A new composite body method for manipulator dynamics — Source link

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A New Composite Body Method for Manipulator Dynamics

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This article presents a new composite body method for numerically forming the inertia matrix and the bias vector of manipulators, which is more efficient than the other two existing types of composite body methods. The main discrepancy of this one from the existing ones is that all points in a manipulator are observed from the origin of the base frame and the distances are all measured from this origin. The required computations of the present method for the inertia matrix and the bias vector of a manipulator with n rotational joints are $(10.5n^2 + 38.5n - 85)$M + $(6n^2 + 39n - 70)$A and $(12.5n^2 + 5.5n + 3)$M + $(9n^2 + n)$A, respectively, where "M" denotes multiplications, "A" does additions. In numerically forming the inertia matrix, the present method is more efficient than other methods in the literature for a manipulator with five or more joints; whereas this method is also superior to the recursive Newton–Euler formulation in computing the bias vector for a manipulator with six or less joints.

この論文では、マニピュレータの慣性マトリックスと傾斜ベクトルを数値的に導くための、新しい複合体法について説明している。　この方法は、他の既存の二つの複合体法と比較して、より効果的である。　他の一つとこの方法の大きな違いは、マニピュレータ内の総ての箇所が、基礎構造の原点から観察され、その総ての原点からの距離が測定されている。　この方法で、回転式ジョイントの付いたマニピュレータの慣性マトリックスと傾斜ベクトルのために必要な計算は、$(10.5n^2+38.5n-85)M+(6n^2+39n-17)A$と$(12.5n^2+5.5n+3)M+(9n^2+n)A$ となり、"M"はマニピュレータを表し、"A" は他の要素を表す。　慣性マトリックスを数値的に形成する場合において、この方法は、５軸以上のジョイントを持つマニピュレータの文献の中で見受けられる他の方法に比較してより効果的である、さらに６軸以下のマニピュレータの傾斜ベクトルの計算で、この方法は、ニュートン・オイラー方程式の再帰呼び出しよりも優れている。

INTRODUCTION

It is well known that the dynamic model of a manipulator can be described with

$$\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{b}(\mathbf{q},\dot{\mathbf{q}}) = \boldsymbol{\tau} \tag{1}$$

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where $\mathbf{H}$ is the positive definite symmetric inertia matrix, $\mathbf{b}$ is often called the bias vector, $\boldsymbol{\tau}$ is the column of actuator forces, and $\mathbf{q}$ is the column of generalized coordinates, i.e., the displacement of joints.

The dynamics of manipulators can be categorized in two parts: the inverse dynamics and the forward dynamics. Many modern control schemes for manipulators require the inverse dynamics that determine the actuator forces for the prescribed joint displacements, velocities and accelerations. Due to the demand of real time control, several efficient algorithms for the inverse dynamics are developed. The recursive Newton–Euler formulation[1] is found the most efficient in the literature.[2] Recently, researchers have been struggling to improve the efficiency of this formulation. Two of the most efficient algorithms based on the recursive Newton–Euler formulation were proposed by Khalil and Kleinfinger[3] and Balafoutis et al.[4]

The forward dynamics are required while the motion simulation of manipulators is performed. The object is to solve the joint accelerations from (1) when the actuator forces are given as input values. The joint velocities and displacements can be obtained by integrating the joint accelerations, and are then used to calculate the bias vector. The research of the forward dynamics focuses on the formulation of the inertia matrix and the bias vector, which is the central topic of this article.

An industrial manipulator is different from a general multibody system in that it mostly has only six links or less. This fact should be taken into account while we investigate the formulation of manipulator dynamics. Our goal is to look for an efficient formulation for a manipulator with six links or less, although this formulation may be less efficient for a manipulator having more than six links.

In the literature, there have been two types of composite body (or generalized body) methods for forming the dynamic model of a manipulator, which are more efficient than the other methods. Walker and Orin[5] developed a composite body method only for numerically forming the inertia matrix $\mathbf{H}$. The bias vector $\mathbf{b}$ was computed using the recursive Newton–Euler formulation[1] by setting $\ddot{\mathbf{q}} = 0$. Another composite body method for forming the inertia matrix was derived from Lagrange's equations by Renaud.[6,7] He also combined the resulting formulation with the reduced Christoffel symbols symbolically to form the bias vector. Based on this method, Burdick[9] established a LISP-based program, EMDEG, to automatically generate the symbolic dynamic model of manipulators. Vukobratovic et al.[10] extended the Renaud's formulation to the bias vector, and established an algorithm for numerically forming the inertia matrix and the bias vector of a manipulator with rotational joints. An alternative algorithm of the Renaud's formulation is recently proposed by Fijany and Bejczy.[11,12] They derived a recursive formulation for the inertia matrix which is similar to that of Walker and Orin[5] in structure, but is equivalent to Renaud's formulation in substance. Nevertheless, it is not more efficient than the original Renaud's formulation (see Appendix).

An important discovery by Renaud is that some combinations of the inertia parameters of the links in a composite body are invariant to the manipulator motion (cf. (A5) and (A6) in Appendix). These invariant terms can be interpreted with the concept of the augmented body[14] as the first moment and the inertia tensor of an augmented body.[4,8] Renaud[8] and Balafoutis et al.[4] adopted this recognition to reformulate the recursive Newton-Euler formulation in terms of these invariant terms, independently, which is found computationally superior to the original one.[1]

For numerically forming the inertia matrix, there is another method, which was developed by the author of this article in his earlier work.[13] The basis of this method is an explicit formulation which relates the entries of the inertia matrix to the partial derivatives of the velocities and angular velocities of links with respect to the joint velocities. The algorithm based on this formulation is only adequate to manipulators with few joints since its computational complexity is of order n^3.

This article presents a new type of composite body method for numerically forming the inertia matrix and the bias vector of manipulators. We try to derive a closed-form formulation from the composite body theory of Walker and Orin[5] since we believe that a closed-form formulation can improve the efficiency. This is achieved by obseving all points in a manipulator from the origin of the base frame. The formulations are derived in the second and third sections. An algorithm and a comparison of efficiency with other methods are presented in the fourth section. It is shown that the present method is more adequate to industrial manipulators than other methods.

FORMULATION OF THE INERTIA MATRIX

The basic theory of the new composite body method is similar to that of Walker and Orin's. However, the forces and torques and the center of mass of a composite body are all observed in an inertia frame, namely the base frame E_0. And we want to derive a closed-form formulation for the inertia matrix, although some terms in the formulation are still computed in a recursive form. Suppose that a manipulator has n low-pair joints, which are labeled as joint 1 to n outward from the base. Assign a body-fixed frame on each joint, i.e., frame E_i is fixed on joint i. The distance from the origin of E_i to that of E_j is designated as ${}_i^j\mathbf{s}$, and the distance from the origin of E_i to the center of mass of link j as ${}_i^j\mathbf{p}$ (Fig. 1). Define a composite body j as the union of link j to link n. The mass of the composite body j is denoted as $\hat{m}_j$, and the distance from the origin of the base frame to the center of mass of the composite body as $\mathbf{r}_j$. Hence

$$\hat{m}_j = \sum_{i=j}^{n} m_i \tag{2}$$

$$\mathbf{r}_j = \frac{\sum_{i=j}^{n} m_i \, {}_0^i\mathbf{p}}{\hat{m}_j} \tag{3}$$

where m_i is the mass link i. The inertia tensor of the composite body, $\mathbf{J}_j$, results by using Huygeno–Steiner formula[14] to obtain

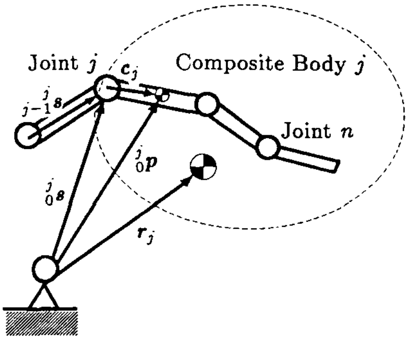


Figure 1. A composite body.

$$\mathbf{J}_j = \sum_{i=j}^{n} \mathbf{I}_i - m_i[({}^{i}_{0}\mathbf{p} - \mathbf{r}_j)\times][({}^{i}_{0}\mathbf{p} - \mathbf{r}_j)\times]$$

$$= \left(\sum_{i=j}^{n} \mathbf{I}_i - m_i[{}^{i}_{0}\mathbf{p}\times][{}^{i}_{0}\mathbf{p}\times]\right) + \hat{m}_j[\mathbf{r}_j\times][\mathbf{r}_j\times] \tag{4}$$

where $\mathbf{I}_i$ is the inertia tensor of link i, and $[\mathbf{a}\times]$ denotes a skew-symmetric matrix representing the vector multiplication $(\mathbf{a}\times)$, i.e., $[\mathbf{a}\times]\mathbf{b} = \mathbf{a} \times \mathbf{b}$.

The acceleration $(\ddot{\mathbf{r}}_j)$ of the center of mass and the angular acceleration $(\boldsymbol{\alpha}_j)$ of the composite body j due to the motion of joint j only (i.e., the other joints are assumed stationary) are

$$\ddot{\mathbf{r}}_j = (K_j^*\mathbf{u}_j \times (\mathbf{r}_j - {}^{j}_{0}\mathbf{s}) + K_j\mathbf{u}_j)\ddot{q}_j \tag{5}$$

$$\boldsymbol{\alpha}_j = K_j^*\mathbf{u}_j\ddot{q}_j \tag{6}$$

where $\mathbf{u}_j$ is the unit vector along joint j, q_j is the displacement of joint j, and

$$K_j^* \equiv (1 - K_j) \equiv \begin{cases} 1, & \text{for rotational joint } j, \\ 0, & \text{for translational joint } j. \end{cases} \tag{7}$$

The inertia force $(\mathbf{f}_{Tj})$ and torque $(\mathbf{t}_{Tj})$ of the composite body j can be obtained using Newton–Euler equations. According to vectorial mechanics, the inertia force and torque can be represented by an equivalent force with an equivalent torque acting at the origin of the base frame, i.e.,

$$\mathbf{f}_{Ej} \equiv \mathbf{f}_{Tj} = -\hat{m}_j\ddot{\mathbf{r}}_j \tag{8}$$

$$\mathbf{t}_{Ej} \equiv \mathbf{t}_{Tj} + \mathbf{r}_j \times \mathbf{f}_{Tj} = -\mathbf{J}_j\alpha_j - \hat{m}_j\mathbf{r}_j \times \ddot{\mathbf{r}}_j \tag{9}$$

Under the situation that only joint j moves and the gravity is neglected, the force and torque exerted on joint m by link m, $m \leq j$, are also an equivalent force and an equivalent torque, but acting on joint m, of the inertia force and torque of the composite body j. The actuator force applied on joint m is to resist the component of the force or torque exerted on joint m by link m along the direction of joint m. Therefore, we get the actuator force of joint m due to $\ddot{q}_j$,

$$\tau_{m,j} = -\mathbf{u}_m \cdot (K_m^*(\mathbf{t}_{Ej} - {}_0^m\mathbf{s} \times \mathbf{f}_{Ej}) + K_m\mathbf{f}_{Ej}), \qquad m \leq j. \tag{10}$$

Recalling (1) while setting $\mathbf{b} = 0$ and all $\ddot{q}_i = 0$, $i \neq j$, we have

$$\tau_{m,j} = H_{mj}\ddot{q}_j, \qquad m \leq j \tag{11}$$

too, where H_{mj} is the (m,j)th-entry of $\mathbf{H}$. Since $\mathbf{H}$ is symmetric, we just need to consider the upper triangular matrix, i.e., $m \leq \boldsymbol{j}$. Substituting (4)–(9) into (10) yields

$$\begin{aligned} H_{mj} = K_m^*K_j^* \Bigg\{\mathbf{u}_m \cdot \Bigg(\Bigg(\sum_{i=j}^{n} \mathbf{I}_i - m_i[{}_0^i\mathbf{p}\times][{}_0^i\mathbf{p}\times]\Bigg) \mathbf{u}_j + (\mathbf{u}_j \times {}_0^j\mathbf{s}) \times (\hat{m}_j\mathbf{r}_j)\Bigg) \\ - (\mathbf{u}_m \times {}_0^m\mathbf{s}) \cdot (\mathbf{u}_j \times ((\hat{m}_j\mathbf{r}_j) - \hat{m}_j{}_0^j\mathbf{s}))\Bigg\} \\ + K_mK_j^* \, \mathbf{u}_m \cdot (\mathbf{u}_j \times ((\hat{m}_j\mathbf{r}_j) - \hat{m}_j{}_0^j\mathbf{s})) \\ + K_m^*K_j\mathbf{u}_j \cdot (\mathbf{u}_m \times (\hat{m}_j\mathbf{r}_j) - \hat{m}_j\mathbf{u}_m \times {}_0^m\mathbf{s}) \\ + K_mK_j\hat{m}_j\mathbf{u}_m \cdot \mathbf{u}_j, \qquad m \leq j. \end{aligned} \tag{12}$$

This is the formulation of the inertia matrix based on the new composite body method. All vectors and the inertia tensors in this formulation are represented with respect to the base frame. Since the result of the formulation is a scalar, it is independent of the choice of the coordinate frame for the representation. Therefore, the transformation from the base frame to another frame for the representations of vectors and inertia tensors in the formulation does not change the result. It can be shown that the representations of vectors and inertia tensors with respect to the local frame E_j can save quite many computations.

For brevity, we denote

$$\hat{\mathbf{J}}_j^{\langle j\rangle} \equiv \sum_{i=j}^{n} \mathbf{I}_i^{\langle j\rangle} - m_i[{}_0^i\mathbf{p}^{\langle j\rangle}\times][{}_0^i\mathbf{p}^{\langle j\rangle}\times] \tag{13}$$

$$\hat{\mathbf{p}}_j^{\langle j\rangle} \equiv \hat{m}_j\mathbf{r}_j^{\langle j\rangle} = \sum_{i=j}^{n} m_i \, {}_0^i\mathbf{p}^{\langle j\rangle} \tag{14}$$

$$\hat{\mathbf{s}}_m^{\langle j\rangle} \equiv \mathbf{u}_m^{\langle j\rangle} \times {}_0^m\mathbf{s}^{\langle j\rangle}, \qquad m \leq j. \tag{15}$$

where superscript "$\langle j \rangle$" denotes the representation of a vector or an inertia tensor with respect to frame E_j. $\hat{\mathbf{J}}_j$ is the inertia tensor of the composite body j about the origin of the base frame in contrast to that about the center of mass of the composite body $\mathbf{J}_j$. $\hat{\mathbf{p}}_j$ is the first moment of the composite body j about the origin of the base frame. Note that

$$\mathbf{a}^{\langle j \rangle} = {}^{j+1}_{j}\mathbf{R}\mathbf{a}^{\langle j+1 \rangle} \tag{16}$$

$$\hat{\mathbf{J}}^{\langle j \rangle}_{j+1} = {}^{j+1}_{j}\mathbf{R}\hat{\mathbf{J}}^{\langle j+1 \rangle}_{j+1}\ {}^{j+1}_{j}\mathbf{R}^T \tag{17}$$

where ${}^{j+1}_{j}\mathbf{R}$ is a 3×3 matrix representing the coordinate transformation from frame E_j to frame E_{j+1}. The proof of (17) is as follows. Suppose $\mathbf{a}^{\langle j+1 \rangle} \equiv \mathbf{A}^{\langle j+1 \rangle}\mathbf{b}^{\langle j+1 \rangle}$, where **a, b** are vectors, **A** is a matrix mapping **b** to **a**. Thus,

$$\mathbf{a}^{\langle j \rangle} = {}^{j+1}_{j}\mathbf{R}\mathbf{A}^{\langle j+1 \rangle}\mathbf{b}^{\langle j+1 \rangle} = {}^{j+1}_{j}\mathbf{R}\mathbf{A}^{\langle j+1 \rangle}\ {}^{j+1}_{j}\mathbf{R}^T({}^{j+1}_{j}\mathbf{R}\mathbf{b}^{\langle j+1 \rangle}) \tag{18}$$

since the coordinate transformation matrix is an orthogonal matrix. This is the so-called similarity transformation.

Let

$$\bar{\mathbf{I}}_i \equiv \mathbf{I}_i - \frac{1}{2}\,\mathrm{tr}[\mathbf{I}_i]\mathbf{E} \tag{19}$$

where **E** is an identity matrix, then

$$\mathbf{I}_i = \bar{\mathbf{I}}_i - \mathrm{tr}[\bar{\mathbf{I}}_i]\mathbf{E} \tag{20}$$

It is easy to show that

$$[\mathbf{a}\times][\mathbf{b}\times] = \mathbf{b}\mathbf{a}^T - \mathrm{tr}[\mathbf{b}\mathbf{a}^T]\mathbf{E} \tag{21}$$

According to (20) and (21), we can compute

$$\bar{\mathbf{J}}^{\langle j \rangle}_j \equiv \sum_{i=j}^{n} \bar{\mathbf{I}}^{\langle j \rangle}_i - m_i\ {}^{i}_{0}\mathbf{p}^{\langle j \rangle}({}^{i}_{0}\mathbf{p}^{\langle j \rangle})^T \tag{22}$$

and then get $\hat{\mathbf{J}}^{\langle j \rangle}_j$ by the relation of

$$\hat{\mathbf{J}}^{\langle j \rangle}_j = \bar{\mathbf{J}}^{\langle j \rangle}_j - \mathrm{tr}[\bar{\mathbf{J}}^{\langle j \rangle}_j]\mathbf{E} \tag{23}$$

Such an arrangement can save a few computations (see the fourth section).

Finally, we establish the following recursive algorithm for the variables in (12)

$$\mathbf{u}^{\langle j \rangle}_m = {}^{j-1}_{j}\mathbf{R}\mathbf{u}^{\langle j-1 \rangle}_m, \qquad m \le j \tag{24}$$

$$ {}^{j}_{0}\mathbf{s}^{\langle j\rangle} = {}^{j-1}_{j}\mathbf{R}({}^{j-1}_{0}\mathbf{s}^{\langle j-1\rangle} + {}_{j-1}^{j}\mathbf{s}^{\langle j-1\rangle}) \tag{25} $$

$$ {}^{j}_{0}\mathbf{p}^{\langle j\rangle} = {}^{j}_{0}\mathbf{s}^{\langle j\rangle} + \mathbf{c}_{j}^{\langle j\rangle} \tag{26} $$

$$ \hat{\mathbf{s}}_{m}^{\langle j\rangle} = {}^{j-1}_{j}\mathbf{R}\hat{\mathbf{s}}_{m}^{\langle j-1\rangle}, \qquad m \le j \tag{27} $$

which are in a forward recursive form, and

$$ \hat{m}_j = m_j + \hat{m}_{j+1} \tag{28} $$

$$ \hat{\mathbf{p}}_{j}^{\langle j\rangle} = m_j \, {}^{j}_{0}\mathbf{p}^{\langle j\rangle} + {}^{j+1}_{j}\mathbf{R}\hat{\mathbf{p}}_{j+1}^{\langle j+1\rangle} \tag{29} $$

$$ \bar{\mathbf{J}}_{j}^{\langle j\rangle} = \bar{\mathbf{I}}_{j}^{\langle j\rangle} - m_j \, {}^{i}_{0}\mathbf{p}^{\langle j\rangle}({}^{j}_{0}\mathbf{p}^{\langle j\rangle})^{T} + {}^{j+1}_{j}\mathbf{R}\bar{\mathbf{J}}_{j+1}^{\langle j+1\rangle} \, {}^{j+1}_{j}\mathbf{R}^{T} \tag{30} $$

which are in a backward recursive form.

FORMULATION OF THE BIAS VECTOR

A formulation for the bias vector based on the same composite body concept is derived in this section. First, we should know the properties of the bias vector. The *m*th component of the bias vector is in the form of

$$ (\mathbf{b})_m = \sum_{j=1}^{n}\sum_{k=1}^{n} D_{mkj}\dot{q}_k\dot{q}_j + \eta_m \tag{31} $$

where η_m contains the gravity, and the first term on the right-hand side is composed of Coriolis and centripetal forces and has the following properties:

$$ D_{mjk} = D_{mkj}, \tag{32} $$

$$ D_{jkj} = D_{jjk} = 0, \qquad j \ge k; \tag{33} $$

$$ D_{jkm} = -D_{mkj}, \qquad m, j \ge k. \tag{34} $$

which can be derived from Christoffel symbol.[6,9] Thus, we are only concerned with D_{mkj}, $k \le j$ and $m < j$; the others can be directly obtained using the above relations (32)–(34). We rewrite (31) as follows,

$$ (\mathbf{b})_m = \sum_{j=1}^{n}\sum_{k=1}^{j-1} 2D_{mkj}\dot{q}_k\dot{q}_j + \sum_{j=1}^{n} D_{mjj}\dot{q}_j\dot{q}_j + \eta_m \tag{35} $$

This equation indicates that the bias vector is influenced by joint velocities in pairs. To derive D_{mkj}, we can just assume that joints k and j, $k < j$, move at constant speed (i.e., $\ddot{q}_k = \ddot{q}_j = 0$) and other joints are kept stationary. Under this situation, the angular velocity and acceleration of the composite body j are, respectively,

$$\boldsymbol{\omega}_j = K_k^*\mathbf{u}_k\dot{q}_k + K_j^*\mathbf{u}_j\dot{q}_j \tag{36}$$

$$\boldsymbol{\alpha}_j = \frac{d\boldsymbol{\omega}_j}{dt} = K_k^*K_j^*(\mathbf{u}_k\dot{q}_k) \times (\mathbf{u}_j\dot{q}_j) \tag{37}$$

In (37), the differential rule for a vector in a rotating frame is applied. The equation for the acceleration of point p fixed in a moving frame is

$$\mathbf{a}_p = \mathbf{a}_0 + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \boldsymbol{\alpha} \times \mathbf{r} + 2\boldsymbol{\omega} \times \mathbf{v}_0 \tag{38}$$

where $\mathbf{a}_0$, $\mathbf{v}_0$ are the acceleration and velocity of the origin of the moving frame; $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$ are the angular velocity and acceleration of the frame, $\mathbf{r}$ is the distance from the origin of the frame to point p. On the right-hand side of (38), the second term is the centripetal acceleration, the third term is the tangential acceleration and the last term is Coriolis acceleration. Applying this equation, we obtain the acceleration of the origin of joint j,

$$\begin{aligned}
\frac{d^2\,{}^{j}_{0}\mathbf{s}}{dt^2} &= \boldsymbol{\omega}_k \times (\boldsymbol{\omega}_k \times ({}^{j}_{0}\mathbf{s} - {}^{k}_{0}\mathbf{s})) + 2\boldsymbol{\omega}_k \times (K_j\mathbf{u}_j\dot{q}_j) \\
&= K_k^*\mathbf{u}_k \times (\mathbf{u}_k \times ({}^{j}_{0}\mathbf{s} - {}^{k}_{0}\mathbf{s}))\dot{q}_k\dot{q}_k + 2K_k^*K_j\mathbf{u}_k \times \mathbf{u}_j\dot{q}_k\dot{q}_j
\end{aligned} \tag{39}$$

and then the acceleration of the center of mass of the composite body j,

$$\begin{aligned}
\ddot{\mathbf{r}}_j &= \frac{d^2\,{}^{j}_{0}\mathbf{s}}{dt^2} + \boldsymbol{\omega}_j \times (\boldsymbol{\omega}_j \times (\mathbf{r}_j - {}^{k}_{0}\mathbf{s})) + \boldsymbol{\alpha}_j \times (\mathbf{r}_j - {}^{k}_{0}\mathbf{s}) \\
&= \frac{d^2\,{}^{j}_{0}\mathbf{s}}{dt^2} + \{K_k^*\mathbf{u}_k \times (\mathbf{u}_k \times (\mathbf{r}_j - {}^{j}_{0}\mathbf{s}))\dot{q}_k\dot{q}_k + K_j^*\mathbf{u}_j \times (\mathbf{u}_j \times (\mathbf{r}_j - {}^{j}_{0}\mathbf{s}))\dot{q}_j\dot{q}_j \\
&\quad + K_k^*K_j^*(\mathbf{u}_k \times (\mathbf{u}_j \times (\mathbf{r}_j - {}^{j}_{0}\mathbf{s})) + \mathbf{u}_j \times (\mathbf{u}_k \times (\mathbf{r}_j - {}^{j}_{0}\mathbf{s})))\dot{q}_k\dot{q}_j\} \\
&\quad + K_k^*K_j^*(\mathbf{u}_k \times \mathbf{u}_j) \times (\mathbf{r}_j - {}^{j}_{0}\mathbf{s})\dot{q}_k\dot{q}_j
\end{aligned} \tag{40}$$

It should be remarked that

$$\boldsymbol{\alpha}_i = 0, \qquad i < j \tag{41}$$

$$\frac{d^2\,{}^{i}_{0}\mathbf{p}}{dt^2} = \begin{cases} 0, & \mathrm{i} < k, \\ K_k^*\mathbf{u}_k \times (\mathbf{u}_k \times ({}^{i}_{0}\mathbf{p} - {}^{k}_{0}\mathbf{s}))\dot{q}_k\dot{q}_k & k \le i < j. \end{cases} \tag{42}$$

For the present case, the bias vector without the gravity term is

$$(\mathbf{b})_{m,(kj)} = 2D_{mkj}\dot{q}_k\dot{q}_j + D_{mkk}\dot{q}_k\dot{q}_k + D_{mjj}\dot{q}_j\dot{q}_j \tag{43}$$

As was mentioned above, we are only concerned with D_{mkj} and D_{mjj}. Therefore, only the terms with $\dot{q}_k\dot{q}_j$ and q_j^2 in (37) and (40) are necessary to be taken in account for the derivation of the bias coefficients. One of Newton–Euler's equations for composite body j in the present case is the same as (8), the other is

$$\begin{aligned}\mathbf{t}_{Tj} &= -\mathbf{J}_j\boldsymbol{\alpha}_j - \boldsymbol{\omega}_j(\mathbf{J}_j\boldsymbol{\omega}_j)\\ &= K_k^*K_j^*(-\mathbf{J}_j(\mathbf{u}_k \times \mathbf{u}_j) - \mathbf{u}_k \times (\mathbf{J}_j\mathbf{u}_j) - \mathbf{u}_j \times (\mathbf{J}_j\mathbf{u}_k))\dot{q}_k\dot{q}_j\\ &\quad - K_k^*\mathbf{u}_k \times (\mathbf{J}_j\mathbf{u}_k)\dot{q}_k\dot{q}_k - K_j^*\mathbf{u}_j \times (\mathbf{J}_j\mathbf{u}_j)\dot{q}_j\dot{q}_j \qquad (44)\end{aligned}$$

Although the inertia forces of links k to $j - 1$ are not zero in the present case, they can be neglected since they have only the terms with $\dot{q}_k^2$. We use (37), (40), and (44) instead of (5), (6), and (9) to repeat the procedure in the previous section, and finally get

$$D_{mkj} = K_k^*\mathbf{u}_k^{\langle j\rangle} \cdot \mathbf{d}_{mj}, \qquad m < j, k \leq j; \qquad (45)$$

$$\begin{aligned}\mathbf{d}_{mj} \equiv{}& K_m^*K_j^*\{-\bar{\mathbf{J}}_j^{\langle j\rangle}(\mathbf{u}_j^{\langle j\rangle} \times \mathbf{u}_m^{\langle j\rangle}) + \hat{\mathbf{s}}_j^{\langle j\rangle} \times (\hat{\mathbf{p}}_j^{\langle j\rangle} \times \mathbf{u}_m^{\langle j\rangle})\\ &+ \hat{\mathbf{s}}_m^{\langle j\rangle} \times (\mathbf{u}_j^{\langle j\rangle} \times (\hat{\mathbf{p}}_j^{\langle j\rangle} - \hat{m}_j\, {}_0^j\mathbf{s}^{\langle j\rangle}))\}\\ &- K_mK_j^*\mathbf{u}_m^{\langle j\rangle} \times (\mathbf{u}_j^{\langle j\rangle} \times (\hat{\mathbf{p}}_j^{\langle j\rangle} - \hat{m}_j\, {}_0^j\mathbf{s}^{\langle j\rangle}))\\ &+ K_m^*K_j\mathbf{u}_j^{\langle j\rangle} \times (\mathbf{u}_m^{\langle j\rangle} \times \hat{\mathbf{p}}_j^{\langle j\rangle} - \hat{m}_j\hat{\mathbf{s}}_m^{\langle j\rangle})\\ &- K_mK_j\hat{m}_j\mathbf{u}_m^{\langle j\rangle} \times \mathbf{u}_j^{\langle j\rangle}, \qquad m < j. \qquad (46)\end{aligned}$$

It follows from (34) that

$$D_{mkj} = -D_{jkm} = -K_k^*\mathbf{u}_k^{\langle m\rangle} \cdot \mathbf{d}_{jm}, \qquad m > j > k. \qquad (47)$$

In the derivation of (45) and (46), the following relations were first substituted into (40) and (44),

$$\begin{aligned}\mathbf{u}_k \times (\mathbf{u}_j \times (\mathbf{r}_j - {}_0^j\mathbf{s})) + \mathbf{u}_j \times (\mathbf{u}_k \times (\mathbf{r}_j - {}_0^j\mathbf{s})) + (\mathbf{u}_k \times \mathbf{u}_j) \times (\mathbf{r}_j - {}_0^j\mathbf{s})\\ = 2\mathbf{u}_k \times (\mathbf{u}_j \times (\mathbf{r}_j - {}_0^j\mathbf{s})) \qquad (48)\end{aligned}$$

$$\mathbf{J}_j(\mathbf{u}_k \times \mathbf{u}_j) + \mathbf{u}_k \times (\mathbf{J}_j\mathbf{u}_j) + \mathbf{u}_j \times (\mathbf{J}_j\mathbf{u}_k) = 2\mathbf{u}_j \times \left(\left(\mathbf{J}_j - \frac{1}{2}\,\mathrm{tr}[\mathbf{J}_j]\mathbf{E}\right)\mathbf{u}_k\right) \quad (49)$$

The final form of (46) is obtained by applying the following further equalities,

$$(\mathbf{u}_m \times \mathbf{r}_j) \times (\mathbf{u}_j \times \mathbf{r}_j) = -\mathbf{r}_j\mathbf{r}_j^T[\mathbf{u}_j\, \times]\mathbf{u}_m \qquad (50)$$

$$\mathbf{J}_j - \frac{1}{2}\,\mathrm{tr}[\mathbf{J}_j]\mathbf{E} = \bar{\mathbf{J}}_j - \hat{m}_j\mathbf{r}_j\mathbf{r}_j^T \qquad (51)$$

$$\mathbf{u}_j \times \left\{\left(\mathbf{J}_j - \frac{1}{2}\,\mathrm{tr}[\mathbf{J}_j]\right)\mathbf{u}_j\right\} = \mathbf{u}_j \times (\mathbf{J}_j\mathbf{u}_j) \qquad (52)$$

Equation (51) follows from (4), (21), (22) and

$$[\mathbf{a}\times][\mathbf{b}\times] - \frac{1}{2}\,\mathrm{tr}[[\mathbf{a}\times][\mathbf{b}\times]]\mathbf{E} = \mathbf{b}\mathbf{a}^T \tag{53}$$

It is worth mentioning that all variables in (46) can also be computed with the recursive algorithm (24)–(30). That means, H_{mj} and d_{mj} are related to the same variables. However, to compute $D_{m_{kj}}$ is still time-consuming. Combining (35), (45), and (47), we can alternatively compute the bias vector without the gravity term in a more efficient form of

$$(\mathbf{b})_m - \eta_m = -\sum_{j=1}^{m-1} (\boldsymbol{\zeta}_j^{\langle m\rangle}\dot{q}_j) \cdot \boldsymbol{d}_{jm} + \sum_{j=m+1}^{n} (\boldsymbol{\zeta}_j^{\langle j\rangle}\dot{q}_j) \cdot \boldsymbol{d}_{mj} \tag{54}$$

where

$$\boldsymbol{\zeta}_j^{\langle j\rangle} \equiv \sum_{k=1}^{j-1} 2K_k^*\mathbf{u}_k^{\langle j\rangle}\dot{q}_k + K_j^*\mathbf{u}_j^{\langle j\rangle}\dot{q}_j. \tag{55}$$

which can be calculated in a forward recursive form of

$$\boldsymbol{\zeta}_{j+1}^{\langle j+1\rangle} = {}_{j+1}^{\;\;j}\mathbf{R}(\boldsymbol{\zeta}_j^{\langle j\rangle} + K_j^*\mathbf{u}_j^{\langle j\rangle}\dot{q}_j) + K_{j+1}^*\mathbf{u}_{j+1}^{\langle j+1\rangle}\dot{q}_{j+1} \tag{56}$$

and

$$\boldsymbol{\zeta}_j^{\langle m\rangle}\dot{q}_j = {}^{m-1}_{\;\;m}\mathbf{R}(\boldsymbol{\zeta}_j^{\langle m-1\rangle}\dot{q}_j), \qquad m > j. \tag{57}$$

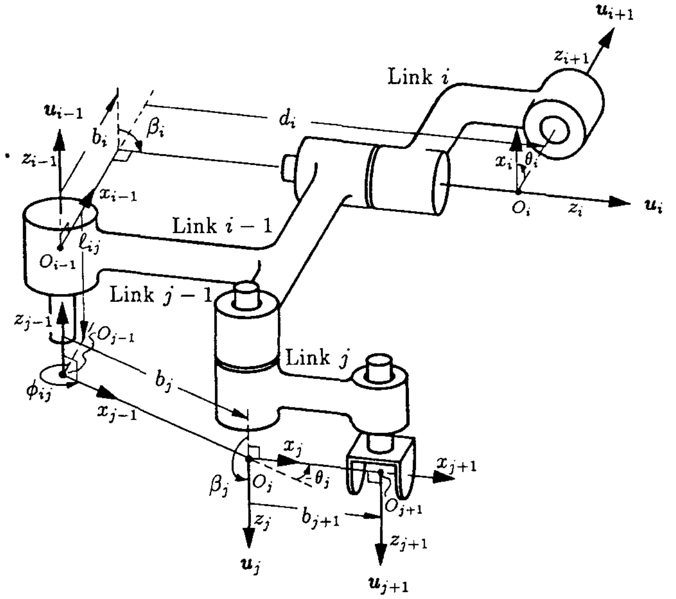
It is apparent that $\boldsymbol{\zeta}_j^{\langle j\rangle} = 0$ for $j < N_{fR}$ if the first $(N_{fR} - 1)$ joints are translational. Therefore, we should keep in mind that $\mathbf{d}_{mj}$ for $j < N_{fR}$ is unnecessary to be calculated in forming the bias vector.

It is much easier to derive the gravity term, η_m, in the bias vector. Assume the manipulator is stationary, i.e., $\ddot{q}_i = \dot{q}_i = 0$ for all i. The forces exerted on the manipulator are gravitational forces only. The gravitational force of the composite body j is $\hat{m}_j\mathbf{g}$ acting at the center of mass, where $\mathbf{g}$ is the gravitational acceleration. The actuator force applied on joint j is to resist the component of the force or torque exerted on joint j by link j along the direction of joint j, i.e.,

$$\eta_j = -K_j^*\mathbf{u}_j^{\langle j\rangle} \cdot ((\mathbf{r}_j^{\langle j\rangle} - {}_0^j\mathbf{s}^{\langle j\rangle}) \times \hat{m}_j\mathbf{g}^{\langle j\rangle}) - K_j\mathbf{u}_j^{\langle j\rangle} \cdot \hat{m}_j\mathbf{g}^{\langle j\rangle} \tag{58}$$

ALGORITHM

The literature[13,15] has shown that the choice of a body-fixed coordinate system plays an essential role in the computation efficiency of kinematic and dynamic problems of manipulators. In the work,[15] the normal driving-axis coordinate system (also called the modified Denavit–Hartenberg notation[16]) was



$$ {}_{i-1}^{\;\;i}\boldsymbol{A} = \begin{bmatrix} {}_{i-1}^{\;\;i}\boldsymbol{R} & {}_{i-1}^{\;\;i}\boldsymbol{s}^{<i-1>} \\ 0\quad 0\quad 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & b_i \\ \cos\beta_i\sin\theta_i & \cos\beta_i\cos\theta_i & -\sin\beta_i & -d_i\sin\beta_i \\ \sin\beta_i\sin\theta_i & \sin\beta_i\cos\theta_i & \cos\beta_i & d_i\cos\beta_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}_{i-1}^{j-1}\boldsymbol{A} = \begin{bmatrix} {}_{i-1}^{j-1}\boldsymbol{R} & {}_{i-1}^{j-1}\boldsymbol{s}^{<i-1>} \\ 0\quad 0\quad 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\phi_{ij} & -\sin\phi_{ij} & 0 & 0 \\ \sin\phi_{ij} & \cos\phi_{ij} & 0 & 0 \\ 0 & 0 & 1 & \ell_{ij} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Figure 2. Illustration of the normal driving-axis coordinate system.

used to reformulate the recursive Newton–Euler Formulation.[1] The resulting formulation is more efficient than the original one based on the Denavit-Hartenberg notation, and made the implementation on an INTEL 8086/8087 microprocessor (5MHz clock) under 17ms for the Stanford manipulator. It is also shown[13] that the reformulated Walker and Orin's composite body method based on the normal driving-axis coordinate system can save 45 multiplications and 28 additions for a general open chain in comparison with the one based on the Denavit–Hartenberg notation. This experience is adopted again to establish an algorithm for the formulations derived above.

In the normal driving-axis coordinate system (Fig. 2), the z-axis of a body-fixed frame is the driving axis of the corresponding link, i.e., $\mathbf{u}_i^{\langle i\rangle} = [0\ 0\ 1]^T$. And the distance from the origin of frame E_{i-1} to frame E_i is ${}_{i-1}^{\;\;i}\mathbf{s}^{\langle i-1\rangle} = [b_i,\ -d_i \sin\beta_i,\ d_i\cos\beta_i]^T$, where b_i, d_i, β_i and θ_i are the parameters of the coordinate system and are shown in Figure 2. Note, $d_i = d_i' + q_i$, $\theta_i = \theta_i'$, if joint i is translational; otherwise $d_i = d_i'$, $\theta_i = \theta_i' + q_i$; i.e., d_i' and θ_i' are the null-position values of d_i and θ_i, respectively.

Since it is recommended to assign the base frame E_0 coincident with frame E_1 in the null-position configuration,[15] the distance between the origins of E_0 and E_1 is then zero. Thus,

$$\hat{\mathbf{s}}_1^{\langle i\rangle} = \mathbf{u}_1^{\langle i\rangle} \times {}_0^1\mathbf{s}^{\langle i\rangle} = 0, \qquad i \geq 1 \tag{59}$$

$${}_0^1\mathbf{p}^{\langle 1\rangle} = \mathbf{c}_1^{\langle 1\rangle} \tag{60}$$

$${}_0^2\mathbf{s}^{\langle 2\rangle} = {}_2^1\mathbf{R}\;{}_1^2\mathbf{s}^{\langle 1\rangle} = \begin{bmatrix} b_2\cos\theta_2 \\ -b_2\sin\theta_2 \\ d_2 \end{bmatrix} \tag{61}$$

The terms, which are constant and can be calculated in advance, are listed in the part "initialization" of the algorithm shown in Figure 3. Examining (12), we find that $\hat{\mathbf{s}}_m^{\langle j\rangle} = \mathbf{u}_m^{\langle j\rangle} \times {}_0^m\mathbf{s}^{\langle j\rangle}$ is just required for $K_m^* = 1$ in forming H_{mj}. If joint N_{fR} is the first rotational joint (i.e., $K_i^* = 0,\ i < N_{fR}$), only the (3,3)th entry of $\hat{\mathbf{J}}_{N_{fR}}^{\langle N_{fR}\rangle}$ is required, and all $\hat{\mathbf{J}}_i^{\langle i\rangle}$, $i < N_{fR}$ are unnecessary since H_{mj} is related to $\hat{\mathbf{J}}_j^{\langle j\rangle}$ only in the case of $K_m^*K_j^* = 1$, $m \leq j$, nor are all $\hat{\mathbf{p}}_i^{\langle i\rangle}$, $i < N_{fR}$. For the case that joint j is a rotational joint, but not the first, we also just need the third column of $\hat{\mathbf{J}}_j^{\langle j\rangle}$ because of $\mathbf{u}_j^{\langle j\rangle} = [0,\ 0,\ 1]^T$. This implies that it is unnecessary to convert $\bar{\mathbf{J}}_j^{\langle j\rangle}$ to $\hat{\mathbf{J}}_j^{\langle j\rangle}$ in the recursive procedure, so that computing $\bar{\mathbf{J}}_j^{\langle j\rangle}$ can save a few of additions. All these conclusions apply to the formulation (46) for $\mathbf{d}_{mj}$, too.

Let

$$\begin{aligned} \mathbf{a}_j &\equiv \hat{\mathbf{J}}_j^{\langle j\rangle}\mathbf{u}_j^{\langle j\rangle} + \hat{\mathbf{s}}_j^{\langle j\rangle} \times \hat{\mathbf{p}}_j^{\langle j\rangle} \\ &= \begin{bmatrix} (\bar{\mathbf{J}}_j^{\langle j\rangle})_{13} \\ (\bar{\mathbf{J}}_j^{\langle j\rangle})_{23} \\ -(\bar{\mathbf{J}}_j^{\langle j\rangle})_{11} - (\bar{\mathbf{J}}_j^{\langle j\rangle})_{22} \end{bmatrix} + \hat{\mathbf{s}}_j^{\langle j\rangle} \times \hat{\mathbf{p}}_j^{\langle j\rangle} \end{aligned} \tag{62}$$

$$\mathbf{e}_j \equiv \mathbf{u}_j^{\langle j\rangle} \times (\hat{\mathbf{p}}_j^{\langle j\rangle} - \hat{m}_j\,{}_0^j\mathbf{s}^{\langle j\rangle}) = \mathbf{u}_j^{\langle j\rangle} \times \hat{\mathbf{p}}_j^{\langle j\rangle} - \hat{m}_j\hat{\mathbf{s}}_j^{\langle j\rangle} \tag{63}$$

$$\mathbf{B}_j \equiv -\bar{\mathbf{J}}_j^{\langle j\rangle}[\mathbf{u}_j^{\langle j\rangle}\times] + [\hat{\mathbf{s}}_j^{\langle j\rangle}\times][\hat{\mathbf{p}}_j^{\langle j\rangle}\times] \tag{64}$$

where $(\mathbf{J})_{ij}$ denotes the $(i,\ j)$th-entry of $\mathbf{J}$. The formulations (12), (46), and (58) can be rewritten as

$$\begin{aligned} H_{mj} = {} & K_m^*K_j^*(\mathbf{a}_j \cdot \mathbf{u}_m^{\langle j\rangle} - \hat{\mathbf{s}}_m^{\langle j\rangle} \cdot \mathbf{e}_j) \\ & + K_mK_j^*\mathbf{e}_j \cdot \mathbf{u}_m^{\langle j\rangle} \\ & + K_m^*K_j((\mathbf{u}_m^{\langle j\rangle} \times \hat{\mathbf{p}}_j^{\langle j\rangle})_z - \hat{m}_j(\hat{\mathbf{s}}_m^{\langle j\rangle})_z) \\ & + K_mK_j\hat{m}_j(\mathbf{u}_m^{\langle j\rangle})_z \qquad m \leq j \end{aligned} \tag{65}$$

$$\begin{aligned} \mathbf{d}_{mj} = {} & K_m^*K_j^*(\mathbf{B}_j\mathbf{u}_m^{\langle j\rangle} + \hat{\mathbf{s}}_m^{\langle j\rangle} \times \mathbf{e}_j) \\ & + K_mK_j^*\mathbf{e}_j \times \mathbf{u}_m^{\langle j\rangle} \end{aligned}$$

```
INITIALIZATION:
  $\hat{m}_n = m_n$
  DO $i = n$ to 1 with increment of $-1$
    IF $(i < n)$, $\hat{m}_i = \hat{m}_{i+1} + m_i$
    $\tilde{I}_i^{<i>} = I_i^{<i>} - \frac{1}{2}\mathrm{tr}[I_i^{<i>}]E$
  END DO $i$
  $N_{fR} \equiv$ No. of the First Rotational Joint
  IF (no Rotational Joints), $N_{fR} = n + 1$
  ${}^{1}_{0}s^{<1>} = 0$
  ${}^{1}_{0}p^{<1>} = c_1^{<1>}$
  $\hat{s}_1^{<1>} = 0$
  $p_1^C \equiv m_1\, {}^{1}_{0}p^{<1>}$
  $(\hat{J}_1^C)_{33} \equiv (I_1^{<1>})_{33} + m_1(({}^{1}_{0}p^{<1>})_x{}^2 + ({}^{1}_{0}p^{<1>})_y{}^2)$
+--------------------------------------------+†
| IF $(N_{fR} > 1)$ THEN                     |
| DO $j = 1$ to $(N_{fR} - 1)$               |
|   $g^{<j>} = {}^{j-1}_{j}R\, g^{<j-1>}$    |
|   $\eta_j = \hat{m}_j(g^{<j>})_z$          |
| END DO $j$                                 |
| END IF $(N_{fR} > 1)$                      |
+--------------------------------------------+
  RETURN
  END
START:                                                              Number of         Times
          ■ FORM ${}^{i-1}_{i}R$, ${}_{i-1}^{\;\;i}s^{<i-1>}$, $g^{<i>}$ ■                Operations
+-------------------------------------------------------------------------------------------------+
| DO $i = 1$ to $n$                                                                               |
|   ${}^{i-1}_{i}R = \begin{bmatrix} \cos\theta_i & \cos\beta_i\sin\theta_i & \sin\beta_i\sin\theta_i \\ -\sin\theta_i & \cos\beta_i\cos\theta_i & \sin\beta_i\cos\theta_i \\ 0 & -\sin\beta_i & \cos\beta_i \end{bmatrix}$      $K_i^*$: 4M 1s/c      $n$ |
|   IF $(i > 1)$, ${}_{i-1}^{\;\;i}s^{<i-1>} = [b_i \;\; -d_i\sin\beta_i \;\; d_i\cos\beta_i]^T$      $K_i$: 2M      $n - 1$ |
|   IF $(i \geq N_{fR})$, $g^{<i>} = {}^{i-1}_{i}R\, g^{<i-1>}$      8M 4A      $n - N_{fR} + 1$ |
| END DO $i$                                                                                      |
+-------------------------------------------------------------------------------------------------+
          ■ FOR THE CASE OF $n = 1$ ■
  IF $(n = 1)$ THEN
    $H_{11} = K_1^*(\hat{J}_1^C)_{33} + K_1\hat{m}_1$                     0
+-------------------------------------------------------------------------------------------------+
|   $(b)_1 = -K_1^*(m_1 c_1^{<1>} \times g^{<1>})_z - K_1\eta_1$          $K_1^*$: 2M 1A      1 for $n = 1$ |
+-------------------------------------------------------------------------------------------------+
    RETURN
  END IF $(n = 1)$
          ■ FORM ${}^{i}_{0}s^{<i>}$, ${}^{i}_{0}p^{<i>}$, $\hat{s}_i^{<i>}$ ■
  ${}^{2}_{0}s^{<2>} = [b_2\cos\theta_2 \;\; -b_2\sin\theta_2 \;\; d_2]^T$      $K_i^*$: 2M      1 for $n \geq 2$
  DO $i = 2$ to $n$
    IF $(i > 2)$, ${}^{i}_{0}s^{<i>} = {}^{i-1}_{i}R({}^{i-1}_{0}s^{<i-1>} + {}_{i-1}^{\;\;i}s^{<i-1>})$      8M 7A      $n - 2$
    ${}^{i}_{0}p^{<i>} = {}^{i}_{0}s^{<i>} + c_i^{<i>}$      3A      $n - 1$
    $\hat{s}_i^{<i>} = [-({}^{i}_{0}s^{<i>})_y \;\; ({}^{i}_{0}s^{<i>})_x \;\; 0]^T$      0
  END DO $i$
          ■ FORM $u_j^{<i>}$, $\hat{s}_j^{<i>}$ FOR $i > j$ ■
  DO $j = 1$ to $n - 1$
    $u_j^{<j+1>} = {}_{j+1}^{\;\;j}R[0 \;\; 0 \;\; 1]^T$      0
```

---

† The terms enclosed within a box should be excluded when it is to form the inertia matrix only.

Figure 3. Algorithm of the new composite body method.

```
    IF ($j \geq 2$ AND $K_j = 0$), $\hat{s}_j^{<j+1>} =$
                ${}^{\ j}_{j+1}R[-({}^{j}_{0}s^{<j>})_y \ \ ({}^{j}_{0}s^{<j>})_x \ \ 0]^T$     $K_j^*$: 5M 2A     $n-2$
    IF ($j < n-1$) THEN
      DO $i = j+2$ to $n$
        $u_j^{<i>} = {}^{i-1}_{\ \ i}R\, u_j^{<i-1>}$                                8M 4A     $(n-2)(n-1)/2$
        IF ($j \geq 2$ AND $K_j = 0$), $\hat{s}_j^{<i>} = {}^{i-1}_{\ \ i}R\, \hat{s}_j^{<i-1>}$    $K_j^*$: 8M 4A     $(n-3)(n-2)/2$
      END DO $i$
    END IF ($j < n-1$)
  END DO $j$
          ■ FORM $\hat{p}_i^{<i>}$, $\bar{J}_i^{<i>}$ ■
  IF ($N_{fR} > n$), GO TO "calculating $H_{mj}$ and $d_{mj}$"
  $\hat{p}_n^{<n>} = m_n\, {}^{n}_{0}p^{<n>}$                                          3M     1 for $N_{fR} \leq n$
  IF ($n > N_{fR}$), $\bar{J}_n^{<n>} \doteq \tilde{I}_n^{<n>} - (m_n\, {}^{n}_{0}p^{<n>})({}^{n}_{0}p^{<n>})^T$     6M 6A     1 for $N_{fR} < n$
  IF ($n = N_{fR}$), $(\hat{J}_n^{<n>})_{33} = (I_n^{<n>})_{33} + (m_n\, {}^{n}_{0}p^{<n>})_x(p^{<n>})_x$
                                    $+(m_n\, {}^{n}_{0}p^{<n>})_y(p^{<n>})_y$     2M 2A     1 for $N_{fR} = n$
  IF ($N_{fR} = n$), GO TO "calculating $H_{mj}$ and $d_{mj}$"
  DO $i = n-1$ to $N_{fR}$ with increment of $-1$
    IF ($i \neq 1$), $\hat{p}_i^{<i>} = m_i\, {}^{i}_{0}p^{<i>} + {}^{i+1}_{\ \ i}R\, \hat{p}_{i+1}^{<i+1>}$     11M 7A     $\begin{cases} n-2 \text{ for } N_{fR} = 1 \\ n - N_{fR} \text{ else} \end{cases}$
    IF ($i = 1$), $\hat{p}_i^{<i>} = p_1^C + {}^{i+1}_{\ \ i}R\, \hat{p}_{i+1}^{<i+1>}$     8M 7A     1 for $N_{fR} = 1$
    IF ($i > N_{fR}$), $\bar{J}_i^{<i>} = \tilde{I}_i^{<i>} - (m_i\, {}^{i}_{0}p^{<i>})({}^{i}_{0}p^{<i>})^T$
                          $+ {}^{i+1}_{\ \ i}R\, \bar{J}_{i+1}^{<i+1>}\, {}^{i+1}_{\ \ i}R^T$     43M 31A     $n - N_{fR} - 1$
    IF ($i = N_{fR}$) THEN
      $\hat{J}_{i+1}^{<i+1>} = \bar{J}_{i+1}^{<i+1>} - \mathrm{tr}[\bar{J}_{i+1}^{<i+1>}]E$     3A     1 for $1 \leq N_{fR} < n$
      IF ($i \neq 1$), $(\hat{J}_i^{<i>})_{33} = (I_i^{<i>})_{33} + (m_i\, {}^{i}_{0}p^{<i>})_x({}^{i}_{0}p^{<i>})_x$
            $+(m_i\, {}^{i}_{0}p^{<i>})_y({}^{i}_{0}p^{<i>})_y + ({}^{i+1}_{\ \ i}R\, \hat{J}_{i+1}^{<i+1>}\, {}^{i+1}_{\ \ i}R^T)_{33}$     14M 11A     1 for $1 < N_{fR} < n$
      IF ($i = 1$), $(\hat{J}_i^{<i>})_{33} = (\hat{J}_1^C)_{33} + ({}^{i+1}_{\ \ i}R\, \hat{J}_{i+1}^{<i+1>}\, {}^{i+1}_{\ \ i}R^T)_{33}$     12M 9A     1 for $N_{fR} = 1$
    END IF ($i = N_{fR}$)
  END DO $i$
          ■ CALCULATE $H_{mj}$ AND $d_{mj}$ ■
  $H_{11} = K_1^*(\hat{J}_1^{<1>})_{33} + K_1 \hat{m}_1$                                  0
  DO $j = 2$ to $n$
    IF ($K_j = 0$) THEN
      $e_j \equiv \begin{bmatrix} -(\hat{p}_j^{<j>})_y - \hat{m}_j(\hat{s}_j^{<j>})_x \\ (\hat{p}_j^{<j>})_x - \hat{m}_j(\hat{s}_j^{<j>})_y \\ 0 \end{bmatrix}$     $K_j^*$: 2M 2A     $n-1$
      IF ($j > N_{fR}$) THEN
        $A = \hat{s}_j^{<j>}(\hat{p}_j^{<j>})^T$                                  $K_j^*$: 6M     $n - N_{fR}$
        $B \equiv \begin{bmatrix} -(\bar{J}_j^{<j>})_{12} & (\bar{J}_j^{<j>})_{11} & 0 \\ -(\bar{J}_j^{<j>})_{22} & (\bar{J}_j^{<j>})_{21} & 0 \\ -(\bar{J}_j^{<j>})_{32} & (\bar{J}_j^{<j>})_{31} & 0 \end{bmatrix}$
            $+ \begin{bmatrix} -(A)_{22} & (A)_{21} & 0 \\ (A)_{12} & -(A)_{11} & 0 \\ (A)_{13} & (A)_{23} & -(A)_{22} - (A)_{11} \end{bmatrix}$     $K_j^*$: 7A     $n - N_{fR}$
```

Figure 3. continued

```
      $\boldsymbol{a}_j \equiv \begin{bmatrix} (\bar{\boldsymbol{J}}_j^{<j>})_{13} \\ (\bar{\boldsymbol{J}}_j^{<j>})_{23} \\ -(\bar{\boldsymbol{J}}_j^{<j>})_{11} - (\bar{\boldsymbol{J}}_j^{<j>})_{22} \end{bmatrix} + \begin{bmatrix} (\boldsymbol{A})_{23} \\ -(\boldsymbol{A})_{13} \\ (\boldsymbol{A})_{12} - (\boldsymbol{A})_{21} \end{bmatrix}$      $K_j^*$: 5A      $n - N_{fR}$
    ELSE ($j = N_{fR}$)
      $(\boldsymbol{a}_j)_z \equiv (\hat{\boldsymbol{J}}_j^{<j>})_{33} + (\hat{\boldsymbol{s}}_j^{<j>})_x(\hat{\boldsymbol{p}}_j^{<j>})_y - (\hat{\boldsymbol{s}}_j^{<j>})_y(\hat{\boldsymbol{p}}_j^{<j>})_x$      2M 2A      1 for $2 \le N_{fR} \le n$
    END IF ($j > N_{fR}$)
  END IF ($K_j = 0$)
  $H_{jj} = K_j^*((\boldsymbol{a}_j)_z - \hat{\boldsymbol{s}}_j^{<j>} \cdot e_j) + K_j \hat{m}_j$      $K_j^*$: 2M 2A      $n - 1$
  DO $m = 1$ to $j - 1$
    $H_{mj} = K_m^* K_j^* \boldsymbol{a}_j \cdot \boldsymbol{u}_m^{<j>}$      $K_m^* K_j^*$: 3M 2A
        $+ K_m K_j^* e_j \cdot \boldsymbol{u}_m^{<j>}$      $K_m K_j^*$: 2M 1A      $n(n-1)/2$
        $+ K_m^* K_j (\boldsymbol{u}_m^{<j>} \times \hat{\boldsymbol{p}}_j^{<j>})_z$      $K_m^* K_j$: 2M 1A
        $+ K_m K_j \hat{m}_j (\boldsymbol{u}_m^{<j>})_z$      $K_m K_j$: 1M
    IF ($j \ge N_{fR}$), $\boldsymbol{d}_{mj} = K_m^* K_j^* \boldsymbol{B} \boldsymbol{u}_m^{<j>} + K_m K_j^* e_j \times \boldsymbol{u}_m^{<j>}$      $K_m^* K_j^*$: 7M 4A
        $+ K_m^* K_j \begin{bmatrix} -(\boldsymbol{u}_m^{<j>} \times \hat{\boldsymbol{p}}_j^{<j>})_y \\ (\boldsymbol{u}_m^{<j>} \times \hat{\boldsymbol{p}}_j^{<j>})_x \\ 0 \end{bmatrix} + K_m K_j \hat{m}_j \begin{bmatrix} -(\boldsymbol{u}_m^{<j>})_y \\ (\boldsymbol{u}_m^{<j>})_x \\ 0 \end{bmatrix}$      $K_m K_j^*$: 4M 1A   $K_m^* K_j$: 4M 2A   $K_m K_j$: 2M      $(n - N_{fR} + 1) \times (n + N_{fR} - 2)/2$
    IF ($m \ne 1$) THEN
      $H_{mj} = H_{mj} - K_m^* K_j^* \hat{\boldsymbol{s}}_m^{<j>} \cdot e_j$      $K_m^* K_j^*$: 2M 2A      $(n-1)(n-2)/2$
          $- K_m^* K_j \hat{m}_j (\hat{\boldsymbol{s}}_m^{<j>})_z$      $K_m^* K_j$: 1M 1A
      IF ($j \ge N_{fR}$), $\boldsymbol{d}_{mj} = \boldsymbol{d}_{mj}$      $(n - N_{fR} + 1) \times (n + N_{fR} - 4)/2$
          $+ K_m^* K_j^* \hat{\boldsymbol{s}}_m^{<j>} \times e_j + K_m^* K_j \hat{m}_j \begin{bmatrix} (\hat{\boldsymbol{s}}_m^{<j>})_y \\ -(\hat{\boldsymbol{s}}_m^{<j>})_x \\ 0 \end{bmatrix}$      $K_m^* K_j^*$: 4M 4A   $K_m^* K_j$: 2M 2A      $+\begin{cases} 1 \text{ for } N_{fR} = 1 \\ 0 \text{ for } N_{fR} \ne 1 \end{cases}$
    END IF ($m \ne 1$)
  END DO $m$
END DO $j$
```

```
        ■ CALCULATE $\eta_j$ ■
IF ($N_{fR} \le n$) THEN
DO $j = N_{fR}$ to $n$
    IF ($j = 1$), $e_1 = \boldsymbol{u}_1^{<1>} \times \hat{\boldsymbol{p}}_1^{<1>} - \hat{m}_1 \hat{\boldsymbol{s}}_1^{<1>}$      2M 2A      1 for $N_{fR} = 1$
    $\eta_j = -K_j^* e_j \cdot \boldsymbol{g}^{<j>}$      $K_j^*$: 2M 1A      $n - N_{fR} + 1$
        $- K_j \hat{m}_j (\boldsymbol{g}^{<j>})_z$      $K_j$: 1M
END DO $j$
END IF ($N_{fR} \le n$)
        ■ CALCULATE $\zeta_j^{<j>}$ AND $\hat{\zeta}_j^{<j>} \equiv \zeta_j^{<j>} \dot{q}_j$ ■
IF ($N_{fR} = n + 1$), GO TO "calculating $(\boldsymbol{b})_m$"
$\zeta_{N_{fR}}^{<N_{fR}>} = \boldsymbol{u}_{N_{fR}}^{<N_{fR}>} \dot{q}_{N_{fR}}$      0
$(\hat{\zeta}_{N_{fR}}^{<N_{fR}>})_z = (\zeta_{N_{fR}}^{<N_{fR}>})_z \dot{q}_{N_{fR}}$      1M      1 for $N_{fR} \le n$
IF ($N_{fR} = n$), GO TO "calculating $(\boldsymbol{b})_m$"
$\zeta_{N_{fR}+1}^{<N_{fR}+1>} = {}^{N_{fR}}_{N_{fR}+1}\boldsymbol{R}(\zeta_{N_{fR}}^{<N_{fR}>} + \zeta_{N_{fR}}^{<N_{fR}>})$      $K_{N_{fR}}^*$: 3M 1A      1 for $N_{fR} < n$
    $+ K_{N_{fR}+1}^* \boldsymbol{u}_{N_{fR}+1}^{<N_{fR}+1>} \dot{q}_{N_{fR}+1}$      $K_{N_{fR}+1}^*$: 1A
$\hat{\zeta}_{N_{fR}+1}^{<N_{fR}+1>} = \zeta_{N_{fR}+1}^{<N_{fR}+1>} \dot{q}_{N_{fR}+1}$      3M      1 for $N_{fR} < n$
```

Figure 3. continued

```
IF ($N_{fR} = n - 1$),  GO TO "calculating $\hat{\zeta}_j^{<m>}$"
DO $j = N_{fR} + 2$ to $n$
   $\zeta_j^{<j>} = {}^{j-1}_{j}R(\zeta_{j-1}^{<j-1>} + K_{j-1}^{*}u_{j-1}^{<j-1>}\dot{q}_{j-1})$       $\begin{cases}8M4A\\ K_{j-1}^{*}: 1A\\ K_{j}^{*}: 1A\end{cases}$      $n - N_{fR} - 1$ for $N_{fR} < n - 1$
           $+K_j^{*}u_j^{<j>}\dot{q}_j$
   $\hat{\zeta}_j^{<j>} = \zeta_j^{<j>}\dot{q}_j$                                          3M        $n - N_{fR} - 1$ for $N_{fR} < n - 1$
END DO $j$
        ■ CALCULATE $\hat{\zeta}_j^{<m>}$ ■
$\dot{q}_{N_{fR}}^2 = \dot{q}_{N_{fR}} \times \dot{q}_{N_{fR}}$                                     1M        1 for $N_{fR} < n$
DO $m = N_{fR} + 1$ to $n$
   $\hat{\zeta}_{N_{fR}}^{<m>} = u_{N_{fR}}^{<m>}\dot{q}_{N_{fR}}^2$                                   3M        $n - N_{fR}$
END DO $m$
IF ($N_{fR} = n - 1$),  GO TO "calculating $(b)_m$"
DO $j = N_{fR} + 1$ to $n - 1$
   DO $m = j + 1$ to $n$
      $\hat{\zeta}_j^{<m>} = {}^{m-1}_{m}R\,\hat{\zeta}_j^{<m-1>}$                                8M 4A     $(n - N_{fR})\times$ $(n - N_{fR} - 1)/2$
   END DO $m$
END DO $j$
        ■ CALCULATE $(b)_m$ ■
DO $m = 1$ to $n$
   $(b)_m = \eta_m$                                                0
   IF ($N_{fR} \le n$) THEN
      IF ($N_{fR} < m$),  $(b)_m = (b)_m - \hat{\zeta}_{N_{fR}}^{<m>} \cdot d_{(N_{fR})m}$              3M 3A     $n - N_{fR}$
      IF ($N_{fR} > m$),  $(b)_m = (b)_m + \hat{\zeta}_{N_{fR}}^{<N_{fR}>} \cdot d_{m(N_{fR})}$          1M 1A     $N_{fR} - 1$
   END IF ($N_{fR} \le n$)
   IF ($N_{fR} < n$) THEN
      DO $j = N_{fR} + 1$ to $n$
         IF ($j < m$),  $(b)_m = (b)_m - \hat{\zeta}_j^{<m>} \cdot d_{jm}$                    3M 3A     $(n - 1)(n - N_{fR})$ for $N_{fR} < n$
         IF ($j > m$),  $(b)_m = (b)_m + \hat{\zeta}_j^{<j>} \cdot d_{mj}$
      END DO $j$
   END IF ($N_{fR} < n$)
END DO $m$
RETURN
END
```

Figure 3. continued

$$+ K_m^{*}K_j \begin{bmatrix} -(\mathbf{u}_m^{\langle j\rangle} \times \hat{\mathbf{p}}_j^{\langle j\rangle})_y + \hat{m}_j(\hat{\mathbf{s}}_m^{\langle j\rangle})_y \\ (\mathbf{u}_m^{\langle j\rangle} \times \hat{\mathbf{p}}_j^{\langle j\rangle})_x - \hat{m}_j(\hat{\mathbf{s}}_m^{\langle j\rangle})_x \\ 0 \end{bmatrix}$$

$$+ K_mK_j\hat{m}_j \begin{bmatrix} -(\mathbf{u}_m^{\langle j\rangle})_y \\ (\mathbf{u}_m^{\langle j\rangle})_x \\ 0 \end{bmatrix}, \qquad m < j \tag{66}$$

$$\eta_m = -K_m^{*}\mathbf{e}_m \cdot \mathbf{g}^{\langle m\rangle} - K_j\hat{m}_m(\mathbf{g}^{\langle m\rangle})_z \tag{67}$$

where $(\cdot)_x$ denotes the x-component and etc.

Using (25)–(30), (54), (56), (57), and (62)–(67), we establish the algorithm for numerically forming the dynamic equations of a general manipulator and show it in Figure 3. It should be noted that the computations of the matrix product of

Table I. Comparison of the efficiency of five algorithms for calculating the inertial matrix.[a]

Method	Number of Rotational Joints $n \geq 2$	2	3	4	5	6	7	8
Walker and Orin[1,b]	M: $11n^2 + 53n - 74$	76	184	314	466	640	836	1054
	A: $8n^2 + 52n - 62$	74	166	274	398	538	694	866
Renaud[6,7,c]	M: $13.5n^2 + 22.5n - 68$	31	121	238	382	553	751	976
	A: $9n^2 + 26n - 64$	24	95	184	291	416	559	720
Lin[13]	M: $(7n^3 + 99n^2 - 214n)/6 + 19$	*23*	*92*	*215*	399	651	978	1387
	A: $n^3 + 13.5n^2 - 28.5n + 14$	*19*	*77*	*180*	334	545	819	1162
Fijany and Bejczy[11,d]	M: $4.5n^2 + 115.5n - 175$	74	212	359	515	680	854	1037
	A: $4n^2 + 88n - 134$	58	166	282	406	538	678	826
Present	M: $10.5n^2 + 38.5n - 85$	34	125	237	*370*	*524*	*699*	*895*
	A: $6n^2 + 39n - 70$	32	101	182	*275*	*380*	*497*	*626*

[a] M: Multiplication, A: Addition/Subtraction.

[b] The estimate of the required computations is based on the reformulated one in the work;[13] but the estimated computations herein are $2(3\text{M} + 3\text{A})(n - 1)$ less than those in the work[13] since $\mathbf{b}\mathbf{a}^T - (\mathbf{a} \cdot \mathbf{b})\mathbf{E} = [\mathbf{a}\times][\mathbf{b}\times]$ was not recognized.

[c] The computational complexity is reestimated in Appendix.

[d] The other one[12] is somewhat less efficient.

${}^{i-1}_{\ i}\mathbf{R}$ with a general vector can be reduced to 8M 4A instead of 8M 5A because the (1,2)th, (1,3)th, (2,2)th, and (2,3)th entries of ${}^{i-1}_{\ i}\mathbf{R}$ can be factorized, where "M" denotes multiplications, "A" does additions. The required computations for a manipulator with n rotational joints are $(23n^2 + 44n - 82)$M + $(15n^2 + 40n - 70)$A. If we delete the terms enclosed with a rectangle in Figure 3, it turns out to be an algorithm only for the inertia matrix while the coordinate transformation matrices are assumed to be calculated in some other method for the bias vector. The operations for the inertia matrix are then $(10.5n^2 + 38.5n - 85)$M + $(6n^2 + 39n - 70)$A. This implies that the algorithm takes $(12.5n^2 + 5.5n + 3)$M + $(9n^2 + n)$A more operations if, in addition to the inertia matrix, the bias vector is also computed with the new composite body formulation. The algorithm is verified by a FORTRAN program, whose numerical results are found the same as those of other methods.

A comparison of the efficiency of this algorithm with the others in numerically forming the inertia matrix is shown in Table I. It is apparent that the present method is the most efficient for computing the inertia matrix of a manipulator with five or more joints, whereas the algorithm described in the work[13] is preferable for a manipulator with 4 or less joints. The present method is also superior to the most efficient algorithms of the recursive Newton–Euler formulation[3,4] in numerically forming the bias vector for a manipulator with six or less joints, which can be seen from Table II.

CONCLUSION

A new efficient composite body method has been derived. It is found that the algorithm of this method to numerically form the inertia matrix is more efficient

Table II. Comparison of the efficiency of three algorithms for calculating the bias vector.[a]

Method	Number of Rotational Joints, $n \geq 2$	2	3	4	5	6	7	8
Khalil and Kleinfinger[3]	M: $101n - 129$	73	174	275	376	477	578	679
	A: $90n - 118$	62	152	242	332	422	512	602
Balafoutis et al.[4]	M: $93n - 69$	117	210	303	396	489	*582*	*675*
	A: $81n - 66$	96	177	258	339	420	*501*	*582*
Present	M: $12.5n^2 + 5.5n + 3$	*64*	*132*	*225*	*343*	*486*	654	847
	A: $9n^2 + n$	*38*	*84*	*148*	*230*	*330*	448	584

[a] M: Multiplication, A: Addition/Subtraction.

than the earlier works for a manipulator with five or more joints. For a manipulator with six or less joints, it is also recommended to use the present method to compute the bias vector. The secret of the present method lies in that all points are referred to the base frame. Since the distances are all measured from the same point (the origin of the base frame), the coefficients in second-order part (n^2) of the required computations are reduced in comparison with the other types of composite body methods.

We have also tried to reformulate the new composite body formulations by changing the reference point from the origin of the base frame to that of any body-fixed frame. However, the required computations increase. Someone would suggest to apply the concept of the identical reference point to Renaud's formulation. Unfortunately, the invariant property of the inertia parameters in Renaud's formulation will be destroyed.

To accomplish the dynamic simulation, we still need a linear equation solver and an integration technique. According to the numerical experiment,[13] it is preferable to use the LDL^T decomposition to solve the linear equations and to apply the fourth order Adams–Bashforth integration method for the dynamic simulation of manipulators.

Kasahara et al.[17] have applied the parallel computation process to Walker and Orin's method.[5] The parallel computation process can also apply to the present method.

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APPENDIX: ALGORITHM OF RENAUD'S FORMULATION

Renaud's formulation for generating the inertia matrix of a manipulator with only rotational joints is presented in references 6 and 7. The extension to a general manipulator is derived by Burdick.[9] Vukobratovic et al.[10] extended Renaud's formulation to the bias vector for a manipulator with only rotational joints, and proposed an algorithm of Renaud's formulation based on the modified Denavit–Hartenberg notation for numerically forming the inertia matrix and the bias vector. However, the distance between the origins of two adjacent frames was mistaken to ${}_{i-1}^{\ \ i}s^{\langle i-1\rangle} = [b_i,\ 0,\ d_i]^T$ (cf. (5.2) in the work[10] and the fourth section of this article). The required computations were, thereafter, underestimated. For the purpose of comparing the efficiency of several methods for numerically forming the inertia matrix, we re-establish an algorithm of Renaud's formulation in a natural programming language in Figure A1.

```
INITIALIZATION:
  $\hat{m}_n = m_n$
  $\hat{c}_n^{<n>} = m_n c_n^{<n>}$
  $\hat{J}_n^{<n>} = I_n^{<n>} - m_n[c_n^{<n>}\times]\,[c_n^{<n>}\times]$
  DO $i = n-1$ to 1 with increment of $-1$
    $\hat{m}_i = \hat{m}_{i+1} + m_i$
    $\tilde{c}_i^{<i>} = m_i c_i^{<i>} + K^*_{i+1}\hat{m}_{i+1}\,{}^{i+1}_{\ \ i}s^{<i>}$
    $\tilde{J}_i^{<i>} = I_i^{<i>} - m_i[c_i^{<i>}\times]\,[c_i^{<i>}\times] - K^*_{i+1}\hat{m}_{i+1}[{}^{i+1}_{\ \ i}s^{<i>}\times]\,[{}^{i+1}_{\ \ i}s^{<i>}\times]$
    $({}^{i+1}_{\ \ i}\hat{s}^{<i+1>})_z = -b_{i+1}\sin\beta_{i+1}$
  END DO $i$
  $N_{fR} \equiv$ No. of the First Rotational Joint
  IF (no Rotational Joints), $N_{fR} = n+1$
  RETURN
  END

START:                                                              Number of        Times
                                                                    Operations
          ■ FOR THE CASE OF $n = 1$ ■
  IF ($n = 1$) THEN
    $H_{11} = K_1^*(\tilde{J}_1^{<1>})_{33} + K_1\hat{m}_1$                         0
    RETURN
  END IF ($n = 1$)
          ■ FORM $u_j^{<i>}$, ${}^{i}_{j}\hat{s}^{<i>}$ FOR $i > j$ ■
  DO $j = 1$ to $n-1$
    $u_j^{<j+1>} = {}_{j+1}^{\ \ \ j}R[0\ \ 0\ \ 1]^T$                               0
    IF ($K_j = 0$), ${}^{j+1}_{\ \ j}\hat{s}^{<j+1>} =$
              ${}_{j+1}^{\ \ \ j}R[-({}^{j+1}_{\ \ j}s^{<j>})_y\ \ ({}^{j+1}_{\ \ j}s^{<j>})_x\ \ 0]^T$      $K_j^*$: 4M 2A       $n-1$
    IF ($j < n-1$) THEN
      DO $i = j+2$ to $n$
        $u_j^{<i>} = {}^{i-1}_{\ \ i}R\,u_j^{<i-1>}$                             8M 4A          $(n-2)(n-1)/2$
        IF ($K_j = 0$), ${}^{i}_{j}\hat{s}^{<i>} = {}^{i-1}_{\ \ i}R\,({}^{i-1}_{\ \ j}\hat{s}^{<i-1>}$
                       $+u_j^{<i-1>}\times\,{}_{i-1}^{\ \ \ i}s^{<i-1>})$            $K_j^*$: 14M 10A     $(n-2)(n-1)/2$
      END DO $i$
    END IF ($j < n-1$)
  END DO $j$
          ■ FORM $\hat{c}_i^{<i>}$, $\hat{J}_i^{<i>}$ ■
  DO $i = n-1$ to 1 with increment of $-1$
    $\hat{c}_{i+1}^{<i>} = {}^{i+1}_{\ \ i}R\,\hat{c}_{i+1}^{<i+1>}$                        8M 4A          $n-1$
    $\hat{c}_i^{<i>} = \tilde{c}_i^{<i>} + \hat{c}_{i+1}^{<i>} + K_{i+1}\hat{m}_{i+1}\,{}^{i+1}_{\ \ i}s^{<i>}$      {$K^*_{i+1}$: 3A       $n-1$
                                                                    {$K_{i+1}$: 3M 6A
    IF ($i > N_{fR}$), $\hat{J}_i^{<i>} = \tilde{J}_i^{<i>} + {}^{i+1}_{\ \ i}R\,\hat{J}_{i+1}^{<i+1>}\,{}^{i+1}_{\ \ i}R^T$     37M 25A
      $-[{}^{i+1}_{\ \ i}s^{<i>}\times]\,[\hat{c}_{i+1}^{<i>}\times] - [\hat{c}_{i+1}^{<i>}\times]\,[{}^{i+1}_{\ \ i}s^{<i>}\times]$       9M 15A         $n - N_{fR} - 1$
      $-K_{i+1}\hat{m}_{i+1}[{}^{i+1}_{\ \ i}s^{<i>}\times]\,[{}^{i+1}_{\ \ i}s^{<i>}\times]$             $K_{i+1}$: 9M 3A
```

Figure A1. Algorithm of Renaud's formulation for the inertia matrix.

```
    IF $(i = N_{fR})$,  $(\hat{J}_i^{<i>})_{33} = (\tilde{J}_i^{<i>})_{33}$
              $+({}^{i+1}_{i}R\,\hat{J}_{i+1}^{<i+1>}\,{}^{i+1}_{i}R^T)_{33}$                          12M 9A
              $+2\left(({}^{i+1}_{i}s^{<i>})_x(\hat{c}_{i+1}^{<i>})_x + (\hat{c}_{i+1}^{<i>})_y({}^{i+1}_{i}s^{<i>})_y\right)$     2M 2A         1 for $N_{fR} \le n$
              $+K_{i+1}\hat{m}_{i+1}\left(({}^{i+1}_{i}s^{<i>})_x^2 + ({}^{i+1}_{i}s^{<i>})_y^2\right)$             $K_{i+1}$: 3M 1A
    END IF $(i = N_{fR})$
  END DO $i$
          ■ CALCULATE $H_{mj}$ ■
  $H_{11} = K_1^*(\hat{J}_1^{<1>})_{33} + K_1\hat{m}_1$                                      0
  DO $j = 2$ to $n$
      $e_j \equiv [-(\hat{p}_j^{<j>})_y \;\; (\hat{p}_j^{<j>})_x \;\; 0]^T$                                   0
      IF $(j > N_{fR}$ AND $K_j = 0)$,  $a_j \equiv$ [Third Column of $\hat{J}_j^{<j>}$]   0
    $H_{jj} = K_j^*(\hat{J}_j^{<j>})_{33} + K_j\hat{m}_j$                                     0
    DO $m = 1$ to $j - 1$
      $H_{mj}$  $= K_m^*K_j^*\,(a_j \cdot u_m^{<j>} + {}^{j}_{m}\hat{s}^{<j>} \cdot e_j)$                  ⎧$K_m^*K_j^*$: 5M 4A
              $+K_mK_j^*\,e_j \cdot u_m^{<j>}$                                      ⎨$K_mK_j^*$: 2M 1A   $n(n-1)/2$
              $+K_m^*K_j\,(\hat{m}_j({}^{j}_{m}\hat{s}^{<j>})_z - e_j \cdot u_m^{<j>})$                ⎨$K_m^*K_j$: 3M 2A
              $+K_mK_j\,\hat{m}_j(u_m^{<j>})_z$                                     ⎩$K_mK_j$: 1M
    END DO $m$
  END DO $j$
  RETURN
  END
```

Figure A1. continued

The general form of Renaud's formulation is[9]

$$
\begin{aligned}
H_{mj} = {} & K_m^*K_j^*\{\mathbf{u}_m^{\langle j\rangle} \cdot (\hat{\mathbf{J}}_j^{\langle j\rangle}\mathbf{u}_j^{\langle j\rangle}) + {}^{j}_{m}\hat{\mathbf{s}}^{\langle j\rangle} \cdot (\mathbf{u}_j^{\langle j\rangle} \times \hat{\mathbf{c}}_j^{\langle j\rangle})\} \\
& + K_mK_j^*\mathbf{u}_m^{\langle j\rangle} \cdot (\mathbf{u}_j^{\langle j\rangle} \times \hat{\mathbf{c}}_j^{\langle j\rangle}) \\
& + K_m^*K_j\{\hat{m}_j\,{}^{j}_{m}\hat{\mathbf{s}}^{\langle j\rangle} \cdot \mathbf{u}_j^{\langle j\rangle} + \mathbf{u}_m^{\langle j\rangle} \cdot (\hat{\mathbf{c}}_j^{\langle j\rangle} \times \mathbf{u}_j^{\langle j\rangle})\} \\
& + K_mK_j\hat{m}_j\mathbf{u}_m^{\langle j\rangle} \cdot \mathbf{u}_j^{\langle j\rangle}, \qquad m \le j.
\end{aligned}
\tag{A1}
$$

where the notation is identical to that in the above text except that

$$\hat{\mathbf{c}}_j^{\langle j\rangle} \equiv \sum_{i=j}^{n} m_i\,{}^{i}_{j}\mathbf{p}^{\langle j\rangle} \tag{A2}$$

$$\hat{\mathbf{J}}_j^{\langle j\rangle} \equiv \sum_{i=j}^{n} \mathbf{I}_i^{\langle j\rangle} - m_i[{}^{i}_{j}\mathbf{p}^{\langle j\rangle}\times][{}^{i}_{j}\mathbf{p}^{\langle j\rangle}\times] \tag{A3}$$

$${}^{j}_{m}\hat{\mathbf{s}}^{\langle j\rangle} \equiv \mathbf{u}_m^{\langle j\rangle} \times {}^{j}_{m}\mathbf{s}^{\langle j\rangle} \tag{A4}$$

$\hat{\mathbf{c}}_j$ and $\hat{\mathbf{J}}_j$ are the first moment and the inertia tensor of the composite body j about the origin of frame E_j, respectively.

Defining the constants concerning the inertia property,

$$\bar{\mathbf{c}}_j^{\langle j\rangle} \equiv m_j \mathbf{c}_j^{\langle j\rangle} + K_{j+1}^{*} \hat{m}_{j+1}\, {}^{j+1}_{\ j}\mathbf{s}^{\langle j\rangle} \tag{A5}$$

$$\bar{\mathbf{J}}_j^{\langle j\rangle} \equiv \mathbf{I}_j^{\langle j\rangle} - m_j[\mathbf{c}_j^{\langle j\rangle}\times][\mathbf{c}_j^{\langle j\rangle}\times] - K_{j+1}^{*}\hat{m}_{j+1}[{}^{j+1}_{\ j}\mathbf{s}^{\langle j\rangle}\times][{}^{j+1}_{\ j}\mathbf{s}^{\langle j\rangle}\times] \tag{A6}$$

then we get

$$\hat{\mathbf{c}}_j^{\langle j\rangle} = \bar{\mathbf{c}}_j^{\langle j\rangle} + {}^{j+1}_{\ j}\mathbf{R}\hat{\mathbf{c}}_{j+1}^{\langle j+1\rangle} + K_{j+1}\hat{m}_{j+1}\, {}^{j+1}_{\ j}\mathbf{s}^{\langle j\rangle} \tag{A7}$$

$$\begin{aligned}\hat{\mathbf{J}}_j^{\langle j\rangle} = {} & \bar{\mathbf{J}}_j^{\langle j\rangle} + {}^{j+1}_{\ j}\mathbf{R}\hat{\mathbf{J}}_{j+1}^{\langle j+1\rangle}\, {}^{j+1}_{\ j}\mathbf{R}^T - [{}^{j+1}_{\ j}\mathbf{s}^{\langle j\rangle}\times][\hat{\mathbf{c}}_{j+1}^{\langle j\rangle}\times] - [\hat{\mathbf{c}}_{j+1}^{\langle j\rangle}\times][{}^{j+1}_{\ j}\mathbf{s}^{\langle j\rangle}\times] \\ & -K_{j+1}\hat{m}_{j+1}[{}^{j+1}_{\ j}\mathbf{s}^{\langle j\rangle}\times][{}^{j+1}_{\ j}\mathbf{s}^{\langle j\rangle}\times]\end{aligned} \tag{A8}$$

$${}^{j}_{m}\hat{\mathbf{s}}^{\langle j\rangle} = {}^{j-1}_{\ j}\mathbf{R}({}^{j-1}_{\ m}\hat{\mathbf{s}}^{\langle j-1\rangle} + \mathbf{u}_m^{\langle j-1\rangle} \times {}_{j-1}^{\ j}\mathbf{s}^{\langle j-1\rangle}), \qquad m \leq j \tag{A9}$$

from (A2)–(A4). It should be remarked that most of works[4,12] dealing with Renaud's invariant terms made a mistake that (A5) and (A6) with "1" replacing K_{j+1}^{*} were still seen as invariant to the manipulator motion for translational joint $j + 1$. In fact, ${}^{j+1}_{\ j}\mathbf{s}^{\langle j\rangle}$ for translational joint $j + 1$ varies with d_{j+1}, the joint displacement of translational joint $j + 1$. The required computations of (A1) for a manipulator with n rotational joints are $(13.5n^2 + 22.5n - 68)$M + $(9n^2 + 26n - 64)$A.

The algorithm proposed by Fijany and Bejczy[11,12] uses the recursive forms (A7) and (A8) to calculate the equivalent force ($\mathbf{f}_{Ej,j}$) and couple ($\mathbf{t}_{Ej,j}$), at the origin of frame j, of the inertia force and torque of the composite-body j due to $\ddot{q}_j$,

$$\begin{aligned}-\mathbf{f}_{Ej,j}^{\langle j\rangle} &= \hat{m}_j\ddot{\mathbf{r}}_j^{\langle j\rangle} \\ &= K_j\hat{m}_j\mathbf{u}_j^{\langle j\rangle}\ddot{q}_j + K_j^{*}\mathbf{u}_j^{\langle j\rangle}\ddot{q}_j \times \hat{\mathbf{c}}_j^{\langle j\rangle}\end{aligned} \tag{A10}$$

$$\begin{aligned}-\mathbf{t}_{Ej,j}^{\langle j\rangle} &= \mathbf{J}_j^{\langle j\rangle}\alpha_j^{\langle j\rangle} + \hat{m}_j(\mathbf{r}_j^{\langle j\rangle} - {}^{j}_{0}\mathbf{s}^{\langle j\rangle}) \times \ddot{\mathbf{r}}_j^{\langle j\rangle} \\ &= K_j\hat{\mathbf{c}}_j^{\langle j\rangle} \times \mathbf{u}_j^{\langle j\rangle}\ddot{q}_j + K_j^{*}\hat{\mathbf{J}}_j^{\langle j\rangle}\mathbf{u}_j^{\langle j\rangle}\ddot{q}_j\end{aligned} \tag{A11}$$

The first terms on the right-hand sides of (A10) and (A11) were erroneously ignored in the work[12] (cf. (54) and (55) in reference 12).

The actuator force applied on joint m is to resist the component of the force or torque exerted on joint m by link m along the direction of joint m. The fact that the (m, j)th entry of the inertia matrix is the part of the actuator force of joint m due to $\ddot{q}_j$ results in that

$$H_{jj} = K_j\hat{m}_j + K_j^{*}\mathbf{u}_j^{\langle j\rangle} \cdot (\hat{\mathbf{J}}_j^{\langle j\rangle}\mathbf{u}_j^{\langle j\rangle}) \tag{A12}$$

$$H_{mj} = K_m\mathbf{u}_m^{\langle k\rangle} \cdot \mathbf{f}_{m,j}^{\langle k\rangle} + K_m^{*}\mathbf{u}_m^{\langle k\rangle} \cdot \mathbf{t}_{m,j}^{\langle k\rangle}, \qquad m < j \tag{A13}$$

where

$$\mathbf{f}_{m,j}^{\langle k\rangle} = -{}_{k}^{j}\mathbf{R}\mathbf{f}_{Ej,j}^{j} \tag{A14}$$

$$\mathbf{t}_{m,j}^{\langle k\rangle} = \mathbf{t}_{m+1,j}^{\langle k\rangle} + {}_{m}^{m+1}\mathbf{s}^{\langle k\rangle} \times \mathbf{f}_{m+1,j}^{\langle k\rangle} \tag{A15}$$

Note that $\mathbf{t}_{j,j}^{\langle k\rangle} = \mathbf{t}_{Ej,j}^{\langle k\rangle}$.

Fijany and Bejczy found that the selection of frame $k = 2$[11] or the frame of the end-effector[12] to represent $\mathbf{f}_{m,j}^{\langle k\rangle}$ and $\mathbf{t}_{m,j}^{\langle k\rangle}$ can make the algorithm more efficient than the selection of other frame. However, H_{jj} and H_{mj}, $m < j$ can be expanded explicitly to be (A1), which is more efficient than Fijang and Bejczy's algorithm (Table I).

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