A Convexity Measurement for Polygons

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Abstract

Convexity estimators are commonly used in the analysis of shape. In this paper we define and evaluate a new easily computable measure of convexity for polygons. Let P be an arbitrary polygon. If $\mathcal{P}_1(P, \alpha)$ denotes the perimeter in the sense of l_1 metrics of the polygon obtained by the rotation of P by angle α with the origin as the center of the applied rotation, and if $\mathcal{P}_2(R(P, \alpha))$ is the Euclidean perimeter of the minimal rectangle $R(P, \alpha)$ having the edges parallel to coordinate axes which includes such a rotated polygon P, then we show that $\mathcal{C}(P)$ defined as

$$\mathcal{C}(P) \;=\; \min_{\alpha \in [0, 2\pi]} \frac{\mathcal{P}_2(R(P, \alpha))}{\mathcal{P}_1(P, \alpha)}$$

can be used as an estimate for the convexity of P. Several desirable properties of $\mathcal{C}(P)$ are proved, as well.

Keywords: Shape, polygons, convexity, measurement.

1 Introduction

Shape is a crucial component in many areas of scientific analysis [4, 5], with examples including geomorphology [10], powder particle characterisation [6], and biology [2]. This paper is concerned with the measurement of the *convexity* of polygons, which can be considered as one of the basic descriptors of shape [14] and has received some attention over the years [3, 15]. A convexity measure can be used for a variety of applications, for instance shape decomposition [7, 13] which in turn can be used to compute shape similarity and has been applied to object indexing [8].

In general, a planar shape S is said to be convex if it has the following property: If points A and B belong to S then all points from the line segment $[AB]$ belong to S as well. The smallest convex set which includes a shape S is called the *convex hull of S* and it is denoted as $CH(S)$ (see Fig. 1). The previous two definitions suggest the following two possibilities for convexity measurements of planar shapes.

Definition 1 *For a given planar shape S its convexity measure $\mathcal{C}_1(S)$ is defined to be the probability that for randomly chosen points A and B from S all points from the line segment $[AB]$ also belong to S, under the assumption that A and B are chosen uniformly.*
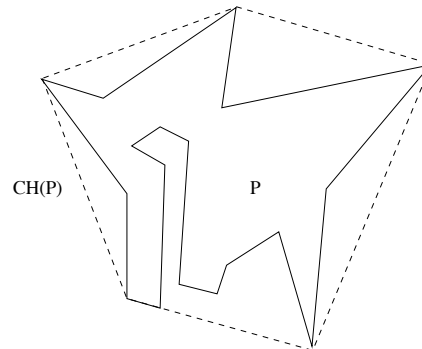
Figure 1: Non convex polygon P and its convex hull $CH(P)$ (dashed line).

Definition 2 *For a given planar shape S, its convexity measure $\mathcal{C}_2(S)$ is defined to be*
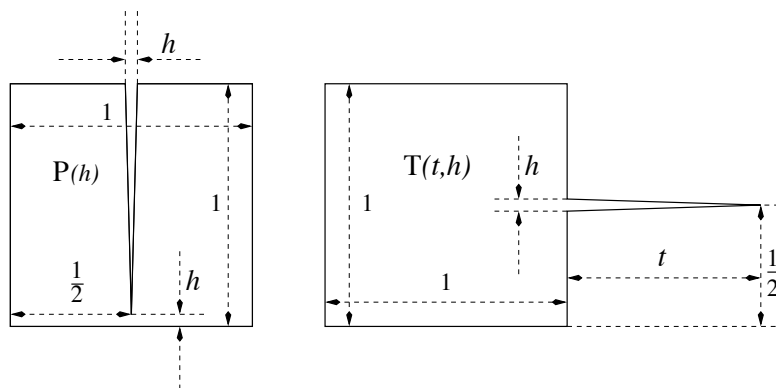
$$\mathcal{C}_2(S) \;=\; \frac{Area(S)}{Area(CH(S))}.$$

Figure 2: $\mathcal{C}_1(P(h)) \to \frac{1}{2}$ as $h \to 0$ is reasonable, but $\mathcal{C}_2(P(h)) \to 1$ as $h \to 0$ is not. $\mathcal{C}_1(T(t,h)) \to 1$ as $h \cdot t \to 0$ is poor for a big t, while it could be acceptable that $\mathcal{C}_2(T(t,h)) \to \frac{2}{2+t}$ with any fixed t and $h \to 0$.

Both convexity measures defined above have the following desirable properties:
i) the convexity measure is a number from $(0,1]$;
ii) the convexity measure of a given shape equals 1 if and only if this shape is convex;
iii) there are shapes whose convexity measure is arbitrary close to 0;
iv) the convexity measure of a shape is invariant under similarity transformations.

But there is also some "bad" properties of the above definitions. The main objection to the Definition 1 is that $\mathcal{C}_1(S)$ is difficult to compute, even if S is a polygon. In practice, the convexity measure given by Definition 2 is the one that is mostly used, and appears in textbooks [14]. $\mathcal{C}_2(S)$ is easy to compute and is very robust with respect to noise. On the other hand, $\mathcal{C}_2(S)$ does not detect huge defects on boundaries of shapes which

have a relatively small impact on the shape areas. Two simple examples are given in Fig. 2. Setting $t = 1 - h$, then for small enough values of h the polygons $P(h)$ and $T(1 - h, h)$ have the same perimeter and almost the same area. Nevertheless, this leads to the following inconsistent $\lim_{h \to 0} \mathcal{C}_2(P(h)) = 1 > \lim_{h \to 0} \mathcal{C}_2(T(1 - h, h)) = 2/3$.

The previously mentioned possibilities for convexity measurement can be understood as "area based" measures, and consequently, they are expected to be robust with respect to boundary defects (caused by a noise for an example). On the other hand, if a convexity measure is based on the shape boundary, then it is likely to be more sensitive to the boundary properties then the measures $\mathcal{C}_1$ and $\mathcal{C}_2$. Such a sensitivity can be a very useful property of the measurement – see again Fig. 2 for an example. The following boundary based convexity measure seems to be a very natural solution.

Definition 3 *If a planar shape S is given and, $\mathcal{P}_2(CH(S))$ and $\mathcal{P}_2(S)$ are the Euclidean perimeters of $CH(S)$ and S, respectively, then convexity measure $\mathcal{C}_3(S)$ is defined as*

$$\mathcal{C}_3(S) = \frac{\mathcal{P}_2(CH(S))}{\mathcal{P}_2(S)}.$$

It seems that $\lim_{h \to 0} \mathcal{C}_3(P(h)) = \frac{2}{3}$ and $\lim_{h \to 0} \mathcal{C}_3(T(1 - h, h)) = \frac{3 + \sqrt{5}}{6}$ are acceptable.

In this paper we define a new (easily computable) convexity measure for polygons which is also computed from the boundaries of measured shapes, which also satisfies the requirements **i)**, **ii)**, **iii)**, and **iv)**, but also has some advantages with respect to the previous convexity measure – particularly in measuring shapes with holes. In Sections 5 and 6 the new measure $\mathcal{C}(S)$ will be compared against $\mathcal{C}_2(S)$ and $\mathcal{C}_3(S)$.

2   Definitions and Denotations

Throughout the paper it will be assumed that all considered shapes are planar bounded compact sets with a non empty interior. A polygon means a compact planar area bounded by a polygonal line.

We will use the following denotations. For a given n-gon P having vertices denoted by $A_0, A_1, \ldots, A_n = A_0$, its edges will be denoted $e_i = [A_{i-1}, A_i]$ for $i = 1, 2, \ldots, n$. $l_2(e)$ is the Euclidean length of an edge $e = [(x_1, y_1), (x_2, y_2)]$, while the length of e according to the l_1 metric is $l_1(e) = |x_1 - x_2| + |y_1 - y_2|$. $\mathcal{P}_2(P)$ will denote the Euclidean perimeter of P, while $\mathcal{P}_1(P)$ will denote the perimeter of P in the sense of the l_1 metric. So, $\mathcal{P}_2(P) = \sum_{e_i \ is \ an \ edge \ of \ P} l_2(e_i)$ and $\mathcal{P}_1(P) = \sum_{e_i \ is \ an \ edge \ of \ P} l_1(e_i)$.

Since isometric polygons do not necessarily have the same perimeter under the l_1 metric, we shall use $\mathcal{P}_1(P, \alpha)$ for the l_1 perimeter of the polygon which is obtained by rotating P by the angle α with the origin as the center of rotation. If the same rotation is applied to the edge e, the l_1 perimeter of the obtained edge will be denoted as $l_1(e, \alpha)$. If the oriented angle between the positively oriented x-axis and the vector $\overrightarrow{A_{i-1}A_i}$ is denoted by ϕ_i ($i = 1, 2, \ldots n$). Obviously $l_1(e_i) = l_2(e_i) \cdot (|\cos \phi_i| + |\sin \phi_i|)$.

The line determined by points A and B will be denoted as $l(A, B)$. The minimal rectangle with edges parallel to the coordinate axes which includes a polygon P will be denoted by $R(P)$. If a polygon P is rotated by an angle α around the origin then $R(P, \alpha)$

denotes the minimal rectangle with edges parallel to coordinate axes which includes the rotated polygon.

We conclude this section with a simple lemma.

Lemma 1 *The inequality* $\mathcal{P}_2(R(P)) \leq \mathcal{P}_1(P)$ *holds for any polygon* P.

Proof. Let A, B, C, and D be four vertices belonging to pairwise different edges of $R(P))$ (some of them may coincide). Let Q be the 4-gon with vertices A, B, C, and D. Then $\mathcal{P}_2(R(P)) = \mathcal{P}_1(R(P)) = \mathcal{P}_1(Q) \leq \mathcal{P}_1(P)$ finishes the proof. ▊

3 A New Convexity Measure

In order to define a new convexity measure for polygons we will exploit the following useful characterization of convex polygons (for another implications of it see [1]).

Theorem 1 *A polygon* P *is convex* **iff** $\mathcal{P}_2(R(P, \alpha)) = \mathcal{P}_1(P, \alpha)$ *for any* $\alpha \in [0, 2\pi]$.

Proof. If a given polygon P is convex then the projections of the edges of P onto the x and y axes exactly covers the boundary of the minimal rectangle whose edges are parallel to the coordinate axes (see Fig. 3 (a)). Since the sum of such projections equals both the l_1 perimeter of the polygon P and the Euclidean perimeter of $R(P)$ the convexity of P implies $\mathcal{P}_1(P) = \mathcal{P}_2(R(P))$ independently on the choice of the coordinate system, or equivalently, $\mathcal{P}_1(P, \alpha) = \mathcal{P}_2(R(P, \alpha))$ for any $\alpha \in [0, 2\pi]$.

On the other hand, if P is not convex, then there exist points A and B from the interior of P such that the line segment $[AB]$ is not completely contained in P. If the line determined by A and B is chosen to be one of coordinate axes, say u, then the projections of edges of P onto the coordinate axis v (perpendicular to u) must overlap (see Fig. 3 (b)). So, $\mathcal{P}_1(P) > \mathcal{P}_2(R(P))$ holds. This completes the proof. ▊
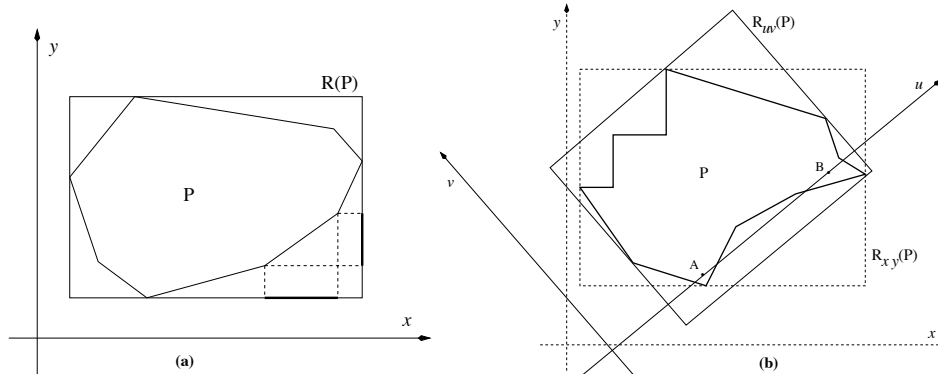

Figure 3: (a) If a given polygon P is convex then $\mathcal{P}_1(P) = \mathcal{P}_2(R(P))$. (b) If x and y are chosen to be the coordinate axes then $\mathcal{P}_2(R_{xy}(P)) = \mathcal{P}_1(P)$. Since P is not convex there is another choice of the coordinate axes, say u and v, such that the strict inequality $\mathcal{P}_2(R_{uv}(P)) < \mathcal{P}_1(P)$ holds.

Theorem 1 gives the basic idea for the polygon convexity measurement described in this paper. In the first stage, the inequality $\mathcal{P}_1(P) \geq \mathcal{P}_2(R(P))$ (see Lemma 1) suggests

that the fraction $\frac{\mathcal{P}_2(R(P))}{\mathcal{P}_1(P)}$ can be used as a convexity measure for polygons since it is a number from $(0, 1]$, it is defined for any polygon P, it can be calculated easily, and for any convex polygon it equals 1. But, on the other hand, this ratio strongly depends of the choice of the coordinate system – which is not a desirable property. Also, this ratio can be equal to 1 for non-convex polygons (see Fig. 3 (b)) which is not acceptable for a convexity measure. The mentioned problems are avoided by the next definition.

Definition 4 *For a polygon P we define its convexity as*

$$\mathcal{C}(P) \;=\; \min_{\alpha \in [0, 2\pi]} \frac{\mathcal{P}_2(R(P, \alpha))}{\mathcal{P}_1(P, \alpha)}.$$

The following theorem summarizes the desirable properties of $\mathcal{C}(P)$.

Theorem 2 *For any polygon P we have:*

- *i) $\mathcal{C}(P)$ is well defined and $\mathcal{C}(P) \in (0, 1]$;*

- *ii) $\mathcal{C}(P) = 1$ if and only if P is convex;*

- *iii) $\inf_{P \in \Pi}(\mathcal{C}(P)) = 0$, where Π denotes the set of all polygons;*

- *iv) $\mathcal{C}(P)$ is invariant under similarity transformations.*

Proof. Since $\frac{\mathcal{P}_2(R(P, \alpha))}{\mathcal{P}_1(P, \alpha)}$ is a continuous function on α (for more details see Section 4) it must reach its minimum on a closed interval $[0, 2\pi]$. So, $\mathcal{C}(P)$ is well defined. Since $\mathcal{C}(P) > 0$ is trivial and because $\mathcal{C}(P) \leq 1$ follows from Lemma 1, item *i)* is proved.
Item *ii)* is a direct consequence of Theorem 1.
To prove item *iii)* we consider the polygon P_n from Fig. 4. Easily, $\quad 0 \leq \lim\limits_{n \to \infty} \mathcal{C}(P_n) =$

$= \lim\limits_{n \to \infty} \min\limits_{\alpha \in [0, 2\pi]} \frac{\mathcal{P}_2(R(P_n, \alpha))}{\mathcal{P}_1(P_n, \alpha)} \leq \lim\limits_{n \to \infty} \frac{4 \cdot perimeter\_of(P_n)}{\mathcal{P}_1(P_n, 0)} \leq \lim\limits_{n \to \infty} \frac{4 \cdot \sqrt{2} \cdot n}{2 \cdot n^2 + 2} = 0.$
Item *iv)* follows from Definition 4 and for the definition of similar transformations. ▯
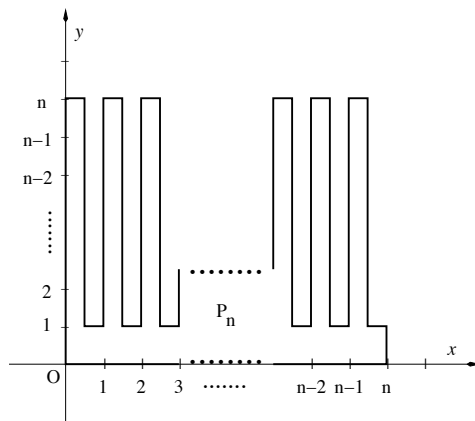

Figure 4: For a big enough n the convexity measure $\mathcal{C}(P_n)$ is arbitrary close to 0.

4 Computation of $\mathcal{C}(P)$

The remaining question is how to effectively and efficiently compute $\mathcal{C}(P)$ from a given polygon P. We need some additional investigation related to the form of the functions $\mathcal{P}_1(P,\alpha)$ and $\mathcal{P}_2(R(P,\alpha))$.

Let us start with a study on $\mathcal{P}_1(P,\alpha)$. Trivially, for any edge e_i of P $\quad l_1(e_i,\alpha) = l_2(e_i)\cdot(|\cos(\phi_i+\alpha)|+|\sin(\phi_i+\alpha)|) = a(\alpha)\cdot l_2(e_i)\cdot\cos(\phi_i+\alpha)+b(\alpha)\cdot l_2(e_i)\cdot\sin(\phi_i+\alpha)$ where $a(\alpha)$ and $b(\alpha)$ take $+1$ or -1 depending on α. Consequently, there is an integer $k \leq 4\cdot n$ and a sequence $0 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_k \leq 2\pi$ such that

$$\mathcal{P}_1(P,\alpha) = \sum_{i=1}^{n} a_{j,i}\cdot l_2(e_i)\cdot\cos(\phi_i+\alpha)+b_{j,i}\cdot l_2(e_i)\cdot\sin(\phi_i+\alpha) \quad \text{if } \alpha \in [\alpha_j,\alpha_{j+1}] \quad (1)$$

where $\quad \alpha_{n+1} = \alpha_1 + 2\pi \quad$ and $\quad \{a_{j,i}, b_{j,i} \mid 1 \leq j \leq k,\ 1 \leq i \leq k\} \subset \{+1,-1\}$. Since $l_2(e_i)$ and ϕ_i are constants we can conclude (from (1)) that there are some numbers c_j and d_j, $(j = 1, 2, \ldots, k)$ such that

$$\mathcal{P}_1(P,\alpha) = c_j\cdot\cos\alpha+d_j\cdot\sin\alpha \quad (\alpha \in [\alpha_j,\alpha_{j+1}],\ 1 \leq j \leq k,\ \alpha_{k+1} = 2\cdot\pi+\alpha_1). \quad (2)$$

Next, we continue with a study on $\mathcal{P}_2(R(P,\alpha))$.

The construction of optimal rectangles which include a polygon P is already well studied in the literature – and various approaches exist [9]. A related problem is the determination of the diameter of a given polygon P, and a very simple algorithm was presented in [11]. The diameter of P is defined to be the greatest distance between parallel lines of support of P. A line L is a line of support of P if the interior of P lies completely to one side of L. A pair of vertices is an antipodal pair if it admits parallel lines of support. In [16] two orthogonal pairs of line supports (called *orthogonal calipers*) are formed around the polygon solving several geometric problem. What is important for us is that the same procedure can be used here in order to obtain the intervals $[\beta_i,\beta_{i+1}]$, $i = 1, 2, \ldots, m$, with $m \leq n$, for which four vertices of P (more precisely, four vertices of $CH(P)$) forming two pairs of antipodal points belonging to the boundary of $R(P,\alpha)$ remains the same for $\alpha \in [\beta_i,\beta_{i+1}]$. That further implies that $\mathcal{P}_2(R(P,\alpha))$ is of the form

$$\mathcal{P}_2(R(P,\alpha)) = e_i\cdot\cos\alpha+f_i\cdot\sin\alpha \quad (\alpha \in [\beta_i,\beta_{i+1}], 1 \leq i \leq m, \beta_{m+1} = 2\cdot\pi+\beta_1). \quad (3)$$

We refer to Fig. 5 for an illustration. Let δ_1 be angle between the positively oriented x-axis and the edge $[AB]$. Also, let $\delta_2 = \min\{\angle(JBC),\ \angle(KDE),\ \angle(GFL),\ \angle(HGI)\}$ (in a situation as in Fig. 6, $\delta_2 = \angle(KDE)$). If the line $l(I,J)$ is chosen to be a support line then B, F and D, G are the antipodal pairs which determine two orthogonal pairs of line supports: $l(I,J), l(L,K)$ and $l(J,K), l(I,L)$. If the support line $l(I,J)$ is rotated into a new position around the vertex B the pairs B, F and D, G remains antipodal until the rotation γ angle varies from 0 to δ_2 (see Fig. 5).

For $\gamma \in [0,\delta_2]$ the width of $R(P,\gamma)$ varies from $l_2([FF'])$ to $l_2([FF''])$. More precisely this width is $l_2([FF']) \cdot \cos(\phi + \gamma) = (l_2([FF']) \cdot \cos\phi) \cdot \cos\gamma + ((-1) \cdot l_2([FF']) \cdot \sin\phi) \cdot \sin\gamma \quad$ for $\quad \gamma \in [0,\delta_2]$. Since $l_2([FF'])$ and ϕ are constants which do not depend on γ, and by noticing that an analogous expression can be derived for the height of $R(P,\gamma)$ we have shown that $\mathcal{P}_2(R(P,\alpha))$ can be expressed as in (3).

Finally, we are able to show that $\mathcal{C}(P)$ can be computed easily by comparing the values of $\frac{\mathcal{P}_2(R(P,\gamma))}{\mathcal{P}_1(P,\gamma)}$ taken into no more than $5\cdot n$ points. Precisely, let $\gamma_1 < \gamma_2 < \ldots < \gamma_l$
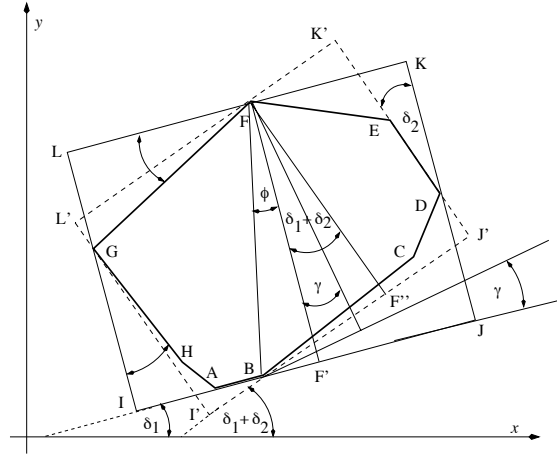
Figure 5: An illustration that $\mathcal{P}_2(R(P,\alpha))$ is of the form described by (3).

be the ordered sequence of angles from the set $\{\alpha_i \mid i = 1, 2, \ldots, k\} \cup \{\beta_i \mid i = 1, 2, \ldots, m\}$ (obviously $l \leq k + m \leq 5 \cdot n$). By using (2) and (3) we have that the first derivative is $\left(\frac{\mathcal{P}_2(R(P,\gamma))}{\mathcal{P}_1(P,\gamma)}\right)' = \left(\frac{\bar{g}_i \cdot \cos \gamma + \bar{f}_i \cdot \sin \gamma}{\bar{c}_i \cdot \cos \gamma + \bar{d}_i \cdot \sin \gamma}\right)' = \frac{\bar{f}_i \cdot \bar{c}_i - \bar{g}_i \cdot \bar{d}_i}{(\bar{c}_i \cdot \cos \gamma + \bar{d}_k \cdot \sin \gamma)^2}$ for some constants $\tilde{c}_i, \tilde{d}_i, \tilde{g}_i, \tilde{f}_i$ for $\gamma \in (\gamma_i, \gamma_{i+1})$ and for $i = 1, 2, \ldots l$. Together with $\mathcal{P}_1(P, \gamma) = \tilde{c}_i \cdot \cos \gamma + \tilde{d}_i \cdot \sin \gamma > 0$ we have that $\frac{\mathcal{P}_2(R(P,\gamma))}{\mathcal{P}_1(P,\gamma)}$ does not have any local extrema inside all intervals (γ_i, γ_{i+1}) where $i = 1, 2, \ldots l$, and consequently it reaches its minimum at some of the points from $\{\gamma_i \mid i = 1, 2, \ldots, l\}$. So, we have proven the following theorem which shows that $\mathcal{C}(P)$ can be computed in a very simple way.

Theorem 3 *If a polygon is given then its convexity measure $\mathcal{C}(P)$ can be computed as*

$$\mathcal{C}(P) = \min \left\{ \frac{\mathcal{P}_2(R(P, \gamma_i))}{\mathcal{P}_1(P, \gamma_i)} \mid i = 1, 2, \ldots, l \right\}.$$

5 Comparison Between Boundary Based Measurements

Let us notice that the new convexity measure can be applied to shapes whose boundaries consist of several polygonal lines (see Fig. 6), i.e., to the shapes which are unions or set differences of polygonal areas. From the view point of practical applications it means that it is possible to measure the convexity of shapes with holes. The perimeter of such shapes is defined to be the sum of the length of all boundary lines.

Let T_1 and T_2 be two isometric polygonal subsets of a given polygon P. Then the set differences $P \setminus T_1$ and $P \setminus T_2$ have the same convexity measured by the $\mathcal{C}_3$-measure, i.e., $\mathcal{C}_3(P \setminus T_1) = \mathcal{C}_3(P \setminus T_2)$, for all isometric (moreover isoperimetric) $T_1 \subset P$ and $T_2 \subset P$. But, $\mathcal{C}(P \setminus T_1) = \mathcal{C}(P \setminus T_2)$ is not always guaranteed. Let us consider the example from Fig. 6. The boundary of the shape S_α consists of two squares. The first one has the vertices $(1, 1)$, $(6, 1)$, $(6, 6)$, and $(1, 6)$ while the second one is obtained by the rotating the square with vertices $(2, 2)$, $(5, 2)$, $(5, 5)$, and $(2, 5)$ for an angle α around the point $A = (3.5, 3.5)$. The equality $\mathcal{C}_3(S_\alpha) = 5/8$ holds for any choice of α but the
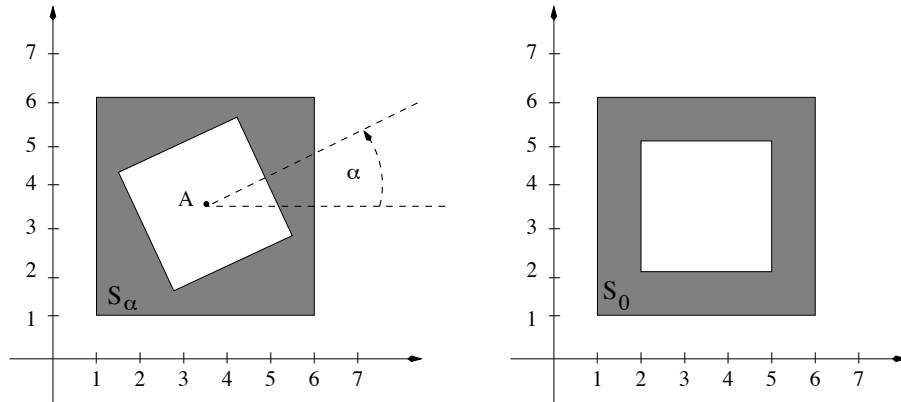
Figure 6: The inequality $\mathcal{C}(S_\alpha) < \mathcal{C}(S_0)$ is more appropriate than the equality $\mathcal{C}_3(S_\alpha) = \mathcal{C}_3(S_0)$.
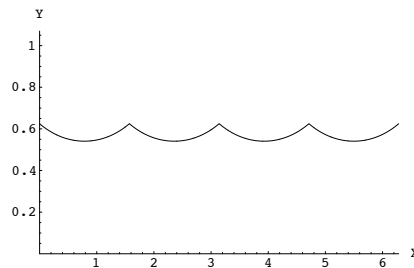

Figure 7: The graph of the convexity measure $\mathcal{C}(S_\alpha)$ when α varies from 0 to 2π.

new convexity measure gives different values for non isometric S_α and S_β (see Fig. 7). Moreover, $\mathcal{C}(S_\alpha)$ reaches its maximum for $\alpha = k \cdot \dfrac{\pi}{2}$, with $k = 0, 1, 2, 3$, and $\mathcal{C}(S_\alpha)$ reaches the minimum for $\alpha = \dfrac{\pi}{4} + k \cdot \dfrac{\pi}{2}$, with $k = 0, 1, 2, 3$, which seems to be natural and which is in accordance with $\mathcal{C}_1$. The previous discussion shows an advantage of the new convexity measure $\mathcal{C}(S)$ with respect to the already known $\mathcal{C}_3(S)$ measure.

6 Concluding Remarks and Experimental Results

In this paper a new convexity measure has been proposed for describing shapes. In contrast to the most common approach (the ratio of the area of the shape to the area of its convex hull) it is based on the shape's boundary perimeter rather than its area. Theoretical and experimental analysis shows that it performs well. Compared to area based approaches it is more sensitive to deep indentations into shapes, especially if they are thin (i.e. of negligible area).

Moreover, compared to the already known boundary based convexity measure $\mathcal{C}_3$ it gives better results if applied to shapes with holes. Namely, while the relative position of the holes inside the shape has no effect on the convexity measured by $\mathcal{C}_3$ there is an
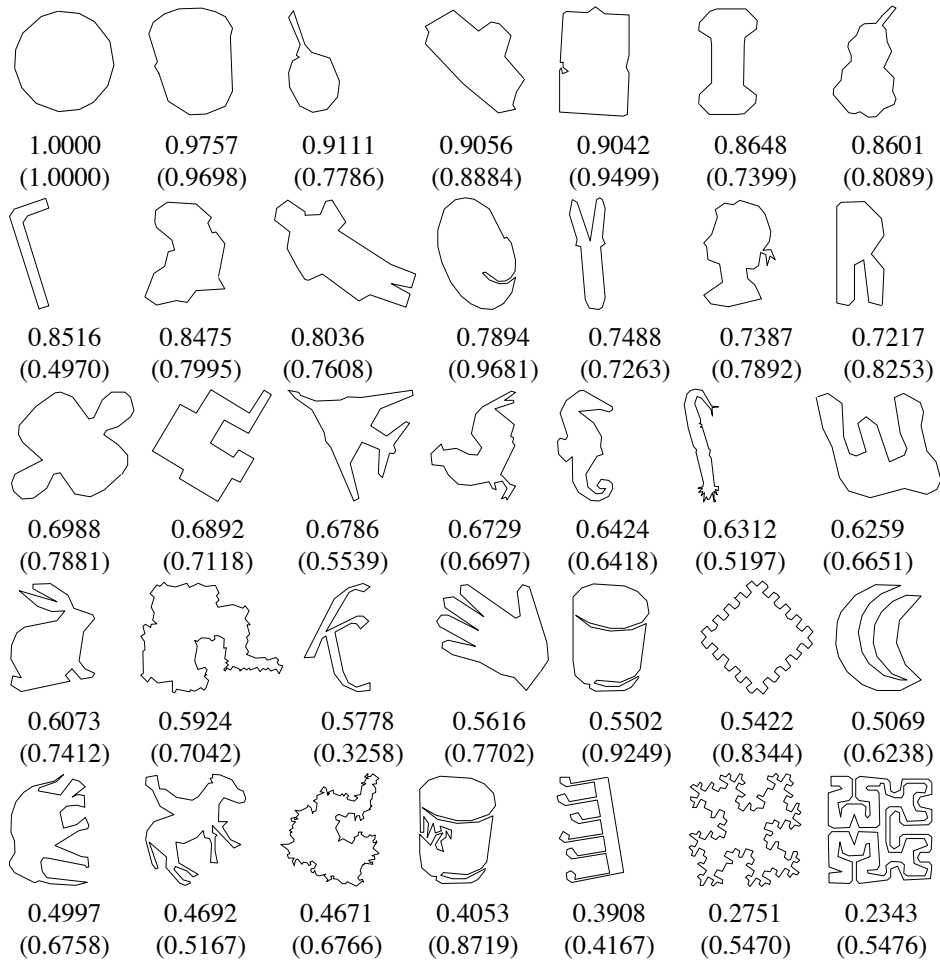
Figure 8: Shapes ranked by the $\mathcal{C}$ measure. They are plotted at the orientation α minimising $\frac{\mathcal{P}_2(R(P,\alpha))}{\mathcal{P}_1(P,\alpha)}$. The results of measure $\mathcal{C}_2$ are shown in brackets.

impact if the convexity is measured by $\mathcal{C}$. That is an essential advantage.

Fig. 8 shows 35 shapes ordered into decreasing convexity by the two measures $\mathcal{C}(P)$ and $\mathcal{C}_2(P)$. To reduce the sensitivity of $\mathcal{C}(P)$ to noise, minor fluctuations have been removed by first simplifying the boundary using Ramer's [12] polygonal approximation algorithm. A threshold of maximum deviation equal to three is used in all cases and the widths of the shapes are between 100–300 pixels. It can be seen that $\mathcal{C}(P)$ is stricter than $\mathcal{C}_2(P)$ regarding shapes that are roughly convex except for relatively narrow indentations. In contrast, protrusions are penalized rather less by $\mathcal{C}(P)$ than $\mathcal{C}_2(P)$, as shown by examples such as the tennis racket (third shape in the first row) and the "L" shape (eighth shape in the first row) in Fig. 8.

References

[1] D. M. Acketa and J. Žunić. On the maximal number of edges of digital convex polygons included into an (m, m)-grid. *Journal of Combinatorial Theory Series A*, 69(2):358–368, 1995.

[2] F. Bookstein. *Morphometric Tools for Landmark Data: Geometry and Biology*. Cambridge University Press, 1991.

[3] L. Boxer. Computing deviations from convexity in polygons. *Pattern Recognition Letters*, 14:163–167, 1993.

[4] L. da F. Costa and R.M. Cesar Jr. *Shape Analysis and Classification*. CRC Press, 2001.

[5] S. Hyde *et al. The Language of Shape*. Elsevier, 1997.

[6] A. E. Hawkins. *The Shape of Powder-Particle Outlines*. J. Wiley and Sons, 1993.

[7] L.J. Latecki and R Lakämper. Convexity rule for shape decomposition based on discrete contour evolution. *Computer Vision and Image Understanding*, 73(3):441–454, 1999.

[8] L.J. Latecki and R. Lakamper. Shape similarity measure based on correspondence of visual parts. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 22(10):1185–1190, 2000.

[9] R.R. Martin and P.C. Stephenson. Putting objects into boxes. *Computer Aided Design*, 20:506–514, 1988.

[10] A.F. Pitty. *Geomorphology*. Blackwell, 1984.

[11] F.P. Preparata and M.I. Shamos. *Computational Geometry*. Springer-Verlag, 1985.

[12] U. Ramer. An iterative procedure for the polygonal approximation of plane curves. *Computer Graphics and Image Processing*, 1:244–256, 1972.

[13] P.L. Rosin. Shape partitioning by convexity. *IEEE Transactions on Systems, Man and Cybernetics*, part A, 30(2):202–210, 2000.

[14] M. Sonka, V. Hlavac, and R. Boyle. *Image Processing, Analysis, and Machine Vision*. Chapman and Hall, 1993.

[15] H.I. Stern. Polygonal entropy: a convexity measure. *Pattern Recognition Letters*, 10:229–235, 1989.

[16] G.T. Toussaint. Solving geometric problems with the rotating calipers. In *Proc. IEEE MELECON '83*, pages A10.02/1–4, 1983.