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A New Description for a Realistic Inhomogeneous Universe in General Relativity

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A new approximation scheme is proposed to describe a realistic inhomogeneous universe in general relativity. This scheme improves the former one and allows one to treat more general situations where the local effect due to self-gravity of the matter dominates the effect of cosmic expansion. A new statistical way of averaging spacetime is also discussed.

§ 1. Introduction

The global structure of the universe is believed to be described by Friedman-Lemaitre-Robertson-Walker (FLRW) geometry based on the assumption of homogeneity and isotropy. The observation of the isotropies of the cosmic microwave background radiation is usually regarded as the evidence for the validity of the assumptions[1),2)] (there is, however, some ambiguity in the interpretation of the dipole anisotropy[3)]). Since there exist local inhomogeneities in various scales, it is expected that an averaged metric over a large volume in some sense may be described by FLRW geometry. However it should be noted that the averaged metric coincides nowhere with the real inhomogeneous metric. This has a fundamental importance in the observational cosmology because the propagation of light rays is governed by the local inhomogeneous metric, not the averaged homogeneous metric.

In fact there have been many attempts to study the light propagation on inhomogeneous universes theoretically[4)~6)] as well as numerically.[7),8)] There are two main problems one encounters in this study. One is how to describe realistic inhomogeneities in the universe, and the other is how to approximate the propagation itself within the framework of general relativity. The latter is somewhat connected to the former problem and a fully satisfactory approximation is not yet available. As far as the former is concerned, rather crude description of inhomogeneities such as the Dyer-Roeder model or Swiss cheese model are employed for the interpretation of the observational data. These models are not derived in the consistent way from the first principle of general relativity. Sometimes the result of linearized perturbation theory is used even in the nonlinear situation without any careful consideration.[9)]

Recently, a consistent approximation scheme for the construction of an inhomogeneous universes in general relativity is developed.[10),11)] The scheme is based on the post-Newtonian type approximation in the cosmological circumstance and thus allows one to treat nonlinear density fluctuation (the pioneering work on the Newtonian approximation in the expanding universe is done by Nariai and Ueno[12)]). Similar post-Newtonian approximation in the cosmological situation is developed for a system of point particles[13)] and the approximation is applied to the problem of light

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propagation in an inhomogeneous universe.[14] Our approach differs from the previous one in the following sense. Namely it allows one to calculate the back reaction due to the growth of inhomogeneities on the global expansion of the universe and vice versa. Also an expression for the deviation from homogeneous and isotropic expansion is explicitly derived in terms of inhomogeneities. This is done by introducing a spatial averaging procedure. The scheme explicitly demonstrates how averaged FLRW metric is generated from an inhomogeneous spacetime.

In the formulation of the approximation it is assumed that the inhomogeneous spacetime may be regarded in a sense as a small deviation away from the averaged smooth spacetime which is not a priori given. This is also what we expect to hold for our universe. To clarify this rather vague idea two small parameters ϵ and κ were introduced in the approximation scheme to characterize our universe. The ϵ is associated with the amplitude of the gravitational potential (ϕ) generated by inhomogeneous distribution of matter, $\phi \sim \epsilon^2$. The κ is the ratio between the typical scale of inhomogeneity (l) and the scale of the horizon (L), $\kappa \sim l/L$. The relative size of κ and ϵ depends on the system we have in mind. Since the density contrast is of order of ϵ^2/κ^2,[10] the linear and nonlinear stages may be characterized by the conditions $\kappa \gg \epsilon$ and $\epsilon \gg \kappa$, respectively. For example, typical values of ϵ and κ for galaxies are $\epsilon \sim 10^{-3}$ and $\kappa \sim 10^{-4.5}$. The ratio ϵ/κ gets larger as we consider smaller regions. The scheme allows one to construct an approximate metric with arbitrary large density contrast as far as $\epsilon^2 \ll \kappa$.

The propagation problem is then treated on the basis of this approximation.[15] It is found that the linearized approximation can be safely used to study the propagation of light rays in an inhomogeneous universe in which the density contrast is much larger than unity as long as one focuses on a region sufficiently smaller than the horizon scale, i.e., z (redshift of the source object)$\ll 1$. Moreover a general expression for the distance-redshift relation is derived in such a universe without and ad hoc assumption. Using the relation we were able to clarify the range of validity of the Dyer-Roeder distance relation.

Unfortunately as mentioned above the validity of the above approximation is restricted in the parameter range $\epsilon^2 \ll \kappa$. Thus the above analysis applies for inhomogeneities whose typical size is larger than galactic scales. However one might be interested in the light propagation in the situation with strong gravity and/or smaller regions where $\epsilon^2 \gg \kappa$. For this purpose one has to construct an approximate metric in such a situation. This is what the author aims at in this paper. It has also some conceptual interest to construct an averaged spacetime in such a situation.

The plan of this paper is as follows. In § 2, the basic equations are presented and then the previous approximation is outlined for the sake of completeness. In § 3, new approximation is introduced. There we present only the logical steps of the approximation without going to details of the calculation. We shall demonstrate explicit calculations of lower order terms in the Appendix. The discussion on the spatial averaging and other type of averaging is given in § 4. Finally some discussions are given in § 5.

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§ 2. The basic equations and the former approximation

In order to make this paper self-contained, we shall first present basic equations for the approximation and then outline the former approximation.[10),11)] As in the previous paper we introduce two small parameters ϵ and κ. The physical meaning of these parameters is explained in the introduction.

Let us make the following ansatz for the metric:

$$g_{\mu\nu}=a^2(\eta)(\gamma_{\mu\nu}+h_{\mu\nu}), \tag{2·1}$$

where a is the scale factor which describes the averaged expansion and is assumed to be a function of the conformal time η. It is also assumed that $a'/a=O(1/L)$ where the prime represents the derivative with respect to η. The h's are supposed to be generated by inhomogeneous distribution of matter and by possible gravitational waves. We neglect the latter contribution and assume that $h_{\mu\nu}=O(\epsilon^2)$ and $h_{\mu\nu,\rho}=O(\epsilon^2/l)$. We assume that the spacetime considered here reduces to the closed, flat, or open FLRW spacetime depending on the curvature of the spatial section $K=+1$, 0 or -1, respectively, when matter distributes homogeneously and $h_{\mu\nu}$ vanish identically.

The above ansatz for the metric and the ordering are used to expand the Einstein equations in terms of ϵ and κ. In the calculation it is convenient to use the trace-reversed perturbation in the harmonic gauge defined by

$$\bar{h}^{\mu\nu}=h^{\mu\nu}-\frac{1}{2}\gamma^{\mu\nu}h,$$

$$\bar{h}^{\mu\nu}{}_{|\nu}=0, \tag{2·2}$$

where $h=\gamma^{\mu\nu}h_{\mu\nu}$. The indices on h are shifted by γ and the bar $|$ indicates the covariant derivative with respect to γ.

The result may be expressed as follows:

$$\left(\frac{a'}{a}\right)^2(4\gamma^{\mu\eta}\gamma^{\nu\eta}-\gamma^{\mu\nu}\gamma^{\eta\eta})-2\left(\frac{a''}{a}\right)(\gamma^{\mu\eta}\gamma^{\nu\eta}-\gamma^{\eta\eta}\gamma^{\mu\nu})+A^{\mu\nu}$$
$$+\left(\frac{a'}{a}\right)\left(2\bar{h}^{\eta(\mu|\nu)}-\bar{h}^{\mu\nu|\eta}-\gamma^{\eta(\mu}\bar{h}^{|\nu)}+\frac{1}{2}\gamma^{\mu\nu}\bar{h}^{|\eta}\right)-\frac{1}{2}\bar{h}^{\mu\nu|\rho}{}_{|\rho}=8\pi G\tau^{\mu\nu}, \tag{2·3}$$

where $A^{\mu\nu}$ is the background spatial curvature term given by $A^{\eta\eta}=-3K\gamma^{\eta\eta}=3K$, $A^{ij}=-K\gamma^{ij}$ and $A^{\eta i}=0$. One may regard $\tau^{\mu\nu}=a^4T^{\mu\nu}+t^{\mu\nu}$ as the effective stress energy tensor, where $T^{\mu\nu}$ is the material stress energy tensor and $t^{\mu\nu}$ is a gravitational stress energy pseudotensor which consists of terms quadratic in $\bar{h}$.

In the above calculation terms like $\bar{h}_{|\rho}\bar{h}_{|\sigma}\bar{h}$, $(a'/a)\bar{h}_{|\rho}\bar{h}$ and $(a''/a)\bar{h}$ have been neglected. These are of the order of $O(\epsilon^6/l^2)$, $O(\epsilon^4/lL)$ and $O(\epsilon^2/L^2)$, respectively, and may safely be neglected because we are interested here in the situation where $\epsilon^2 \ll \kappa$.

We shall take a perfect fluid as an example of the material source:

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$$T^{\mu\nu}=[\rho+p(\rho)]u^\mu u^\nu+p(\rho)g^{\mu\nu}\,. \tag{2·4}$$

It is more convenient to work with conformally rescaled variables:

$$\tilde{u}^\mu=au^\mu\,,\quad \tilde{g}^{\mu\nu}=a^2g^{\mu\nu}\,. \tag{2·5}$$

Then the total effective stress energy tensor may be written as follows:

$$\tau^{\mu\nu}=a^2\tilde{T}^{\mu\nu}+t^{\mu\nu}\,, \tag{2·6}$$

where

$$\tilde{T}^{\mu\nu}=[\rho+p]\tilde{u}^\mu u^\nu+p\tilde{g}^{\mu\nu}=a^2T^{\mu\nu}\,. \tag{2·7}$$

In the previous approach the spatial averaging is directly taken to the above equations (2·3) assuming spatial periodicity of the material initial data as well as of the free data for the gravitational field. The spatial averaging over a volume V is defined as usual.

$$\langle Q\rangle=V^{-1}\int_V QdV\,. \tag{2·8}$$

The spatial periodicity implies $\langle Q_{|i}\rangle=0$. Moreover we have assumed $\langle\tau^{\eta i}\rangle=0$ expressing no coherent motion over the volume to be averaged. The spatial average of the field equation (2·3) shows that $\langle\bar{h}^{\eta i}\rangle=0$ under this requirement. The spatial average of the gauge condition (2·2) implies that $\langle\bar{h}_{\eta\eta}\rangle$ is constant and a suitable redefinition of the time variable and the scale factor allows one to put the constant zero without loss of generality. Furthermore one may put $\langle\bar{h}^k{}_k\rangle$ zero also because it expresses an additional isotropic expansion and the effect is absorbed into the scale factor by an appropriate redefinition of time and the scale factor.

Under these conditions the averaged Einstein equation (2·3) may be written as

$$\left(\frac{a'}{a}\right)^2=\frac{8\pi G}{3}\langle\tau^{\eta\eta}\rangle-K\,, \tag{2·9}$$

$$\frac{a''}{a}=\frac{4\pi G}{3}\langle\tau^{\eta\eta}-\tau^k{}_k\rangle-K\,, \tag{2·10}$$

$$\frac{1}{a^2}(a^2\langle\bar{h}^{ij}\rangle_{|\eta})_{|\eta}=16\pi G\langle\hat{\tau}^{ij}\rangle\,, \tag{2·11}$$

where $\hat{\tau}^{ij}=\tau^{ij}-\frac{1}{3}\gamma^{ij}\tau^k{}_k$ is the trace free part of τ^{ij}. The averaged line element takes the following form:

$$\langle ds^2\rangle=a^2[-d\eta^2+(\gamma_{ij}+\langle\bar{h}_{ij}\rangle)dx^idx^j]\,. \tag{2·12}$$

Thus $\langle\bar{h}_{ij}\rangle$ express the deviation from the isotropic expansion due to the inhomogeneities $\langle\hat{\tau}^{ij}\rangle$[16] and the averaged spacetime expands anisotropically except if $\langle\bar{h}_{ij}\rangle$ vanishes identically. Equations (2·9) and (2·10) are the same as the equations of the FLRW model except that the source terms are replaced by the total effective stress energy pseudotensor including gravitational contribution. Thus the effect of local inhomogeneity on the global expansion may be expressed partly by the effective density $\rho_{\rm eff}=a^2\langle\tau^{\eta\eta}\rangle$ and the effective pressure $p_{\rm eff}=\frac{1}{3}a^2\langle\tau^k{}_k\rangle$.

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§ 3. New approximation scheme

Now we shall introduce a new approximation scheme which is free from the restriction $\epsilon^2 \ll \kappa$. To do this we first point out how one gets the restriction in the former approach. When one takes the spatial averaging over the right-hand side of (2·3), one naturally expects that the contribution from $t^{\mu\nu}$ should be smaller than that from $T^{\mu\nu}$. Since the spatial average of $T^{\mu\nu}$ generates the global expansion of the universe, their order should be $O(1/L^2)$. On the other hand, the spatial average of $t^{\mu\nu}$ is of the order $O(\epsilon^4/l^2)$. Therefore the condition $\langle T^{\mu\nu}\rangle \gg \langle t^{\mu\nu}\rangle$ brings about the restriction $\epsilon^2 \ll \kappa$ to the above approximation.

This situation may be improved by noting that the main part of $t^{\mu\nu}$ remains the same when the static limit is taken. The static limit here means that one takes the limit where the mean density vanishes and thus there is no global expansion. Equation (2·3) reduces to the usual reduced Einstein equation for the isolated system[17] and thus $t^{\mu\nu}$ should be balanced with higher order contribution in $\bar{h}^{\mu\nu|\rho}{}_{|\rho}$ in (2·3). Thus if one solves the local problem first, namely the equation without cosmological expansion, it is expected that the main contribution of the term $t^{\mu\nu}$ disappears from (2·3) and thus the restriction $\epsilon^2 \ll \kappa$ will disappear as well.

For the clarity of the argument, we presented here only the logical steps of the approximation without any details of the calculation. For the convenience of the reader we shall demonstrate explicit calculations of lower order terms in the present scheme in the Appendix.

To put the above idea in the mathematical base, we introduce the following notations for ρ and $\bar{h}$:

$$\rho = \rho_b + \delta\rho\,, \tag{3·1}$$

$$\bar{h}^{\mu\nu} = \bar{h}_0{}^{\mu\nu} + l^{\mu\nu}\,, \tag{3·2}$$

where ρ_b is the averaged background density whose evolution is not specified a priori and is of the order of $O(1/L^2)$ since it generates the global expansion. The static limit means $L\to\infty$ or $\kappa\to 0$. Thus $\rho_b \to 0$ in this limit. On the other hand the density perturbation $\delta\rho$ expresses local inhomogeneities and is of the order of $O(\epsilon^2/l^2)$. The stress energy tensor may be then decomposed into the two parts:

$$\tilde{T}^{\mu\nu} = \tilde{T}_b{}^{\mu\nu} + \delta\tilde{T}^{\mu\nu}\,, \tag{3·3}$$

where $T_b{}^{\mu\nu}$ is the stress energy tensor generated by the background homogeneous density given by

$$\tilde{T}_b{}^{\mu\nu} = \rho_b \delta_0{}^\mu \delta_0{}^\nu\,. \tag{3·4}$$

In writing this expression for $\tilde{T}_b{}^{\mu\nu}$, we have implicitly assumed the existence of a particular spatial hypersurface on which the 4 velocity of the homogeneous fluid has no spatial components. Here is a silent point in our approach. We shall discuss this point in detail in the next section and for the moment we shall simply assume the above decomposition.

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In (3·2) $\bar{h}_0^{\mu\nu}$ express the metric in the static limit and thus are generated purely by $\delta\rho$ and the proper 3-velocity of the fluid. Namely, $\bar{h}_0$ solves the equation (2·3) without the cosmological expansion, i.e., $a=1$.

$$\bar{h}_0{}^{\mu\nu,\rho}{}_{,\rho}=-16\pi G(\delta\tilde{T}_0{}^{\mu\nu}+t_0{}^{\mu\nu}), \tag{3·5}$$

where the subscript 0 in T and t represents the corresponding quantities evaluated in the static limit and thus $t_0{}^{\mu\nu}$ is the gravitational stress energy pseudotensor constructed from $\bar{h}_0{}^{\mu\nu}$. Equation (3·5) insures that $\bar{h}_0{}^{\mu\nu}$ are of the order of $O(\epsilon^2)$. On the other hand the $l^{\mu\nu}$ are supposed to be generated by the coupling of the expansion with the local gravitational effect and they will be of the order of $O(\kappa\epsilon^2)$ as shown below.

It should be noted that Eq. (3·5) is the leading order expression for the expansion of the full field equation (2·3) as far as $\epsilon\gg\kappa$ which always holds if we consider the nonlinear situations. Thus our approach is similar to Isaacson's work in the vacuum case in spirit[18] and the following method applies not only to the situation $\epsilon^2\gg\kappa$ but also to any situation where $\epsilon\gg\kappa$. For simplicity we shall only present the actual calculation for the situation $\epsilon^2\gg\kappa$. In this situation there may be in general two typical scales, one for the region in which $\epsilon^2\gg\kappa$ and the other for the region in which $\epsilon^2\ll\kappa$. Then one has probably to introduce two spatial averaging. If we take one scale as a stellar size and the other as the size of a galaxy, then one might first average over the scale of galaxy and then average over some large volume. However we will not consider such a complication in this paper and we will idealize the situation where there is one scale for the averaging.[19]

Our scheme consists of the following steps. First solve the leading local equation (3·5). Second subtract (3·5) from (2·3) to obtain the equation to be averaged over.

$$\begin{aligned}&\left(\frac{a'}{a}\right)^2(4\gamma^{\mu\eta}\gamma^{\nu\eta}-\gamma^{\mu\nu}\gamma^{\eta\eta})-2\left(\frac{a''}{a}\right)(\gamma^{\mu\eta}\gamma^{\nu\eta}-\gamma^{\mu\nu}\gamma^{\eta\eta})+A^{\mu\nu}\\&\quad+\left(\frac{a'}{a}\right)\left(2\bar{h}_0{}^{\eta(\mu|\nu)}-\bar{h}_0{}^{\mu\nu|\eta}-\gamma^{\eta(\mu}\bar{h}_0{}^{|\nu)}+\frac{1}{2}\gamma^{\mu\nu}\bar{h}_0{}^{|\eta}\right)-\frac{1}{2}l^{\mu\nu|\rho}{}_{|\rho}\\&\quad=8\pi G(a^2\tilde{T}_b{}^{\mu\nu}+\varDelta\tilde{T}^{\mu\nu}+\varDelta t^{\mu\nu}),\end{aligned} \tag{3·6}$$

where we have neglected higher order terms and $\varDelta\tilde{T}^{\mu\nu}$ is given by

$$\varDelta\tilde{T}^{\mu\nu}=a^2\delta\tilde{T}^{\mu\nu}-\delta\tilde{T}_0{}^{\mu\nu}. \tag{3·7}$$

We need explicit expression for $\varDelta\tilde{T}^{\mu\nu}$ below.

$$\begin{aligned}&\varDelta\tilde{T}^{\mu\nu}=a^2\rho_b(\tilde{u}^\mu\tilde{u}^\nu-\delta_0{}^\mu\delta_0{}^\nu)+(a^2-1)\delta\tilde{T}_0{}^{\mu\nu},\\&\delta\tilde{T}_0{}^{\mu\nu}=(\delta\rho+p)\tilde{u}^\mu\tilde{u}^\nu+p\tilde{g}_0{}^{\mu\nu},\end{aligned} \tag{3·8}$$

where $\tilde{g}_0{}^{\mu\nu}$ is the metric in the static limit and thus does not contain $l^{\mu\nu}$. On the other hand $\varDelta t^{\mu\nu}$ is given by

$$\varDelta t^{\mu\nu}=t^{\mu\nu}-t_0{}^{\mu\nu} \tag{3·9}$$

and is of the order of $O(\epsilon^2/lL)$ because the leading terms are the product of $\bar{h}_0{}^{\mu\nu}{}_{|\sigma}=O(\epsilon^2/l)$ and $l^{\mu\nu}{}_{|\sigma}=O(\epsilon^2/L)$.

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Third we take the spatial average (3·6) over a volume V.

$$\left(\frac{a'}{a}\right)^2(4\gamma^{\mu\eta}\gamma^{\nu\eta}-\gamma^{\mu\nu}\gamma^{\eta\eta})-2\left(\frac{a''}{a}\right)(\gamma^{\mu\eta}\gamma^{\nu\eta}-\gamma^{\mu\nu}\gamma^{\eta\eta})+A^{\mu\nu}$$

$$+\left(\frac{a'}{a}\right)\left\langle\left(2\bar{h}_0{}^{\eta(\mu|\nu)}-\bar{h}_0{}^{\mu\nu|\eta}-\gamma^{\eta(\mu}\bar{h}_0{}^{|\nu)}+\frac{1}{2}\gamma^{\mu\nu}\bar{h}_0{}^{|\eta}\right)\right\rangle-\frac{1}{2}\langle l^{\mu\nu|\rho}{}_{|\rho}\rangle$$

$$=8\pi G(a^2\langle\tilde{T}_b{}^{\mu\nu}\rangle+\langle\Delta\tilde{T}^{\mu\nu}+\Delta t^{\mu\nu}\rangle)\,. \tag{3·10}$$

The rest of the calculation of the left-hand side of (3·10) is the same as in the former approach under the condition $\langle\bar{h}_0{}^{\eta\eta}\rangle=\langle\bar{h}_0{}^{\eta i}\rangle=\langle\bar{h}_{0k}^{k}\rangle=0$. In particular $\langle l^{\mu\nu|\rho}{}_{|\rho}\rangle=0$ because we have also assumed the periodic boundary condition as well as the above condition.

Thus the averaged Einstein equation (3·10) may be written as follows:

$$\left(\frac{a'}{a}\right)^2=\frac{8\pi G}{3}(a^2\rho_b+\langle\Delta\tau^{\eta\eta}\rangle)-K\,, \tag{3·11}$$

$$\frac{a''}{a}=\frac{4\pi G}{3}(a^2\rho_b+\langle\Delta\tau^{\eta\eta}-\Delta\tau^k{}_k\rangle)-K\,, \tag{3·12}$$

$$\frac{1}{a^2}(a^2\langle\bar{h}^{ij}\rangle_{|\eta})_{|\eta}=16\pi G\langle\Delta\hat{\tau}^{ij}\rangle\,, \tag{3·13}$$

where $\Delta\tau^{\mu\nu}=\Delta\tilde{T}^{\mu\nu}+\Delta t^{\mu\nu}$ and $\Delta\hat{\tau}^{ij}=\Delta\tau^{ij}-(1/3)\gamma^{ij}\Delta\tau^k{}_k$ is the trace free part of τ^{ij}.

The equation for $l^{\mu\nu}$ is obtained by subtracting the above averaged equations from (3·10).

$$l^{\mu\nu|\rho}{}_{|\rho}=2\left(\frac{a'}{a}\right)\left(2\bar{h}_0{}^{\eta(\mu|\nu)}-\bar{h}_0{}^{\mu\nu|\eta}-\gamma^{\eta(\mu}\bar{h}_0{}^{|\nu)\eta}+\frac{1}{2}\gamma^{\mu\nu}\bar{h}_0{}^{|\eta}\right). \tag{3·14}$$

As is easily seen from (3·11) and (3·12), our approximation is valid as far as the contribution from background density ρ_b dominates that from the post-Newtonian correction like $\langle\delta\rho v^i v^j\rangle$ as well as from the gravitational stress energy tensor $\langle\Delta t^{\mu\nu}\rangle$. Thus it is valid as far as $\epsilon^4\ll\kappa$. In the local metric the terms associated with the cosmological expansion appear between the first and the second post-Newtonian terms when $\epsilon^2\ll\kappa$, between the second and the third post Newtonian terms when $\epsilon^2\gg\kappa$.

§ 4. On the averaging

In this section we shall discuss the averaging. First of all we should note that there is no covariant definition of spatial averaging available in general spacetime; we thus have to restrict our spacetime to those in which there is a well defined meaning of the spatial averaging. Since we would like to describe our universe as a perturbation of a FLRW spacetime, it seems natural to assume that we can choose spatial slices on which the metric deviation away from the FLRW metric remains everywhere small. We define our averaging in one of such geometrically preferred slices.

Since we have introduced the spatial averaging, we cannot treat the situation where there are singularities. Even if our universe contains singularities, one still

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expects that FLRW metric will be a very good approximation far away from singularities. Thus one would like to have some averaging scheme which applies also to a spacelike slice with singularities.

One possible method would be a statistical averaging. There we consider a statistical ensemble which contains all possible density and velocity distribution of fluid elements (galaxies) with some constraints. These constraints will characterize the universe we wish to approximate. If we choose the particular ensemble in which the density and velocity distribution satisfy the condition $\langle\delta\rho\rangle=\langle v^i\rangle=0$ and the averaging of any quantity with spatial derivative vanishes, then the calculation of the averaging will be identically the same as was done here by means of the spatial averaging, but it would avoid the difficulty mentioned above.

Since the light propagation is somewhat statistical phenomena, the statistical averaging seems to have another advantage at least conceptually to treat such a problem consistently. Much work will be necessary to clarify this respect.

§ 5. Conclusion

We have discussed in this paper a new approximation scheme for constructing a realistic inhomogeneous universe within the framework of general relativity. The present approach solves the local equations first and then takes an average of the Einstein equations. It allows us to approximate the universe as far as $\epsilon^4 \ll \kappa$ and thus improves the former one.

When $\epsilon^2 \gg \kappa$, the first post-Newtonian metric is larger than the metric due to the cosmic expansion. Thus one has to take the effect of cosmic expansion into account in the study of light propagation on such a spacetime only if one is interested in higher order effects than the first post-Newtonian effects.

We have also discussed another possibility of averaging, namely, the statistical averaging. It may well be possible that this way of averaging is able to treat more general spacetime with singularities.

The present approximation scheme may be used as the basis of the investigation of light propagation in an inhomogeneous universe. For example, it was found that the effect of shear along light rays on the distance-redshift relation may be negligible as far as $\epsilon^2 \leq \kappa$, but no consistent treatment of the shear is known when $\epsilon^2 \gg \kappa$.[6)]

One of the reasons is the lack of the approximate metric for such a spacetime. It is expected that our approximation plays an essential role for such a study. We leave such applications of the present scheme in future publications.

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Appendix

For the sake of convenience for the reader we shall present here explicit calculatios for lower order terms according to the present scenario. Here we follow basically a geometrical formulation of the post-Newtonian approximation.[17] There a sequence of solutions of Einstein equation parametrized by ϵ is formally constructed and then the Newtonian limit is defined as the limit $\epsilon \to 0$ along the congruence on which the Newtonian dynamical time ($\epsilon\eta$ in our case) stays constant. This means physically that the typical time-scale gets longer as ϵ^{-1} as the velocity goes to zero ($\epsilon \to 0$) as it should be in Newtonian physics. All this is automatically taken into account by introducing Newtonian dynamical time as the time coordinate. For notational simplicity, we simply replace η by $\epsilon^{-1}\eta$ and regard η as the Newtonian time. This has the effect of replacing $g^{\eta\eta} \to \epsilon^2 g^{\eta\eta}$, $g_{\eta\eta} \to \epsilon^{-2} g_{\eta\eta}$. We shall restrict ourselves to the $K=0$ case. As one can see from this expression, the limiting metric is degenerate. This is the Newtonian spacetime and our zeroth order spacetime. As usual in the post-Newtonian formalism, we shall explicitly introduce the following order for the material variables:

$$\delta\rho \sim O(\epsilon^2)\,, \quad p \sim O(\epsilon^4)\,, \quad v \sim O(\epsilon)\,. \tag{A·1}$$

For simplicity, we shall replace $\rho \to \epsilon^2\rho$, $p \to \epsilon^4 p$ and $v^i \to \epsilon v^i$. Then the $\delta\tilde{T}_0{}^{\mu\nu}$ in the lowest order are as follows:

$$\begin{aligned}
&\delta\tilde{T}_0{}^{\eta\eta} = \epsilon^4\delta\rho\,,\\
&\delta\tilde{T}_0{}^{\eta i} = \epsilon^4\delta\rho v^i\,,\\
&\delta\tilde{T}_0{}^{ij} = \epsilon^4(\delta\rho v^i v^j + p\delta^{ij})\,.
\end{aligned} \tag{A·2}$$

All these components become the same order because of our choice of coordinate (Newtonian time as the time coordinate). In this ordering, the Newtonian order is the order ϵ^4 in $\bar{h}^{\eta\eta}$ and the first post-Newtonian orders are the order ϵ^6 in $\bar{h}^{\eta\eta}$ and the order ϵ^4 in $\bar{h}^{\eta i}$ and $\bar{h}^{ij}$. Using these expressions, one obtains the lowest order equations for metric perturbation in the static limit:

$$\begin{aligned}
&\Delta\bar{h}_0{}^{\eta\eta} = -16\pi G\epsilon^4\delta\rho\,,\\
&\Delta\bar{h}_0{}^{\eta i} = -16\pi G\epsilon^4\delta\rho v^i\,.
\end{aligned} \tag{A·3}$$

Using these expressions, we then calculate the gravitational stress energy pseudotensor in the lowest order:

$$\begin{aligned}
&t_0{}^{\eta\eta} = -\epsilon^6(8\pi G)^{-1}[4\phi\Delta\phi + 3(\nabla\phi)^2]\,,\\
&t_0{}^{\eta i} = -\epsilon^6(8\pi G)^{-1}(-6\phi^{,i}V^k_{,k} + 4\phi_{,k}V^{k,i} + 4V^k\phi_k{}^{i} - 4\pi G\rho V^i - 4\pi G v^i\phi)\,,\\
&t_0{}^{ij} = \epsilon^4(8\pi G)^{-1}[-2\phi^{,i}\phi^{,j} + 4\phi\phi^{,ij} + \delta^{ij}(4\phi\nabla\phi + 3(\nabla\phi)^2)]\,,
\end{aligned} \tag{A·4}$$

where ϕ and V^i are defined as follows:

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$$\bar{h}_0{}^{\eta\eta}=\epsilon^4 4\phi\,,\quad \bar{h}_0{}^{\eta i}=\epsilon^4 4\,V^i\,. \tag{A·5}$$

Thus the lowest order of $\bar{h}^{ij}$ obeys the following equation:

$$\Delta\bar{h}_0{}^{ij}=-16\pi G\epsilon^4[\delta\rho v^i v^j+p\delta^{ij} +(8\pi G)^{-1}\{-2\phi^{,i}\phi^{,j}+4\phi\phi^{,ij}+\delta^{ij}(4\phi\nabla\phi+3(\nabla\phi)^2\}]\,. \tag{A·6}$$

Next we calculate the first post-Newtonian order, i.e., order ϵ^6 in $\tilde{T}^{\eta\eta}$. For the calculation we need an explicit expression for $\tilde{u}^\mu$:

$$\tilde{u}^\eta=1+\epsilon^2\left(\frac{1}{2}v^2+\phi\right)+O(\epsilon^4)\,,$$

$$\tilde{u}^i=v^i\tilde{u}^\eta\,, \tag{A·7}$$

where $v^2=\gamma_{ij}v^i v^j$, $v^i=dx^i/d\eta$. Using this expression, we have

$$\tilde{T}_0{}^{\eta\eta}=\epsilon^4\delta\rho+\epsilon^6(v^2+2\phi)+O(\epsilon^8)\,. \tag{A·8}$$

We thus obtain the following expressions for the total effective stress-energy pseudotensor:

$$\tau_0{}^{\eta\eta}=\epsilon^4\delta\rho+\epsilon^6[(v^2+2\phi)-(8\pi G)^{-1}(4\phi\Delta\phi+3(\nabla\phi)^2)]+O(\epsilon^8)\,,$$

$$\tau_0{}^{ij}=G\epsilon^4[\delta\rho v^i v^j+p\delta^{ij} +(8\pi G)^{-1}\{-2\phi^{,i}\phi^{,j}+4\phi\phi^{,ij}+\delta^{ij}(4\phi\nabla\phi+3(\nabla\phi)^2)\}]+O(\epsilon^6)\,. \tag{A·9}$$

Up to this point the calculation is exactly the same with the usual post Newtonian approximation.

Now we calculate $\Delta\tilde{T}^{\mu\nu}$ and $\Delta t^{\mu\nu}$. In this case we have a non-vanishing background density ρ_b which generates the global expansion. Using the definition (3·8),

$$\Delta\tilde{T}^{\eta\eta}=\epsilon^2 a^2\rho_b(v^2+2\phi)+\epsilon^4(a^2-1)\delta\rho+\epsilon^6(a^2-1)\delta\rho(v^2+2\phi)+O(\rho_b\epsilon^4,\,\epsilon^6)\,,$$

$$\Delta\tilde{T}^{ij}=\epsilon^2\rho_b v^i v^j+\epsilon^4(a^2-1)\delta\rho(v^i v^j+p\delta^{ij})+O(\rho_b\epsilon^4,\,\epsilon^6)\,. \tag{A·10}$$

Before the calculation of $\Delta t^{\mu\nu}$, we need equations for $l^{\mu\nu}$, i.e., (3·14) which becomes in the lowest order:

$$\Delta l^{\eta\eta}=\epsilon^4 4\left(\frac{a'}{a}\right)(V^i_{,i}-Z^i_{i,\eta})\,,$$

$$\Delta l^{\eta i}=\epsilon^4 4\left(\frac{a'}{a}\right)(\phi^{,i}+Z^{k,i}_k)\,,$$

$$\Delta l^{ij}=\epsilon^4 4\left(\frac{a'}{a}\right)(4\,V^{i,j}-2Z^{ij,\eta}-\delta^{ij}(V_k{}^k-Z_k{}^{k,\eta}))\,, \tag{A·11}$$

where we define $\bar{h}_0{}^{ij}=4Z^{ij}$.

The expressions for $t^{\mu\nu}$ are obtained from (A4) by substituting $\bar{h}_0{}^{\mu\nu}+l^{\mu\nu}$ instead of $\bar{h}_0{}^{\mu\nu}$. We need only $\eta\eta$ and ij components for our purpose since we assume $\langle\tau^{\eta i}\rangle=0$. Using the above expression for $l^{\eta\eta}\equiv(a'/a)\Sigma$, we obtain

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$$\Delta t^{\eta\eta} = -\epsilon^6 (8\pi G)^{-1} \frac{a'}{a} [4(\phi\Delta\Sigma + \Sigma\Delta\phi) + 6(\Delta\phi\Delta\Sigma)],$$

$$\Delta t^{ij} = \epsilon^4 (8\pi G)^{-1} \frac{a'}{a} [-\phi^{(,i}\Sigma^{,j)} + 4(\phi\Sigma^{,ij} + \Sigma\phi^{ij}) + \delta^{ij} 4(\phi\Delta\Sigma + \Sigma\Delta\phi) + 6(\Delta\phi\Delta\Sigma)],$$
$$\tag{A·12}$$

where we have neglected $O(\Sigma^2)$. The average of $\Delta\tau^{\mu\nu}$ is now easily calculated by noting $\langle\delta\rho\rangle = \langle v^i\rangle = \langle\phi\rangle = 0$.

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