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A new derivation of the displacement potentials for motion in a homogeneous isotropic elastic medium

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Summary. We give a derivation of the displacement potentials and the wave equations which they satisfy. The derivation is similar to one given by Richards but is more general and yields explicit formulas for the source terms. This generality is retained when the moment tensor representation of the source is used. Formulas for the source terms are given in both spherical and cylindrical coordinate systems and are evaluated for the particular case of a point source with second order moment tensor.

Introduction

The momentum equation for a non-gravitating isotropic elastic medium is

$$\rho\ddot{\mathbf{u}} = (\lambda + 2\mu)\,\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) - \mu\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{u} + \mathbf{f} + (\boldsymbol{\nabla}\cdot\mathbf{u})\boldsymbol{\nabla}\lambda + (\boldsymbol{\nabla}\mu)\cdot(\boldsymbol{\nabla}\mathbf{u} + \mathbf{u}\boldsymbol{\nabla}) \tag{1}$$

in which $\mathbf{u}$ is displacement, λ and μ are Lamé parameters, ρ is density and $\mathbf{f}$ is body force per unit volume. Specializing to a homogeneous medium and taking the Fourier transform of (1) yields

$$-\omega^2\mathbf{u} = \alpha^2\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) - \beta^2\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{u} + \rho^{-1}\mathbf{f}. \tag{2}$$

Richards (1974) noted that in a region where $\mathbf{f}$ vanishes and $\omega \neq 0$ one can divide (2) by ω^2 and write it in the form

$$\mathbf{u} = \boldsymbol{\nabla}\Phi + \boldsymbol{\nabla}\times\mathbf{s} \tag{3}$$

in which

$$\Phi = -\frac{\alpha^2}{\omega^2}\,\boldsymbol{\nabla}\cdot\mathbf{u}; \qquad \mathbf{s} = \frac{\beta^2}{\omega^2}\,\boldsymbol{\nabla}\times\mathbf{u}. \tag{4}$$

Substitution of (3) into (4) then yields the wave equations:

$$\left(\nabla^2 + \frac{\alpha^2}{\omega^2}\right)\Phi = 0; \qquad \left(\nabla^2 + \frac{\omega^2}{\beta^2}\right)\mathbf{s} = 0. \tag{5}$$

He later derived the source free wave equations for *SH* and *SV* by means of the operator Λ^2 which we introduce in the sequel. Here we will consider the equation (2) when $\mathbf{f}$ is twice continuously differentiable, and $|\mathbf{f}| = O(1/|\mathbf{r}|^{2+\epsilon})$ for some $\epsilon > 0$ as $|\mathbf{r}| \to \infty$. These conditions ensure that the operator ∇^{-2} (defined below) commutes with $\nabla\cdot$, $\nabla\times$, ∇^2 and $\boldsymbol{\Lambda}\cdot\nabla\times$ when applied to $\mathbf{f}$, and that $|\mathbf{f}|$ is bounded on $\mathbb{R}^3$.

Potentials for spherical coordinates

We begin by finding scalars S_p, S_H, S_v such that

$$\mathbf{f} = \nabla S_p + \nabla\times \mathbf{r}S_H + \nabla\times\nabla\times\mathbf{r}S_v. \tag{6}$$

To determine S_H and S_v it will be convenient to introduce the infinitesimal rotation operator $\boldsymbol{\Lambda} = \mathbf{r}\times\nabla$ whose properties are summarized by Backus (1958). Application of the operators $\nabla\cdot$, $\boldsymbol{\Lambda}\cdot$ and $\boldsymbol{\Lambda}\cdot\nabla\times$ to both sides of (6) yields the equations $\nabla\cdot\mathbf{f} = \nabla^2 S_p$, $\boldsymbol{\Lambda}\cdot\mathbf{f} = -\Lambda^2 S_H$, and $\boldsymbol{\Lambda}\cdot\nabla\times\mathbf{f} = \Lambda^2\nabla^2 S_v$ respectively and we are thus led to define S_p, S_H and S_v by

$$\begin{aligned} S_p &= \nabla^{-2}(\nabla\cdot\mathbf{f}) \\ S_H &= -\Lambda^{-2}(\boldsymbol{\Lambda}\cdot\mathbf{f}) \\ S_v &= \nabla^{-2}\Lambda^{-2}(\boldsymbol{\Lambda}\cdot\nabla\times\mathbf{f}). \end{aligned} \tag{7}$$

The operator ∇^{-2} is defined by

$$\nabla^{-2} = \int_{\mathbb{R}^3} dV' K(\mathbf{r}, \mathbf{r}')$$

in which $K(\mathbf{r}, \mathbf{r}') = -1/4\pi|\mathbf{r} - \mathbf{r}'|$ and $\mathbb{R}^3$ is Euclidean 3-space. The domain of ∇^{-2} is the set of functions continuous on $\mathbb{R}^3$ which vanish faster than $|\mathbf{r}|^{-2}$ as $|\mathbf{r}| \to \infty$. The operator Λ^{-2} is defined by

$$\Lambda^{-2} = \int_{\Omega'} d\Omega' B(\hat{\mathbf{r}}, \hat{\mathbf{r}}')$$

in which $B(\hat{\mathbf{r}}, \hat{\mathbf{r}}') = \ln(1 - \hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')/4\pi$ and Ω is the unit sphere. The domain of Λ^{-2} is the set of functions continuous on Ω whose average over Ω is zero. If $\delta(\mathbf{r} - \mathbf{r}')$ is the generalized function with support $\mathbf{r} = \mathbf{r}'$ and the replication property

$$\int_{\mathbb{R}^3} dV' \phi(\mathbf{r}')\,\delta(\mathbf{r} - \mathbf{r}') = \phi(\mathbf{r})$$

then

$$\delta(\mathbf{r} - \mathbf{r}') = \frac{\delta(r - r')}{r^2}\,\delta(\hat{\mathbf{r}}, \hat{\mathbf{r}}')$$

where $\delta(\hat{\mathbf{r}}, \hat{\mathbf{r}}')$ is the generalized function with support $\hat{\mathbf{r}} = \hat{\mathbf{r}}'$ and the replication property

$$\int_{\Omega'} d\Omega' \phi(\hat{\mathbf{r}}')\,\delta(\hat{\mathbf{r}}, \hat{\mathbf{r}}') = \phi(\hat{\mathbf{r}}).$$

Also $\nabla^2 K(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$ and $\Lambda^2 B(\hat{\mathbf{r}}, \hat{\mathbf{r}}') = \delta(\hat{\mathbf{r}}, \hat{\mathbf{r}}') - 1/4\pi$. To prove that the quantities defined by (7) actually satisfy (6) we need the theorem stated below. This theorem is an immediate consequence of a theorem of Backus (1958) which states that for any vector field $\mathbf{v}$ if $\mathbf{r}\cdot\mathbf{v} = \boldsymbol{\Lambda}\cdot\mathbf{v} = \nabla\cdot\mathbf{v} = 0$ then $\mathbf{v} = 0$.

Theorem 1. For any twice differentiable vector field $\mathbf{v}$ defined on $\mathbb{R}^3$, if $|\mathbf{v}| \to 0$ as $|\mathbf{r}| \to \infty$ and $\nabla \cdot \mathbf{v} = \mathbf{\Lambda} \cdot \mathbf{v} = \mathbf{\Lambda} \cdot \nabla \times \mathbf{v} = 0$ then $\mathbf{v} = 0$.

Proof. Since $\mathbf{\Lambda} \cdot \mathbf{v} = \mathbf{\Lambda} \cdot \nabla \times \mathbf{v} = 0$ we have that $\nabla \times \mathbf{v} = 0$ by Backus theorem. Since $\nabla \cdot \mathbf{v} = \nabla \times \mathbf{v} = 0$ we have that $\nabla^2 \mathbf{v} = 0$. Thus in any fixed Cartesian axis system $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ we have $\nabla^2 v_x = \nabla^2 v_y = \nabla^2 v_z = 0$. Let S_r be the sphere of radius r about the origin. Since v_x is harmonic the maximum value of v_x in S_r is taken on the boundary ∂S_r. But this maximum is smaller than the maximum of $|\mathbf{v}|$ on ∂S_r which by hypothesis approaches zero as $|\mathbf{r}| \to \infty$. Also since $|\mathbf{v}|$ is bounded on $\mathbb{R}^3$ so is $|v_x|$ hence $|v_x| = 0$. Similarly $v_y = v_z = 0$ hence $\mathbf{v} = 0$ in $\mathbb{R}^3$.

To prove equation (6) we apply Theorem 1 taking $\mathbf{v} = \mathbf{f} - (\nabla S_\mathrm{p} + \nabla \times \mathbf{r} S_\mathrm{H} + \nabla \times \nabla \times \mathbf{r} S_\mathrm{v})$ with S_p, S_H, and S_v given by (7). Since $\mathbf{f}$ is twice continuously differentiable, and $O(1/|\mathbf{r}|^{2+\epsilon})$ for some $\epsilon > 0$ as $|\mathbf{r}| \to \infty$ we have that

$$\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{f} - \nabla^2 (\nabla^{-2} \nabla \cdot \mathbf{f}) = 0$$

$$\mathbf{\Lambda} \cdot \mathbf{v} = \mathbf{\Lambda} \cdot \mathbf{f} + \Lambda^2 (-\Lambda^{-2} \mathbf{\Lambda} \cdot \mathbf{f}) = 0$$

$$\mathbf{\Lambda} \cdot \nabla \times \mathbf{v} = \mathbf{\Lambda} \cdot \nabla \times \mathbf{f} - \Lambda^2 \nabla^2 \nabla^{-2} \Lambda^{-2} (\mathbf{\Lambda} \cdot \nabla \times \mathbf{f}) = 0$$

and (6) follows.

We now find displacement potentials for $\mathbf{u}$. In case $\omega \neq 0$ equation (2) may be written in the form

$$\mathbf{u} = -\frac{\alpha^2}{\omega^2} \nabla (\nabla \cdot \mathbf{u}) + \frac{\beta^2}{\omega^2} \nabla \times \nabla \times \mathbf{u} - \frac{\mathbf{f}}{\rho \omega^2}. \tag{8}$$

The technique used above to find displacement potentials for $\mathbf{f}$ is quite general and if we now apply it to the entire right hand side of (8), including the source term, we find that the right hand side of (8) is given exactly by $\nabla P + \nabla \times \mathbf{r} H + \nabla \times \nabla \times \mathbf{r} V$ where

$$P = -\frac{\alpha^2}{\omega^2} \nabla \cdot \mathbf{u} - \frac{S_\mathrm{p}}{\rho \omega^2}$$

$$H = \frac{\beta^2}{\omega^2} \nabla^2 \Lambda^{-2} (\mathbf{\Lambda} \cdot \mathbf{u}) - \frac{S_\mathrm{H}}{\rho \omega^2} \tag{9}$$

$$V = -\frac{\beta^2}{\omega^2} \Lambda^{-2} (\mathbf{\Lambda} \cdot \nabla \times \mathbf{u}) - \frac{S_\mathrm{v}}{\rho \omega^2}.$$

Therefore by the equality in (8)

$$\mathbf{u} = \nabla P + \nabla \times \mathbf{r} H + \nabla \times \nabla \times \mathbf{r} V. \tag{10}$$

To obtain the wave equations we substitute (10) into each of (9) and there results exactly

$$\left(\nabla^2 + \frac{\omega^2}{\alpha^2}\right) P = -\frac{S_\mathrm{p}}{\lambda + 2\mu}$$

$$\left(\nabla^2 + \frac{\omega^2}{\beta^2}\right) H = -\frac{S_\mathrm{H}}{\mu} \tag{11}$$

$$\left(\nabla^2 + \frac{\omega^2}{\beta^2}\right) V = -\frac{S_\mathrm{v}}{\mu}.$$

In case $\omega = 0$ equation (2) becomes

$$0 = \alpha^2 \nabla(\nabla\cdot \mathbf{u}) - \beta^2 \nabla\times \nabla\times \mathbf{u} + \frac{\mathbf{f}}{\rho}. \tag{12}$$

Applying $\mathbf{\Lambda}\cdot$, $\nabla\cdot$ and $\mathbf{\Lambda}\cdot\nabla\times$ to (12) proves that equations (11) will be satisfied for $\omega = 0$ if

$$P = \nabla^{-2}(\nabla\cdot \mathbf{u})$$
$$H = -\Lambda^{-2}(\mathbf{\Lambda}\cdot \mathbf{u}) \tag{13}$$
$$V = \nabla^{-2}\Lambda^{-2}(\mathbf{\Lambda}\cdot \nabla\times \mathbf{u}).$$

Equation (10) now follows immediately from Theorem 1 using $\mathbf{v} = \mathbf{u} - (\nabla P + \nabla\times \mathbf{r}H + \nabla\times \nabla\times \mathbf{r}V)$ with P, H, V given by (3). Thus we have shown that for any value of ω in any region of space not necessarily source free, there exist potentials P, H, V such that equations (11) are satisfied and $\mathbf{u}$ given by (10) is a particular solution of (2).

The point source in spherical coordinates

We now take $\mathbf{f}$ to be the equivalent body force density associated with the point source having moment tensor $\underline{\mathbf{M}}$ and time dependence $g(t)$. $\underline{\mathbf{M}}$ is a constant symmetric second order tensor and (Burridge & Knopoff 1964)

$$\mathbf{f}(\mathbf{r}, t) = -g(t)\,\underline{\mathbf{M}}\cdot\nabla\delta(\mathbf{r} - \mathbf{r}_0). \tag{14}$$

Using (14), the divergence theorem, the symmetry and constancy of $\underline{\mathbf{M}}$ and $\nabla K(\mathbf{r},\mathbf{r}') = -\nabla' K(\mathbf{r},\mathbf{r}')$ it is straightforward to show that

$$\nabla^{-2}\mathbf{f} = -g(t)\,\underline{\mathbf{M}}\cdot\nabla K(\mathbf{r},\mathbf{r}_0). \tag{15}$$

Now taking the divergence of both sides of (15) and using $\nabla\nabla K(\mathbf{r},\mathbf{r}') = \nabla'\nabla' K(\mathbf{r},\mathbf{r}')$ yields the P motion source potential in the form

$$S_{\mathrm{p}} = -g(t)\,\underline{\mathbf{M}} : \nabla^0\nabla^0 K(\mathbf{r},\mathbf{r}_0). \tag{16}$$

To compute the shear source potentials we introduce the spherical resolutions (Backus 1967) $\nabla = \hat{\mathbf{r}}\partial_r + (1/r)\nabla_1$ and $\underline{\mathbf{M}} = \hat{\mathbf{r}}\hat{\mathbf{r}}M_{rr} + \hat{\mathbf{r}}\mathbf{M}_{rs} + \mathbf{M}_{sr}\hat{\mathbf{r}} + \underline{\mathbf{M}}_{ss}$. $\nabla_1 = -\hat{\mathbf{r}}\times\mathbf{\Lambda}$ is the surface gradient operator on the unit sphere. In consequence of these definitions we have the result that if $\mathbf{v}(r\hat{\mathbf{r}})$ is any continuous vector field tangent to Ω then by the divergence theorem

$$\int_{\Omega'} d\Omega'\,\mathbf{v}(r\hat{\mathbf{r}}')\cdot\nabla_1'\delta(\hat{\mathbf{r}},\hat{\mathbf{r}}') = -\nabla_1\cdot\mathbf{v}(r\hat{\mathbf{r}}). \tag{17}$$

Also if $\mathbf{v}(r\hat{\mathbf{r}})$ is any continuous vector field and $\phi(r\hat{\mathbf{r}})$ any continuous scalar field then by Stokes theorem

$$\int_\Omega d\Omega\,\phi\mathbf{\Lambda}\cdot\mathbf{v} = -\int_\Omega d\Omega(\mathbf{\Lambda}\phi)\cdot\mathbf{v}. \tag{18}$$

Substitution of the spherical resolutions into (14) yields

$$\mathbf{f}(\mathbf{r},t) = -g(t)\frac{\partial}{\partial r}\left\{\frac{\delta(r - r_0)}{r^2}\right\}[\hat{\mathbf{r}}M_{rr} + \mathbf{M}_{sr}]\,\delta(\hat{\mathbf{r}},\hat{\mathbf{r}}_0) - g(t)\frac{\delta(r - r_0)}{r^3}[\hat{\mathbf{r}}\mathbf{M}_{rs} + \underline{\mathbf{M}}_{ss}]\cdot\nabla_1\delta(\hat{\mathbf{r}},\hat{\mathbf{r}}_0).$$

To compute S_H by (7) we write

$$
\begin{aligned}
S_H &= -\int_{\Omega'} d\Omega' B(\hat{\mathbf{r}}, \hat{\mathbf{r}}')\, \mathbf{\Lambda}' \cdot \mathbf{f}(r\hat{\mathbf{r}}', t) \\
&= g(t) \frac{\partial}{\partial r}\left\{\frac{\delta(r-r_0)}{r^2}\right\} \int_{\Omega'} d\Omega' B(\hat{\mathbf{r}}, \hat{\mathbf{r}}')\, \mathbf{\Lambda}' \cdot [\mathbf{M}_{sr}(\hat{\mathbf{r}}')\, \delta(\hat{\mathbf{r}}', \hat{\mathbf{r}}_0)] \\
&\quad + g(t) \frac{\delta(r-r_0)}{r^3} \int_{\Omega'} d\Omega' B(\hat{\mathbf{r}}, \hat{\mathbf{r}}')\, \mathbf{\Lambda}' \cdot [\underline{\mathbf{M}}_{ss}(\hat{\mathbf{r}}') \cdot \mathbf{\nabla}_1' \delta(\hat{\mathbf{r}}', \hat{\mathbf{r}}_0)] \\
&= -g(t) \frac{\partial}{\partial r}\left\{\frac{\delta(r-r_0)}{r^2}\right\} \mathbf{\Lambda}^0 B(\hat{\mathbf{r}}, \hat{\mathbf{r}}_0) \cdot \mathbf{M}_{sr}(\hat{\mathbf{r}}_0) \\
&\quad + g(t) \frac{\delta(r-r_0)}{r^3} \mathbf{\nabla}_1^0 \cdot [\mathbf{\Lambda}^0 B(\hat{\mathbf{r}}, \hat{\mathbf{r}}_0) \cdot \underline{\mathbf{M}}_{ss}(\hat{\mathbf{r}}_0)].
\end{aligned}
\tag{19}
$$

The last line of (19) is obtained by two uses of (18) and then one use of (17).

To compute S_v by (7) we first make use of the commutativity of ∇^{-2} with Λ^{-2}, $\mathbf{\Lambda}\cdot$, and $\mathbf{\nabla}\times$ to write S_v in the form

$$S_v = \Lambda^{-2}(\mathbf{\Lambda} \cdot \mathbf{\nabla} \times \nabla^{-2}\mathbf{f}). \tag{20}$$

But for any vector field $\mathbf{v}$, $\mathbf{\Lambda} \cdot \mathbf{\nabla} \times \mathbf{v} = r\partial_r(\mathbf{\nabla} \cdot \mathbf{v}) - \mathbf{r} \cdot \nabla^2 \mathbf{v}$ so according to the first of (7)

$$S_v = r\partial_r(\Lambda^{-2} S_p) - \Lambda^{-2}(\mathbf{r} \cdot \mathbf{f}). \tag{21}$$

Substitution of the spherical resolutions for $\underline{\mathbf{M}}$, $\delta(\mathbf{r} - \mathbf{r}_0)$ and $\mathbf{\nabla}$ into formula (14) yields

$$-\mathbf{r} \cdot \mathbf{f} = r g(t) M_{rr} \frac{\partial}{\partial r}\left\{\frac{\delta(r-r_0)}{r^2}\right\} \delta(\hat{\mathbf{r}}, \hat{\mathbf{r}}_0) + g(t) \frac{\delta(r-r_0)}{r^2} \mathbf{M}_{rs} \cdot \mathbf{\nabla}_1 \delta(\hat{\mathbf{r}}, \hat{\mathbf{r}}_0).$$

Thus by one use of (17) we obtain

$$\Lambda^{-2}(-\mathbf{r} \cdot \mathbf{f}) = g(t)\, r \frac{\partial}{\partial r}\left\{\frac{\delta(r-r_0)}{r^2}\right\} M_{rr}(\hat{\mathbf{r}}_0) B(\hat{\mathbf{r}}, \hat{\mathbf{r}}_0) - g(t) \frac{\delta(r-r_0)}{r^2} \mathbf{\nabla}_1^0 \cdot \{B(\hat{\mathbf{r}}, \hat{\mathbf{r}}_0)\, \mathbf{M}_{rs}(\hat{\mathbf{r}}_0)\}.$$

Therefore

$$
\begin{aligned}
S_v &= g(t)\, \Lambda^{-2}\left\{-\underline{\mathbf{M}} : \mathbf{\nabla}^0 \mathbf{\nabla}^0 r \frac{\partial}{\partial r} K(\mathbf{r}, \mathbf{r}_0)\right\} + g(t)\, r \frac{\partial}{\partial r}\left\{\frac{\delta(r-r_0)}{r^2}\right\} M_{rr}(\hat{\mathbf{r}}_0) B(\hat{\mathbf{r}}, \hat{\mathbf{r}}_0) \\
&\quad - g(t) \frac{\delta(r-r_0)}{r^2} \mathbf{\nabla}_1^0 \cdot \{B(\hat{\mathbf{r}}, \hat{\mathbf{r}}_0)\, \mathbf{M}_{rs}(\hat{\mathbf{r}}_0)\}.
\end{aligned}
\tag{22}
$$

The expansions of S_p, S_H, and S_v in series of surface spherical harmonics can now be obtained by substitution in (16), (19), and (22) of the well known formulas for $K(\mathbf{r}, \mathbf{r}_0)$ and $B(\hat{\mathbf{r}}, \hat{\mathbf{r}}_0)$ included here for reference.

$$
\begin{aligned}
K(\mathbf{r}, \mathbf{r}_0) &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(\frac{-1}{2l+1}\right) \frac{r_<^l}{r_>^{l+1}}\, Y_l^m(\hat{\mathbf{r}})\, Y_l^{*m}(\hat{\mathbf{r}}_0) \\
B(\hat{\mathbf{r}}, \hat{\mathbf{r}}_0) &= \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \frac{-1}{l(l+1)}\, Y_l^m(\hat{\mathbf{r}})\, Y_l^{*m}(\hat{\mathbf{r}}_0).
\end{aligned}
\tag{23}
$$

In equations (22) and (23) $r_> = \max(r, r_0)$, $r_< = \min(r, r_0)$ and the harmonics Y_l^m have been normalized so that $Y_l^{-m} = (-1)^m Y_l^{*m}$ and

$$\int_\Omega d\Omega\, Y_l^m Y_{l'}^{m'} = \delta_{ll'}\delta_{mm'}.$$

To evaluate the first term in (22) the relation

$$\Lambda^{-2} Y_l^m = \frac{-1}{l(l+1)} Y_l^m$$

is needed.

Potentials for cylindrical coordinates

The derivations and formulas for cylindrical coordinates are similar to those given above for spherical coordinates. Analogous to $\mathbf{\Lambda}$ is the operator $\mathbf{L} = \hat{\mathbf{z}} \times \mathbf{\nabla}$. For a scalar field ϕ, $\mathbf{L}\phi = \hat{\mathbf{z}} \times \mathbf{\nabla}\, \phi = -\,\mathbf{\nabla} \times \hat{\mathbf{z}}\phi$ while for a vector field $\mathbf{v}$, $\mathbf{L} \cdot \mathbf{v} = \hat{\mathbf{z}} \cdot \mathbf{\nabla} \times \mathbf{v}$. Also $\nabla^2 = \partial^2/\partial z^2 + L^2$ and the operator L^{-2} is defined by

$$L^{-2} = \int_{\mathbb{R}^2} dA' C(\bar{\mathbf{\rho}}, \bar{\mathbf{\rho}}')$$

in which $C(\bar{\mathbf{\rho}}, \bar{\mathbf{\rho}}') = \ln|\mathbf{\rho} - \mathbf{\rho}'|/2\pi$ and $\mathbb{R}^2$ $\mathbf{\rho} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$. The domain of L^{-2} is the set of functions continuous on $\mathbb{R}^3$ which vanish faster than $|\mathbf{\rho}|^{-2}$ as $|\mathbf{\rho}| \to \infty$.

We also need a cylindrical Backus theorem the statement of which is that if a vector field $\mathbf{v}$ is defined for $z_1 < z < z_2$ and in that range $v_z = 0$ while v_x and v_y are continuously differentiable and if v_x and v_y approach zero as $|\mathbf{\rho}| \to \infty$ and $\mathbf{\nabla} \cdot \mathbf{v} = \mathbf{L} \cdot \mathbf{v} = 0$ then $\mathbf{v} = 0$. The proof of this is similar to the spherical case and we give it here. For fixed $z \in [z_1, z_2]$ let $\phi(x, y) = v_x - iv_y$. Then $\mathbf{\nabla} \cdot \mathbf{v} = \mathbf{L} \cdot \mathbf{v} = 0$ means that ϕ satisfies the Cauchy-Riemann conditions so $\phi(x + iy)$ is entire. But by hypothesis $|\phi|$ is bounded so by Liouvilles theorem ϕ is constant. Thus since $|\phi| \to 0$ as $|\mathbf{\rho}| \to \infty$, $\phi = 0$.

A consequence of the cylindrical Backus theorem is the following theorem whose proof is similar to the proof of Theorem 1.

Theorem 2. For any twice differentiable vector field $\mathbf{v}$ defined on $\mathbb{R}^3$ if $|\mathbf{v}| \to 0$ as $|\mathbf{r}| \to \infty$ and $\mathbf{\nabla} \cdot \mathbf{v} = \mathbf{L} \cdot \mathbf{v} = \mathbf{L} \cdot \mathbf{\nabla} \times \mathbf{v} = 0$ then $\mathbf{v} = 0$.

If we now define

$$\begin{aligned} S_p &= \nabla^{-2}(\mathbf{\nabla} \cdot \mathbf{f}) \\ S_H &= L^{-2}(-\mathbf{L} \cdot \mathbf{f}) \\ S_v &= \nabla^{-2} L^{-2}(\mathbf{L} \cdot \mathbf{\nabla} \times \mathbf{f}) \end{aligned} \qquad (24)$$

then using Theorem 2 it follows that

$$\mathbf{f} = \mathbf{\nabla} S_p + \mathbf{\nabla} \times \hat{\mathbf{z}} S_H + \mathbf{\nabla} \times \mathbf{\nabla} \times \hat{\mathbf{z}} S_v. \qquad (25)$$

Similarly if $\omega \neq 0$ we can use Theorem 2 to show that the entire right hand side of (8) including the source term is given exactly by $\nabla P + \nabla \times \hat{z} H + \nabla \times \nabla \times \hat{z} V$ where

$$P = -\frac{\alpha^2}{\omega^2} \nabla \cdot \mathbf{u} - \frac{S_P}{\rho\omega^2}$$

$$H = \frac{\beta^2}{\omega^2} \nabla^2 L^{-2}(\mathbf{L} \cdot \mathbf{u}) - \frac{S_H}{\rho\omega^2} \tag{26}$$

$$V = -\frac{\beta^2}{\omega^2} L^{-2}(\mathbf{L} \cdot \nabla \times \mathbf{u}) - \frac{S_V}{\rho\omega^2}.$$

So that by the equality in (8) we have

$$\mathbf{u} = \nabla P + \nabla \times \hat{z} H + \nabla \times \nabla \times \hat{z} V. \tag{27}$$

To obtain the wave equations we substitute (27) into each of (26) and there results exactly

$$\left(\nabla^2 + \frac{\omega^2}{\alpha^2}\right) P = \frac{-S_P}{(\lambda + 2\mu)}$$

$$\left(\nabla^2 + \frac{\omega^2}{\beta^2}\right) H = \frac{-S_H}{\mu} \tag{28}$$

$$\left(\nabla^2 + \frac{\omega^2}{\beta^2}\right) V = \frac{-S_V}{\mu}.$$

In case $\omega = 0$ equation (2) becomes equation (12) and application of the operators $\nabla \cdot$, $\mathbf{L} \cdot$ and $\mathbf{L} \cdot \nabla \times$ to (12) proves that equations (28) will be satisfied for $\omega = 0$ if

$$P = \nabla^{-2}(\nabla \cdot \mathbf{u})$$

$$H = -L^{-2}(\mathbf{L} \cdot \mathbf{u}) \tag{29}$$

$$V = \nabla^{-2} L^{-2}(\mathbf{L} \cdot \nabla \times \mathbf{u}).$$

Equation (27) now follows from Theorem 2 using $\mathbf{v} = \mathbf{u} - (\nabla P + \nabla \times \hat{z} H + \nabla \times \nabla \times \hat{z} V)$ with *P, H,* and *V* given by (29). The above remarks show that for any value of ω in any region of space, not necessarily source free, there exist potentials *P, H,* and *V* such that equations (28) are satisfied and **u** given by (27) is a particular solution of (2).

The point source in cylindrical coordinates

We now specialize to the case where **f** is given by (14). The formula (16) for S_P is valid in any coordinate system. To compute the shear source potentials we introduce the cylindrical resolutions $\nabla = \hat{z}\partial/\partial z + \nabla_c$, $\underline{\mathbf{M}} = \hat{z}\hat{z} M_{zz} + \hat{z}\mathbf{M}_{zc} + \mathbf{M}_{cz}\hat{z} + \underline{\mathbf{M}}_{cc}$ and $\delta(\mathbf{r} - \mathbf{r}') = \delta(z - z')\,\delta(\boldsymbol{\rho} - \boldsymbol{\rho}')$. If **v** is any continuous vector field whose $\hat{z}$ component is zero then the divergence theorem in $\mathbb{R}^2$ yields

$$\int_{\mathbb{R}^2} dA'\, \mathbf{v}(z, \boldsymbol{\rho}') \cdot \nabla_c' \delta(\boldsymbol{\rho} - \boldsymbol{\rho}') = -\nabla_c \cdot \mathbf{v}(z, \boldsymbol{\rho}). \tag{30}$$

Also if **v** is any continuous vector field and ϕ any continuous scalar field such that $|\phi \mathbf{v}| \to 0$ faster than $|\boldsymbol{\rho}|^{-2}$ as $|\boldsymbol{\rho}| \to \infty$ then Stokes theorem in $\mathbb{R}^2$ yields

$$\int_{\mathbb{R}^2} dA\, \phi \mathbf{L} \cdot \mathbf{v} = -\int_{\mathbb{R}^2} dA\, (\mathbf{L}\phi) \cdot \mathbf{v}. \tag{31}$$

Substitution of the cylindrical resolutions into (14) yields

$$\mathbf{f}(\mathbf{r},t) = -g(t)\,\delta'(z-z_0)\,[\hat{\mathbf{z}}M_{zz} + \mathbf{M}_{cz}]\,\delta(\boldsymbol{\rho}-\boldsymbol{\rho}_0) - g(t)\,\delta(z-z_0)\,[\hat{\mathbf{z}}\mathbf{M}_{zc} + \mathbf{M}_{cc}]\cdot\boldsymbol{\nabla}_c\,\delta(\boldsymbol{\rho}-\boldsymbol{\rho}_0).$$

To compute S_{H} by means of (24) we write

$$\begin{aligned} S_{\mathrm{H}} &= -\int_{\mathbb{R}^2} dA'C(\boldsymbol{\rho},\boldsymbol{\rho}')\,\mathbf{L}'\cdot\mathbf{f}(z,\boldsymbol{\rho}',t) \\ &= g(t)\,\delta'(z-z_0)\int_{\mathbb{R}^2} dA'C(\boldsymbol{\rho},\boldsymbol{\rho}')\,\mathbf{L}'\cdot[\mathbf{M}_{cz}(\boldsymbol{\rho}')\,\delta(\boldsymbol{\rho}'-\boldsymbol{\rho}_0)] \\ &\quad + g(t)\,\delta(z-z_0)\int_{\mathbb{R}^2} dA'C(\boldsymbol{\rho},\boldsymbol{\rho}')\,\mathbf{L}'\cdot[\underline{\mathbf{M}}_{cc}(\boldsymbol{\rho}')\cdot\boldsymbol{\nabla}'_c\delta(\boldsymbol{\rho}'-\boldsymbol{\rho}_0)] \\ &= -g(t)\,\delta'(z-z_0)\,\mathbf{L}^0C(\boldsymbol{\rho},\boldsymbol{\rho}_0)\cdot\mathbf{M}_{cz}(\boldsymbol{\rho}_0) + g(t)\,\delta(z-z_0)\,\boldsymbol{\nabla}^0_c\cdot[\mathbf{L}^0C(\boldsymbol{\rho},\boldsymbol{\rho}_0)\cdot\underline{\mathbf{M}}_{cc}(\boldsymbol{\rho}_0)]. \end{aligned} \tag{32}$$

The last line of (32) is obtained by two uses of (31) and one use of (30).

To compute S_{v} by (24) we first make use of the commutativity of $\boldsymbol{\nabla}^{-2}$ with L^{-2}, $\mathbf{L}\,\cdot$, and $\boldsymbol{\nabla}\times$ to write S_{v} in the form

$$S_{\mathrm{v}} = L^{-2}(\mathbf{L}\cdot\boldsymbol{\nabla}\times\boldsymbol{\nabla}^{-2}\mathbf{f}). \tag{33}$$

But for any vector field $\mathbf{v}$,

$$\mathbf{L}\cdot\boldsymbol{\nabla}\times\mathbf{v} = \frac{\partial}{\partial z}(\boldsymbol{\nabla}\cdot\mathbf{v}) - \hat{\mathbf{z}}\cdot\boldsymbol{\nabla}^2\mathbf{v}$$

so according to the first of (24)

$$S_{\mathrm{v}} = \frac{\partial}{\partial z}(L^{-2}S_{\mathrm{p}}) - L^{-2}(\hat{\mathbf{z}}\cdot\mathbf{f}). \tag{34}$$

Substitution of the cylindrical resolutions of $\underline{\mathbf{M}}$, $\delta(\mathbf{r}-\mathbf{r}_0)$ and $\boldsymbol{\nabla}$ into formula (14) yields

$$-\hat{z}\cdot\mathbf{f} = g(t)\,\delta'(z-z_0)\,M_{zz}\,\delta(\boldsymbol{\rho}-\boldsymbol{\rho}_0) + g(t)\,\delta(z-z_0)\,\mathbf{M}_{zc}\cdot\boldsymbol{\nabla}_c\,\delta(\boldsymbol{\rho}-\boldsymbol{\rho}_0).$$

Thus by one use of (30) we obtain

$$L^{-2}(-\mathbf{L}\cdot\mathbf{f}) = g(t)\,\delta'(z-z_0)\,M_{zz}(\boldsymbol{\rho}_0)\,C(\boldsymbol{\rho},\boldsymbol{\rho}_0) - g(t)\,\delta(z-z_0)\,\boldsymbol{\nabla}^0_c\cdot[C(\boldsymbol{\rho},\boldsymbol{\rho}_0)\,\mathbf{M}_{zc}(\boldsymbol{\rho}_0)].$$

And therefore

$$\begin{aligned} S_{\mathrm{v}} &= g(t)\,L^{-2}\left\{-\underline{\mathbf{M}}:\boldsymbol{\nabla}^0\boldsymbol{\nabla}^0\frac{\partial}{\partial z}K(\mathbf{r},\mathbf{r}_0)\right\} + g(t)\,\delta'(z-z_0)\,M_{zz}(\boldsymbol{\rho}_0)\,C(\boldsymbol{\rho},\boldsymbol{\rho}_0) \\ &\quad - g(t)\,\delta(z-z_0)\,\boldsymbol{\nabla}^0_c\cdot[C(\boldsymbol{\rho},\boldsymbol{\rho}_0)\,\mathbf{M}_{zc}(\boldsymbol{\rho}_0)]. \end{aligned} \tag{35}$$

The expansions of S_{p}, S_{H} and S_{v} in series of cylindrical harmonics can now be obtained by substitution into (16), (32), and (35) of the formulas for $K(\mathbf{r},\mathbf{r}_0)$ and $C(\boldsymbol{\rho},\boldsymbol{\rho}_0)$ which are included here for reference.

$$K(\mathbf{r},\mathbf{r}_0) = \frac{-1}{4\pi}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk\cdot k\,\exp[\mathrm{i}m(\theta-\theta_0)]\,J_m(k\rho)\,J_m(k\rho_0)\,\exp(-k\,|z-z_0|) \tag{36}$$

$$A(\bar{\rho},\bar{\rho}_0) = \frac{-1}{2\pi}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\int_0^{\infty} dk\cdot k^{-1}\,\exp[\mathrm{i}m(\theta-\theta_0)]\,J_m(k\rho)\,J_m(k\rho_0). \tag{37}$$

To evaluate the first term in (35) the well known relation

$$L^{-2}[J_m(k\rho)\exp(\mathrm{i}m\theta)] = \frac{-1}{k^2} J_m(k\rho)\exp(\mathrm{i}m\theta) \tag{38}$$

is needed.

Discussion

The research above grew out of an attempt to incorporate source terms into the coupled wave equations for P and SV motion in a smoothly stratified inhomogeneous medium (Richards 1974). If this were done one would expect to find some function of the P wave part of the source, down by ω^{-1}, as an additional source term in the SV wave equation and some function of the SV part of the source, down by ω^{-1}, as an additional source term in the P wave equation. Such terms would necessarily be important whenever the source is located in a region where velocity gradients are sufficiently large to warrant the use of coupled wave equations.

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