1}{\theta}}$, where N_0 is the technology level N at time 0. This means that in the long run, total per capita product and capital per worker *stabilize*, and the economy *never* reaches the asymptotic BGP. From (35), and using the fact that with time, the CES distribution parameter ζ approaches zero, we obtain the steady-state value of capital per worker, $k^* = \frac{s\bar{A}}{\delta_K+n}$, and the steady-state level of product $y^* = s\bar{A}$. The growth rate of b^* is not constant and is certainly not zero; it approaches $\hat{b}^* \to \frac{g}{\alpha\delta}\left(\frac{\alpha\delta-\theta+\alpha\delta\theta}{(\alpha\delta-\theta)\alpha\delta}\right)$. The paradoxical result is that in the CES case, under technical progress through shifts in γ_b, a steady-state economy and continued improvements in unit capital productivity go together.

If $0 > \theta \geq \frac{\alpha\delta-(\alpha\delta)^2}{1-2\alpha\delta}$, i.e. if θ is close enough to zero, then $\hat{b}^*$ is smaller than growth rate of γ_b. If θ is greater than this threshold value, however, we ultimately get a corner solution with $\hat{\gamma}_b = \hat{b} = \frac{g}{\alpha\delta}$. This is the same paradox as in the Cobb-Douglas cases discussed above.

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