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BNL 30152
AMD 879

CONF-810954--2

A NEW COMPUTATIONAL APPROACH FOR THE LINEARIZED SCALAR POTENTIAL FORMULATION OF THE MAGNETOSTATIC FIELD PROBLEM†

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†The submitted manuscript has been authored under contract DE-AC02-76H00016.

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Abstract

We consider the linearized scalar potential formulation of the magnetostatic field problem in this paper. Our approach involves a reformulation of the continuous problem as a parametric boundary problem. By the introduction of a spherical interface and the use of spherical harmonics, the infinite boundary condition can also be satisfied in the parametric framework. The reformulated problem is discretized by finite element techniques and a discrete parametric problem is solved by conjugate gradient iteration. This approach decouples the problem in that only standard Neumann type elliptic finite element systems on separate bounded domains need be solved. The boundary conditions at infinity and the interface conditions are satisfied during the boundary parametric iteration.

1. Introduction

We describe an algorithm for approximating the solution of the linearized scalar potential formulation of the magnetostatic field problem in this paper. Our approach is novel in the way that the infinite boundary conditions and the interface conditions are imposed. The boundary condition at infinity is handled by the introduction of a spherical interface and the use of spherical harmonics for approximation. The interface conditions are satisfied by boundary parametric techniques. Our method is based on rigorous mathematical analysis in that asymtotic (as the mesh size tends to zero) error estimates and stability results have been proven. In addition, conditioning estimates for the discrete boundary parametric problem have been proven which guarantee rapid iterative convergence rates.

The problem of imposing the infinite boundary conditions has been addressed by many researchers [4, 8, 11, 12, 14]. Approaches include boundary integral and finite element coupling, mesh grading and the introduction of "infinite elements" with a variational formulation. Spherical harmonics are "infinite elements" with approximation properties of infinite order in the class of harmonic functions on the complement of the sphere. Our approach, which uses subspaces of spherical harmonics in a boundary parametric framework, is computationally superior to the variational approach.

The outline of this paper is as follows. In Section 2, we reformulate the magnetostatic field problem in the desired parametric framework. The reformulated problem is discretized in Section 3. Section 4 describes some of the details for iterative solution of the parametric problem. Finally Section 5 gives some numerical results for an annular test problem.

The authors are with Cornell University and Brookhaven National Laboratory, respectively.

The submitted manuscript has been authored under contract DE-AC02-76H00016 with the U.S. Dept. of Energy.

2. The Linearized Magnetostatic Field Problem and its Reformulation

Scalar potential formulation of the magnetostatic field problem leads to elliptic boundary interface problems in two and three dimensions [12]. Typical problems involve an iron region Ω_I submersed in a field produced by current carrying conductors. The magnetostatic field problem is the computation of the fields due to conductors and those due to magnetized iron. The fields due to conductor sources H_s are given by the integral

$$H_s = \frac{1}{4\pi} \int_{R^3} J \times \nabla\left(\frac{1}{|x-x'|}\right) dx' \qquad (2.1)$$

where J is the current density. Although nontrivial for complicated current geometries, the problem of computing the conductor fields can be solved by analytical or numerical integration of (2.1). Thus we shall focus on the problem of finding the fields due to magnetized iron.

The fields due to magnetized iron can be computed from a scalar potential ϕ satisfying the differential interface equations

$$-\nabla\mu\nabla\phi = f_1 \qquad \text{in } \Omega_I \qquad (2.2)$$

$$\Delta\phi = 0 \qquad \text{in } \Omega_I^c \qquad (2.3)$$

$$\mu\frac{\partial\phi^-}{\partial n} - \frac{\partial\phi^+}{\partial n} = g_1 \qquad \text{on } \Gamma_1 \qquad (2.4)$$

$$\phi^- - \phi^+ = g_2 \qquad \text{on } \Gamma_1 \qquad (2.5)$$

$$\phi(x) \to 0 \qquad \text{as } |x| \to \infty \qquad (2.6)$$

Here Γ_1 is the boundary of Ω_I and n is the outward normal on Γ_1. ϕ^-(resp ϕ^+) is the limit of ϕ as Γ_1 is approached from the inside (resp. outside) of Ω_I. The functions f_1, g_1 and g_2 depend upon which potential formulation is being used as well as H_s and its derivatives [12].

The basic theme of the numerical method developed in this paper is the reduction of (2.2 - 2.6) to the solution of Neumann problems on bounded domains. We shall first introduce the Neumann solution operators $T_I f = u$ and $G_I g = w$ solving the boundary value problems

$$\begin{aligned} -\nabla\mu\nabla u &= f \qquad \text{in } \Omega_I \\ \mu\frac{\partial u}{\partial n} + \alpha_1 u &= 0 \qquad \text{on } \Gamma_1 \end{aligned} \qquad (2.7)$$

and

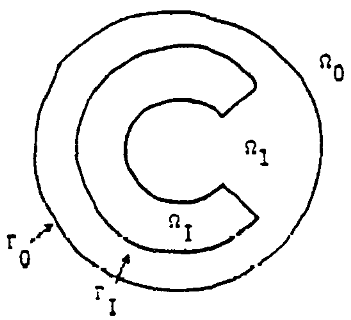
$$-\nabla\mu\nabla w = 0 \qquad \text{in } \Omega_I$$
$$\mu\frac{\partial w}{\partial n} + \alpha_1 w = g \qquad \text{on } \Gamma_1 \tag{2.8}$$

The nonnegative constant α_1 is introduced so that the corresponding form

$$A_\alpha(u,w) \equiv \int_{\Omega_I} \mu\nabla u\cdot\nabla w dx + \int_{\Gamma_1} \alpha_1 uw\, ds$$

is positive definite. Later we shall require that α_1 satisfy additional hypothesis. We note that problems (2.7) and (2.8) have natural boundary conditions and are readily approximated using Galerkin projections with conforming elements.

To handle the boundary condition at infinity we introduce a sphere Ω_s of radius R_0 properly containing the iron region Ω_I. Let Γ_0 denote the boundary of the sphere and set $\Omega_1 = \Omega_s/\Omega_I$ and $\Omega_0 = \Omega_s^c$ where c denotes complement.



We next change variables and reduce the problem to the solution of an interface problem with the desired form. Let

$$u_I = \phi - T_I f_1 - G_I(g_1+\alpha_1 g_2) \qquad \text{on } \Omega_I \tag{2.9}$$

$$u_1 = \phi \qquad \text{on } \Omega_1 \tag{2.10}$$

$$u_0 = \phi \qquad \text{on } \Omega_0 \tag{2.11}$$

then the functions u_I, u_1, and u_0 satisfy the differential interface equations

$$-\nabla\mu\nabla u_I = 0 \qquad \text{in } \Omega_I \tag{2.12}$$

$$\Delta u_1 = 0 \qquad \text{in } \Omega_1 \tag{2.13}$$

$$\Delta u_0 = 0 \qquad \text{in } \Omega_0 \tag{2.14}$$

$$\mu\frac{\partial u_I}{\partial n} + \alpha_1 u_I - \left(\frac{\partial u_1}{\partial n} + \alpha_1 u_1\right) = 0 \text{ on } \Gamma_1 \tag{2.15}$$

$$\frac{\partial u_1}{\partial n} + \alpha_0 u_1 - \left(\frac{\partial u_0}{\partial n} + \alpha_0 u_0\right) = 0 \quad \text{on } \Gamma_0 \tag{2.16}$$

$$u_I - u_1 = \overline{g} \equiv g_1 - T_I f_1 - G_I(g_1 + \alpha_1 g_2) \text{on } \Gamma_I \tag{2.17}$$

$$u_1 - u_0 = 0 \qquad \text{on } \Gamma_0 \tag{2.18}$$

$$u_0(x) \to 0 \qquad \text{as } |x| \to \infty \tag{2.19}$$

Here α_0 is a positive constant less than R_0^{-1}. The constant α_1 is chosen small enough (depending on α_0) so that the form $B_\alpha(\cdot,\cdot)$ to be later defined is positive definite. In practice one sets α_0 and finds that for α_1 below a certain threshold everything works fine.

The reformulation (2.12 - 2.19) leads to a parameterization of the solution (u_I, u_1, u_0) in terms of the parameters

$$\sigma_1 = \mu\frac{\partial u_I}{\partial n} + \alpha_1 u_I \qquad \text{on } \Gamma_1 \tag{2.20}$$

$$\sigma_0 = -\frac{\partial u_0}{\partial n} - \alpha_0 u_0 \qquad \text{on } \Gamma_0 \tag{2.21}$$

We shall next formulate a boundary problem which determines σ_0 and σ_1. First we define solution operators for additional problems. Let G_0 denote the solution operator defined by $G_0\delta = w$ where w is the solution of the boundary value problem

$$\Delta w = 0 \qquad \text{in } \Omega_0$$
$$-\frac{\partial w}{\partial n} - \alpha_0 w = \delta \qquad \text{on } \Gamma_0 \tag{2.22}$$
$$w(x) \to 0 \qquad \text{as } |x| \to \infty$$

We also consider the following two boundary value problems on Ω_1.

$$\Delta w = 0 \qquad \text{in } \Omega_1$$
$$-\frac{\partial w}{\partial n} - \alpha_1 w = \delta_1 \qquad \text{on } \Gamma_1 \tag{2.23}$$
$$\frac{\partial w}{\partial n} + \alpha_0 w = 0 \qquad \text{on } \Gamma_0$$

and

$$\Delta u = 0 \qquad \text{in } \Omega_1$$
$$-\frac{\partial u}{\partial n} - \alpha_1 u = 0 \qquad \text{on } \Gamma_1 \tag{2.24}$$
$$\frac{\partial u}{\partial n} + \alpha_0 u = \delta_0 \qquad \text{on } \Gamma_0$$

The quadratic form $B_\alpha(\cdot,\cdot)$ corresponding to the boundary value problems (2.23) and (2.24) is given by

$$B_\alpha(v_1,v_2) = \int_{\Omega_1} \nabla v_1 \cdot \nabla v_2 dx - \int_{\Gamma_1} \alpha_1 v_1 v_2 ds + \int_{\Gamma_0} \alpha_0 v_1 v_2 ds$$

The solution operator for problem (2.23) (resp (2.24)) shall be denoted by G_2 (resp G_1). The operators G_2 and G_1 are of course defined by $G_2\delta_1 = w$ and $G_1\delta_0 = u$ where w and u are solutions of (2.23) and (2.24) respectively. From the definition of the parameters (2.20 - 2.21) and (2.12 - 2.16), it is obvious that the functions u_I, u_1 and u_0 are given by

$$u_I = G_I\,\sigma_1 \tag{2.25}$$

$$u_1 = -G_2\sigma_1 - G_1\sigma_0 \tag{2.26}$$

$$u_0 = G_0\,\sigma_0 \tag{2.27}$$

Using (2.17) and (2.18) we also have

$$(G_I + G_2)\,\sigma_1 + G_1\sigma_0 = \overline{g} \quad \text{on } \Gamma_1 \tag{2.28}$$

$$G_2\,\sigma_1 + (G_1 + G_0)\,\sigma_0 = 0 \quad \text{on } \Gamma_0. \tag{2.29}$$

The boundary problem (2.28 - 2.29) defines (σ_1, σ_0) from data $\overline{g}$.

Thus we have reformulated the magnetic field problem as a boundary parametric problem. That is we could solve (2.2 - 2.6) by first computing $\overline{g}$ and then solving for the parameters σ_1 and σ_0 from the equations (2.28 - 2.29). Finally ϕ could be reconstructed using (2.9 - 2.11) and (2.25 - 2.27). We base our numerical method on the above approach, replacing the continuous operators T_I, G_I, G_2, G_1 and G_0 by discrete approximations.

3. The Discrete Problem

In this section we shall discretize problem (2.28 - 2.29). First we use finite element approximation for the operators T_I, G_I, G_2 and G_1. G_0 is approximated by using subspaces of spherical harmonics H_N on Γ_0. Finally, a finite element subspace S_k of boundary functions on Γ_1 is defined and the discrete version of (2.28 - 2.29) is posed on $S_k \oplus H_N$.

To define the finite element discretization of T_I, G_I, G_2 and G_1, we shall need subspaces of approximating functions on Ω_I and Ω_1. These subspaces can be constructed by, for example, "triangulating" the respective domains and considering subspaces of piecewise polynomials on the triangles. For more details on the definition and approximation properties of these and other finite element type subspaces see [1, 5]. The finite element approximation subspaces on Ω_I (resp. Ω_1) shall be denoted $S_{I,h}$ (resp $S_{1,h}$). Here h is a mesh parameter which is related to the approximation assumptions on the subspace.

To define the discrete operators approximating T_I, G_I, G_2 and G_1 we use the standard Galerkin projection. For example, applying Green's identity to (2.7) gives that for $u = T_I f$,

$$A_\alpha(u,\chi) = \int_{\Omega_I} f\,\chi\,dx.$$

The discrete operator $T_{I,h}$ approximating T_I is defined by $T_{I,h}f = U$ where U is the unique function in $S_{I,h}$ satisfying

$$A_\alpha(U,\Phi) = \int_{\Omega_I} f\Phi dx \quad \text{for all } \Phi\varepsilon S_{I,h}. \tag{3.1}$$

Similarly, the operator $G_{I,h}$ is defined by $G_{I,h}\sigma_1 = W$, where W satisfies

$$A_\alpha(W,\Phi) = \int_{\Gamma_1} \sigma_1\Phi\, ds \text{ for all } \Phi\varepsilon S_{I,h} \tag{3.2}$$

The definition of $G_{2,h}$ and $G_{1,h}$ are analogous to the definition of $G_{I,h}$ except that one uses $B_\alpha(\cdot,\cdot)$ and $S_{1,h}$. Note that the boundary conditions for T_I, G_I, G_2 and G_1 are natural and thus the subspaces $S_{I,h}$ and $S_{1,h}$ need not satisfy boundary conditions. The solution of any one of the above operators involves the solution of a sparse positive definite system of linear equations. For our method we assume that such systems can be solved economically [7,13].

Next we discretize G_0. The subspace H_N is defined to be the subspace of harmonic polynomials of degree less than or equal to N. For the definition of spherical harmonics and their properties see [6,10]. In two dimensional calculations, H_N is replaced by the subspace of trigonometric polynomials of degree less than or equal to N. On H_N, the operator G_0 can be computed exactly. Indeed if

$$g = \sum_{i=1}^{N} g_i$$

where g_i is a homogeneous harmonic polynomial of degree i, then

$$G_0 g = \sum_{i=1}^{N} (i/R_0 - \alpha_0)^{-1} g_i. \qquad (3.3)$$

Finally, we shall need subspaces of approximating functions $\{\dot{S}_k\}$ on Γ_1. We shall assume $\dot{S}_k$ satisfy inverse assumptions [1,5]. For two dimensional calculations Γ_1 is one dimensional and examples of finite element subspaces on Γ_1 can be constructed by using smooth splines on a uniform grid parameterized by arclength.

Let P_1 and P_0 denote the L^2 projection operators onto $\dot{S}_k$ and H_N respectively. The discrete analogue of (2.28 - 2.29) is given by

$$P_1[(G_{I,h} + G_{2,h})\sigma_{1,h} + G_{1,h}\sigma_{0,h}] = P_1\overline{g}_h \quad \text{on } \Gamma_1 \qquad (3.4)$$

$$P_0[G_{2,h}\sigma_{1,h} + (G_{1,h} + G_0)\sigma_{0,h}] = 0 \text{ on } \Gamma_0 \qquad (3.5)$$

where

$$\overline{g}_h \equiv g_1 - T_{I,h} f_1 - G_{I,h}(g_1 + \alpha_1 g_2).$$

The system (3.3 - 3.4) defines a matrix operator M_h operating on $\dot{S}_k \oplus H_N \subseteq L^2(\Gamma \equiv \Gamma_1 \cup \Gamma_2)$ with range $\dot{S}_k \oplus H_N$. In the next section we shall discuss the solution of (3.4 - 3.5). Of course after $\sigma_{1,h}$ and $\sigma_{0,h}$ are computed, the final approximation to (2.2 - 2.6) is given by

$$\phi_h = G_{I,h}\sigma_{1,h} + T_{I,h} f_1 + G_{I,h}(g_1 + \alpha_1 g_2) \text{ in } \Omega_I$$
$$\phi_h = -G_{2,h}\sigma_{1,h} - G_{1,h}\sigma_{0,h} \qquad \text{in } \Omega_1$$
$$\phi_h = G_0\sigma_{0,h} \qquad \text{in } \Omega_0.$$

We have analytically investigated the stability and convergence of the approximation method described in this paper [3]. Specifically, we have shown that if $h < \varepsilon\{k + N^{-1}\}$ then (3.4 - 3.5) has a unique solution. In fact, we have the stability result: For $\delta_h \in \dot{S}_k \oplus H_N$,

$$C_0 \,|\delta_h|^2_{-1/2} \le \int_\Gamma (M_h\delta_h)\,\delta_h ds \le C_1\,|\delta_h|^2_{-1/2} \qquad (3.6)$$

where $|\ \ |_{-1/2}$ denotes the Sobolev norm of order $-1/2$ on Γ [9] and the constants C_0 and C_1 are independent of k, N and h.

We have also shown that under sufficient smoothness assumptions on the domain Ω_I, the error satisfies

$$\|\phi - \phi_h\|_{H^1(\Omega_I)} + \|\phi - \phi_h\|_{H^1(\Omega_1)} + \|\phi - \phi_h\|_{H^1(\Omega_0)} \le C\{h^{r-1} + k^{\dot{r}-1/2} + \tau^N N^{-j}\} \qquad (3.7)$$

where r and $\dot{r}$ are the approximation orders of $S_{I,h}$ and $\dot{S}_k$ respectively. $\tau \equiv R_s/R_0$ where R_s is the radius of the smallest sphere containing Ω_I with the same center as Ω_s. The constant C in (3.7) depends on certain norms of the data f_1, g_1 and g_2 and on r, $\dot{r}$ and j but is independent of h, k, and N.

4. Iterative Solution of (3.4 - 3.5)

A straightforward calculation using the definitions of the operators defining M_h shows that M_h is symmetric and positive definite on $\dot{S}_k \oplus H_N$. We propose solving (3.4 - 3.5) by conjugate gradient iteration. Note that to apply the conjugate gradient method to (3.4 - 3.5) one need only evaluate the action of M_h on functions δ_h in $\dot{S}_k \oplus H_N$. Thus the matrix for M_h need never be computed.

To evaluate $M_h\delta_h$ we must evaluate three types of operators. First, we must calculate the action of $G_{I,h}$, $G_{2,h}$ and $G_{1,h}$. These operators involve the solution of matrix problems corresponding to standard elliptic finite element problems and are the most time consuming part of the M_h calculation. Next, G_0 can be evaluated by using (3.3). Finally, P_0 and P_1 must be evaluated. Both P_1 and P_0 require the evaluation of L^2 inner products. In addition P_1 requires the solution of another sparse system with fewer unknowns than the systems corresponding to $T_{I,h}$, $G_{I,h}$, $G_{2,h}$ or $G_{1,h}$.

We note that from (3.6) it readily follows that the condition number[1] for the matrix M_h is bounded by $C \max(k^{-1}, N)$. If k^{-1} and N are not too large, conjugate gradient iteration converges fast enough. If k^{-1} or N is large, equivalent well conditioned problems can be defined by introducing discrete boundary operators, see [2] for details.

5. A Numerical Example

As an example, we consider the computation of magnetic fields on an annular region in two dimensions with constant permeability. We assume that the field H_s is given by the Fourier series

$$H_s = \sum_{j=0}^{\infty} c_j \, r^j (\cos j\theta, -\sin j\theta). \qquad (5.1)$$

Then the total scalar potential ϕ can be analytically calculated by Fourier series analysis.

For this geometry, we use subspaces of trigonometric polynomials on both iron-air interfaces and finite element subspaces in the interior of the annulus. Our program runs with second, third or fourth order elements in the interior which are isoparametric images of smooth splines on the periodic strip. Our numerical results illustrated in Table 1 are in basic agreement with the theoretical error bounds of [3]. For our examples, we found that because of the relatively small number of trigonometric polynomials employed, iterative solution of (3.4 - 3.5) converged rapidly without conditioning.

In Table 1, we give the L^2 error in the annular region and on the two boundary interfaces as a function of h and N. The coefficients C_j defining H_s were given by $C_j \equiv e^{-3j}$. The boundary of the annulus was at r=1 and 2 and the permeability was 10.

TABLE 1: L^2 Error for the Annular Calculation

h	Order	N	$e_{L^2}(r=1)$	$e_{L^2}(R=2)$	$e_{L^2}(\Omega_I)$
.05	2	3	2.3×10^{-4}	3.5×10^{-4}	$1. \times 10^{-3}$
.04	2	4	1.5×10^{-4}	2.3×10^{-4}	6.6×10^{-4}
.0666	3	4	9.4×10^{-6}	2.5×10^{-5}	6.6×10^{-5}
.05	3	4	3.9×10^{-6}	$1. \times 10^{-5}$	2.7×10^{-5}

[1] The condition number for a positive definite symmetric matrix M_h is defined to be the ratio of the largest to smallest eigenvalue.

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