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A new description of the Bowen–Margulis measure

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Abstract. The Bowen–Margulis measure on the unit tangent bundle of the universal covering of a compact manifold of negative curvature is determined by its restriction to the leaves of the strong unstable foliation. We describe this restriction to any strong unstable manifold W as a spherical measure with respect to a natural distance on W.

Let M be a compact connected Riemannian manifold of negative curvature $-\infty < -b^2 \leq K \leq -a^2 < 0$ and fundamental group Γ. The *geodesic flow* g^t acts on the unit tangent bundle $S\tilde{M}$ of the universal covering $\tilde{M}$ of M. $S\tilde{M}$ admits foliations W^{ss}, W^s, W^{su}, W^u which are invariant under g^t and the action of Γ on $S\tilde{M}$. The leaves of W^{ss} (resp. W^s, W^{su}, W^u) are called the *strong stable* (resp. *stable, strong unstable, unstable*) manifolds of $S\tilde{M}$ (see [**6**]) We write $A \subset W^i$ if $A \subset S\tilde{M}$ is contained in a leaf of W^i ($i = ss, s, u, su$).

The *Bowen–Margulis measure* $\tilde{\mu}$ on $S\tilde{M}$ is the lift to $S\tilde{M}$ of the unique g^t-invariant Borel-probability measure on SM of maximal entropy ([**2**], [**6**]). $\tilde{\mu}$ has natural restrictions to measures $\tilde{\mu}^i$ on the leaves of W^i ($i = ss, s, u, su$) and is determined by $\tilde{\mu}^{su}$.

The purpose of this paper is to show that for every $v \in S\tilde{M}$ the measure $\tilde{\mu}^{su}$ on the leaf $W^{su}(v)$ of W^{su} containing v is a spherical measure with respect to a natural distance on $W^{su}(v)$. In order to define this distance we have to fix some notations:

For $v \in S\tilde{M}$ let φ_v be the geodesic line in $\tilde{M}$ with initial direction $\varphi'_v(0) = v$. φ_v determines a point $\varphi_v(-\infty) = \xi$ of the *ideal boundary* $\partial\tilde{M}$ of $\tilde{M}$. $W^u(v)$ then consists of all unit tangent vectors of geodesic lines γ in $\tilde{M}$ which satisfy $\gamma(-\infty) = \xi$. In particular the restriction to $W^u(v)$ of the canonical projection $P: S\tilde{M} \to \tilde{M}$ is a diffeomorphism of $W^u(v)$ onto $\tilde{M}$.

$v \in S\tilde{M}$ determines a *Busemann function* θ_v at ξ which is normalized by $\theta_v\varphi_v(0) = 0$. For $t \in \mathbb{R} \cup \{\infty\}$ denote by $\pi_{t,v}$: $\tilde{M} \cup (\partial\tilde{M} - \xi) \to \theta_v^{-1}(t)$ the projection along the geodesics which are asymptotic to ξ. Then for every $y \in \partial\tilde{M} - \xi$ the curve γ: $t \to \pi_{t,v}(y)$ is the unique unit-speed geodesic in $\tilde{M}$ with $\gamma'(0) \in W^{su}(v)$ and $\gamma(\infty) = y$.

The projection $\pi: S\tilde{M} \to \partial\tilde{M}$, $w \to \varphi_w(\infty)$ maps $W^{su}(v)$ homeomorphically onto $\partial\tilde{M} - \xi$ and $\pi(w) = \pi_{\infty,w} \circ P(w)$ for all $w \in S\tilde{M}$. If $w \in W^{su}(v)$ then $\varphi_w(-\infty) = \varphi_v(-\infty)$ and $\theta_w = \theta_v$, hence $\pi_{t,w} = \pi_{t,v}$ for all $t \in \mathbb{R}$.

In the sequel we will suppress the index v of the various objects depending on $v \in S\tilde{M}$ whenever v is arbitrarily fixed.

The level sets $\theta^{-1}(t)$ of $\theta(t\in\mathbb{R})$ are C^1-manifolds, the *horospheres* through ξ. The restriction of the Riemannian metric to $\theta^{-1}(t)$ induces a distance $d_{t,v}=d_t$ on $\theta^{-1}(t)$. Fix $R>0$ and define, for $x, y\in\partial\tilde{M}-\xi$, $f(x,y)=\sup\{t\in\mathbb{R}\mid d_t(\pi_t(x),\pi_t(y))\le R\}$. The function $\eta=\eta_{v,R}:(\partial\tilde{M}-\xi)\times(\partial\tilde{M}-\xi)\to\mathbb{R}_+$, $(x,y)\to e^{-f(x,y)}$ is symmetric and $\eta(x,y)=0$ if and only if $x=y$.

Using the upper curvature bound $-a^2$ on $\tilde{M}$, η can be estimated as follows:

LEMMA 1. *If $x, y\in\partial\tilde{M}-\xi$ and $d_t(\pi_t(x),\pi_t(y))=\varepsilon\le R$, then*

$$\eta(x,y)\ge e^{-t}\left(\frac{\varepsilon}{R}\right)^{1/a}.$$

Proof. Let $\tau=a^{-1}(\log R/\varepsilon)$; then $K\le -a^2$ implies by the estimates in [4] that $d_{t+\tau}(\pi_{t+\tau}(x),\pi_{t+\tau}(y))\ge R$. Thus $\eta(x,y)\ge e^{-t}(\varepsilon/R)^{1/a}$. □

As a corollary we find how $\eta_{v,R}$ varies with $R>0$:

COROLLARY 2. *If $0<r<R$ then $\eta_{v,R}\le\eta_{v,r}\le(R/r)^{1/a}\eta_{v,R}$.*

Proof. Let $x, y\in\partial\tilde{M}-\xi$ and $t=-\log(\eta_{v,r}(x,y))$. Then $d_t(\pi_t(x),\pi_t(y))=r$ hence $\eta_{v,R}(x,y)\ge\eta_{v,r}(x,y)(r/R)^{1/a}$ by Lemma 1. Moreover clearly $\eta_{v,R}\le\eta_{v,r}$. □

COROLLARY 3. *$\eta^a:(x,y)\to(\eta(x,y))^a$ is a distance on $\partial\tilde{M}-\xi$.*

Proof. We have to check the triangle inequality. For this let $x, y, z\in\partial\tilde{M}-\xi$ and $t=-\log(\eta(x,y))$, i.e. $d_t(\pi_t(x),\pi_t(y))=R$. Then

$$\eta^a(x,y)\le e^{-at}(d_t(\pi_t(x),\pi_t(z))+d_t(\pi_t(z),\pi_t(y)))/R$$

hence the claim follows from Lemma 1. □

Using the identification of $W^{su}(v)$ with $\partial\tilde{M}-\varphi_v(-\infty)$ via the map π, $\eta^a_{v,R}$ can be viewed as a distance on $W^{su}(v)$. Let h be the topological entropy of the geodesic flow on SM. Our aim is to prove the following

THEOREM. *The measure $\tilde{\mu}^{su}$ on $W^{su}(v)$ equals up to a constant the h/a-dim. spherical measure associated to $\eta^a_{v,R}$.*

It will be convenient to show first the analogous theorem for a slightly different function $\rho=\rho_{v,R}:(\partial\tilde{M}-\xi)\times(\partial\tilde{M}-\xi)\to\mathbb{R}_+(v\in S\tilde{M}, R>0)$ which is defined as $\eta_{v,R}$ but using the distance d on $\tilde{M}$ which is induced by the Riemannian metric: For $x, y\in\partial\tilde{M}-\xi$ let $\bar{f}(x,y)=\sup\{t\in\mathbb{R}\mid d(\pi_t(x),\pi_t(y))\le R\}$ and $\rho(x,y)=e^{-\bar{f}(x,y)}$. Clearly $\rho_{v,R}=\rho_{w,R}$ if $w\in W^{su}(v)$.

ρ is related to η as follows:

LEMMA 4. *There is a number $\nu>0$ such that $\nu\eta\le\rho\le\eta$ on $\partial\tilde{M}-\xi$.*

Proof. If $x, y\in\partial\tilde{M}-\xi$ and $d(\pi_t(x),\pi_t(y))=R$ for some $t\in\mathbb{R}$, then $d_t(\pi_t(x),\pi_t(y))\ge R$ which implies $\rho\le\eta$. To show the first inequality, assume again $d(\pi_t(x),\pi_t(y))=R$. Since the curvature K on $\tilde{M}$ is bounded from below by $-b^2$, it follows from [4] that $d_t(\pi_t(x),\pi_t(y))\le 2/b\sinh(\frac{1}{2}bR)$, i.e. if we define $r=2b^{-1}\sinh(\frac{1}{2}bR)$, then $\eta_{v,r}\le\rho_{v,R}$. The claim now follows from Corollary 2. □

COROLLARY 5. *There is a number $c>0$ such that $\rho(x,z)\le\varepsilon$, $\rho(z,y)\le\varepsilon$ implies $\rho(x,y)\le c\varepsilon$.*

Proof. If $\rho(x,z)\le\varepsilon$ and $\rho(z,y)\le\varepsilon$, then by Lemma 4 $\eta(x,z)$ and $\eta(z,y)$ are not larger than ε/ν. Since η^a satisfies the triangle inequality, this implies $\eta^a(x,y)\le 2(\varepsilon/\nu)^a$. Thus by Lemma 4 $\rho(x,y)\le 2^{1/a}\varepsilon/\nu$. □

LEMMA 6. *If $0<r<R$ then $\rho_{v,R}\le\rho_{\nu,r}\le((\sinh\frac{1}{2}aR)/(\sinh\frac{1}{2}ar))^{1/a}\rho_{v,R}$.*

Proof. Assume that for all $x,y\in\partial\tilde{M}-\varphi_v(-\infty)$ and all $t\in\mathbb{R}$, $s\ge 0$

$$(*)\qquad d(\pi_{t+s}(x),\pi_{t+s}(y))\ge\frac{2}{a}\sinh^{-1}\left(e^{as}\sinh\frac{a}{2}d(\pi_t(x),\pi_t(y))\right)$$

(here again $\pi_t=\pi_{t,v}$). With

$$\tau=\frac{1}{a}\log\left(\left(\sinh\frac{a}{2}R\right)\Big/\left(\sinh\frac{a}{2}r\right)\right)$$

we then obtain $d(\pi_{t+\tau}(x),\pi_{t+\tau}(y))\ge R$ whenever $d(\pi_t(x),\pi_t(y))\ge r$, i.e. $\rho_{v,r}\le e^{\tau}\rho_{v,R}$. Since $\rho_{v,R}\le\rho_{v,r}$ is obvious, it rests to prove formula (*). Consider a comparison situation in the hyperbolic plane H_a of constant curvature $-a^2$, given by a point $\bar{\xi}\in\partial H_a$, a Busemann function $\bar{\theta}$ at $\bar{\xi}$ and geodesic lines $\bar{\gamma}$, $\bar{\varphi}$ in H_a such that $\bar{\gamma}(-\infty)=\bar{\xi}=\bar{\varphi}(-\infty)$, $\bar{\theta}\bar{\gamma}(0)=0=\bar{\theta}\bar{\varphi}(0)$ and $d(\bar{\gamma}(0),\bar{\varphi}(0))=d(\pi_t(x),\pi_t(y))$. Then

$$d(\bar{\gamma}(s),\bar{\varphi}(s))=\frac{2}{a}\sinh^{-1}\left(e^{as}\sinh\frac{a}{2}d(\bar{\gamma}(0),\bar{\varphi}(0))\right)$$

(see [4]) and the comparison arguments in [4] show $d((\pi_{t+s}(x),\pi_{t+s}(y))\ge d(\bar{\gamma}(s),\bar{\varphi}(s))$. □

LEMMA 7. *Let $v\in S\tilde{M}$, $\Omega\subset\partial\tilde{M}-\varphi_v(-\infty)$ compact and $\varepsilon>0$. Then there is a neighbourhood U of v in $S\tilde{M}$ such that*

$$(1-\varepsilon)\rho_{w,R}(x,y)\le\rho_{v,R}(x,y)\le(1+\varepsilon)\rho_{w,R}(x,y)\quad\text{for all } w\in U \text{ and } x,y\in\Omega.$$

Proof. Choose an open, relative compact neighbourhood D of Ω in $\partial\tilde{M}-\varphi_v(-\infty)$ and an open neighbourhood V of v in $S\tilde{M}$. Since $\Omega\subset\partial\tilde{M}-\varphi_v(-\infty)$ is compact, $\mu=\sup\{\rho_v(x,y)\,|\,x,y\in\Omega\}$ is finite. Let $\tau=\log(1/\mu)$ and define $\Psi:D\times V\to\tilde{M}$ by $\Psi(x,w)=\pi_{\tau,w}(x)$. Since Ψ is clearly continuous there is for a fixed number $\delta>0$ and every $y\in\Omega$ an open neighbourhood $D(y)$ of y in D and an open neighbourhood $U(v)$ of v in V such that $d(\Psi(z,w),\Psi(y,v))<\delta/2$ for every $z\in D(y)$ and $w\in U(v)$. By compactness Ω can be covered by finitely many of the sets $D(y)$, say $\Omega\subset\bigcup_{i=1}^k D(y_i)$ for some $y_i\in\Omega$. $U=\bigcap_{i=1}^k U(y_i)$ is an open neighbourhood of v in V. If $y\in\Omega$, then $y\in D(y_i)$ for some $i\in\{1,\dots,k\}$, hence

$$d(\pi_{\tau,v}(y),\pi_{\tau,w}(y))\le d(\pi_{\tau,v}(y),\pi_{\tau,v}(y_i))+d(\pi_{\tau,v}(y_i),\pi_{\tau,w}(y))<\delta\ \text{for all}\ w\in U\subset U(y_i).$$

Now for all $y\in\Omega$ and $w\in U$ the function $t\to d(\pi_{t,v}(y),\pi_{t,w}(y))$ is decreasing. Thus given $y,z\in\Omega$ and $t\ge\tau$ we have

$$d(\pi_{t,v}(y),\pi_{t,v}(z))-2\delta\le d(\pi_{t,w}(y),\pi_{t,w}(z))\le d(\pi_{t,v}(y),\pi_{t,v}(z))+2\delta$$

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and consequently

$$\rho_{w,R+2\delta}(y,z)\le\rho_{v,R}(y,z)\le\rho_{w,R-2\delta}(y,z)\quad\text{for all } y,z\in\Omega\quad\text{such that } \rho_v(y,z)\le e^{-\tau},$$

i.e. for all $y, z\in\Omega$ by the choice of τ. Since $\delta>0$ was arbitrary, the claim now follows from Lemma 6. □

The function $\rho=\rho_{v,R}$ determines a family of balls $B_\rho(x,\varepsilon)=\{y\in\partial\tilde M-\xi \mid \rho(x,y)<\varepsilon\}$ $(x\in\partial\tilde M-\xi, \varepsilon>0)$. Our aim is to show that these balls together with their radii give rise by Carathéodory's construction (see [3]) to a Borel measure on $\partial\tilde M-\xi$ which is finite on compact and positive on nontrivial open subsets of $\partial\tilde M-\xi$. This fact is derived from the analogous property of an auxiliary function $\beta=\beta^{v,R}$ which is defined on the subsets of $\partial\tilde M-\xi$ in the following way: For a compact set $\Omega\subset\partial\tilde M-\xi$ and $\varepsilon>0$ let $q_\varepsilon(\Omega)$ be the maximal cardinality of a subset E of Ω with the property that $B_\rho(x,\varepsilon)\cap B_\rho(y,\varepsilon)=\emptyset$ if $x,y\in E$ and $x\ne y$. As above denote by h the topological entropy of the geodesic flow on SM and define $\beta_\varepsilon(\Omega)=q_\varepsilon(\Omega)\cdot\varepsilon^h$ and $\beta(\Omega)=\limsup_{\varepsilon\to0}\beta_\varepsilon(\Omega)$. If $\Omega_1,\Omega_2\subset\partial\tilde M-\xi$ are compact and $\Omega_1\subset\Omega_2$, then $\beta\Omega_1\le\beta\Omega_2$. Thus for $A\subset\partial\tilde M-\xi$ arbitrary we can define $\beta A=\sup\{\beta\Omega\mid\Omega\subset A \text{ compact}\}$.

Notice that β may not be subadditive, i.e. β may not be a measure on $\partial\tilde M-\xi$. However β has the following properties:

(1) If $A\subset B$, then $\beta A\le\beta B$.

(2) If $\Omega_i(i\in\mathbb{Z})$ are compact and $\Omega\subset\bigcup_i\Omega_i$, then $\beta\Omega\le\sum_i\beta\Omega_i$.

We will need the following lemma which is due to Margulis (it is essentially proved in [6]):

LEMMA 8. *For every $r>0$ there are numbers $0<\alpha_1(r)<\alpha_2(r)<\infty$ such that $\alpha_1(r)\le\tilde\mu^u\{w\in W^u(v)\mid d(Pw,Pv)<r\}\le\alpha_2(r)$ for all $v\in S\tilde M$.*

Lemma 8 shows in particular that $\tilde\mu^u$ is finite on compact and positive on nontrivial open subsets of $W^u(v)$.

For $p\in\tilde M$ let $B_d(p,r)$ be the open r-ball around p in $(\tilde M,d)$.

LEMMA 9. *If $p\in\theta^{-1}(t)$ then $\pi_\infty B_d(p,R/2)\subset\pi_\infty(B_d(p,R)\cap\theta^{-1}(t))$.*

Proof. Let $y\in\partial\tilde M-\xi$ such that $d(p,\pi_t(y))\ge R$. Determine a number $\tau\in\mathbb{R}$ with the property that $d(p,\pi_\tau(y))$ realizes the distance of p to the geodesic $s\to\pi_s(y)$. Then $\pi_\tau(y)\in\theta^{-1}(\tau)$, hence $d(p,\pi_\tau(y))\ge|t-\tau|=d(\pi_t(y),\pi_\tau(y))$ and $2d(p,\pi_\tau(y))\ge d(p,\pi_\tau(y))+d(\pi_\tau(y),\pi_t(y))\ge R$. But this shows $y\notin\pi_\infty B_d(p,R/2)$ which is the claim. □

Recall that the geodesic flow g^t on $S\tilde M$ transforms $\tilde\mu^u$ by $\tilde\mu^u\circ g^t=e^{ht}\tilde\mu^u$ (h as in the theorem). This and Lemma 9 is used in the proof of

LEMMA 10. *β is finite on compact subsets of $\partial\tilde M-\xi$.*

Proof. Identify $\tilde M$ with $W^u(v)$, the set of all unit tangent vectors of geodesics γ in $\tilde M$ with $\gamma(-\infty)=\varphi_v(-\infty)=\xi$. With respect to this identification the geodesic flow g^t acts on $\tilde M$ by $w\in\theta^{-1}(s)\to g^tw=\pi_{s+t}w\in\theta^{-1}(s+t)$. The restriction of $\tilde\mu^u$ to $W^u(v)$ can be viewed as a measure on $\tilde M$.

Let $\Omega \subset \partial\tilde{M} - \xi$ be compact and $B_1 = \{y \in \partial\tilde{M} - \xi \mid \rho(y, z) \leq 1 \text{ for some } z \in \Omega\}$. Then

$$B_2 = \left\{\pi_t(w) \mid w \in B_1, -\frac{R}{2} \leq t \leq \frac{R}{2}\right\}$$

is a compact subset of $\tilde{M}$, hence $\lambda = \tilde{\mu}^u B_2 < \infty$.

Let $\varepsilon \in (0, 1)$ and $\{x_1, \ldots, x_q\} \subset \Omega$ be a set of maximal cardinality such that the balls $B_\rho(x_i, \varepsilon)$ are pairwise disjoint. This means

$$B_d(\pi_{\log 1/\varepsilon}(x_i), R) \cap B_d(\pi_{\log 1/\varepsilon}(x_j), R) \cap \theta^{-1}\left(\log\frac{1}{\varepsilon}\right) = \varnothing \quad \text{for } i \neq j.$$

By Lemma 9, the balls $B_d(\pi_{\log 1/\varepsilon}(x_i), R/2)$ are pairwise disjoint and moreover they are contained in $g^{\log 1/\varepsilon}B_2$ by the definition of B_2. With $\alpha = \alpha_1(R/2)$ as in Lemma 8 this implies $q\alpha \leq \tilde{\mu}^u g^{\log 1/\varepsilon} B_2 = (1/\varepsilon)^R \cdot \lambda$ and $q \cdot \varepsilon^h \leq \lambda/\alpha$. Since $\varepsilon \in (0, 1)$ was arbitrary, this is the claim. □

LEMMA 11. *β is positive on nontrivial open subsets of $\partial\tilde{M} - \xi$.*

Proof. It suffices to show that β is positive on compact sets B with nonempty interior. Define $B_3 = \{\pi_t y \mid y \in B, -R \leq t \leq 0\}$ and $\lambda = \tilde{\mu}^u B_3 > 0$. For $\varepsilon > 0$ let $\{x_1, \ldots, x_q\} \subset B$ be a subset of maximal cardinality such that the balls $B_\rho(x_i, \varepsilon)$ are pairwise disjoint. By the definition of ρ this means that the balls $B_d(\pi_{\log 1/\varepsilon}(x_i), 2R)$ cover $\pi_{\log 1/\varepsilon}B$, hence the balls $B_d(\pi_{\log 1/\varepsilon}(x_i), 3R)$ cover $g^{\log 1/\varepsilon}B_3$. If $\alpha = \alpha_2(3R)$ is chosen as in Lemma 9, then $q\alpha \geq (1/\varepsilon)^h\lambda$ and $q \cdot \varepsilon^h \geq \lambda/\alpha$ which yields the lemma. □

Remark. In fact we have shown that $\liminf_{\varepsilon \to 0} \beta_\varepsilon \Omega > 0$ for all nontrivial open subsets Ω of $\partial\tilde{M} - \xi$.

For a fixed number $R > 0$ we investigate now how $\beta^v = \beta^{v,R}$ varies with $v \in S\tilde{M}$.

LEMMA 12. *Let $\Omega \subset \partial\tilde{M}$ be a compact subset with nonempty complement. Then the map $v \to \beta^v\Omega$ is continuous on $S\tilde{M} - \{w \mid \varphi_w(-\infty) \in \Omega\}$.*

Proof. We show first that $v \to \beta^v\Omega$ is upper semi-continuous on its domain of definition.

Let $v \in S\tilde{M} - \{w \mid \varphi_w(-\infty) \in \Omega\}$; since $\beta^v\Omega < \infty$ by Lemma 10 it suffices to find for every $\delta > 0$ an open neighbourhood U of v in $S\tilde{M} - \{w \mid \varphi_w(-\infty) \in \Omega\}$ such that $\beta^w\Omega \leq (1+\delta)\beta^v\Omega$ for all $w \in U$.

Since $\Omega \subset \partial\tilde{M} - \varphi_v(-\infty)$ is compact, $A = \{y \in \partial\tilde{M} \mid \rho_{v,R}(x, y) \leq 1 \text{ for some } x \in \Omega\}$ is a compact subset of $\partial\tilde{M} - \varphi_v(-\infty)$. By Lemma 7 there is for $\lambda = (1/(1+\delta))^{1/h}$ a neighbourhood U of v in $S\tilde{M}$ such that for all $x, y \in A$ and all $w \in U$ $\lambda\rho_{w,R}(x, y) \leq \rho_v(x, y)$. Let $\varepsilon \in (0, 1)$ and $\{y_1, \ldots, y_m\} \subset \Omega$ be a subset of maximal cardinality with the property that the balls $B_{\rho_{w,R}}(y_i, \varepsilon)$ are pairwise disjoint. Then the sets $B_{\rho_{v,R}}(y_i, \lambda\varepsilon) \cap A$ are pairwise disjoint. But by the choice of A for every $y \in \Omega$ the $\rho_{v,R}$-ball of radius $\lambda\varepsilon < 1$ centred at y is contained in A. This implies $\beta^w_\varepsilon(\Omega) \leq (1+\delta)\beta^v_{\lambda\varepsilon}(\Omega)$ and since $\varepsilon \in (0, 1)$ was arbitrary, $\beta^w(\Omega) \leq (1+\delta)\beta^v(\Omega)$. The lower semi-continuity of the map is shown similarly. □

Remark. The proof of Lemma 12 yields the following fact: If $\Omega \subset \partial\tilde{M} - \varphi_v(-\infty)$ is compact and $\beta^v(\Omega) = 0$, then $\beta^w(\Omega) = 0$ for all $w \in \partial\tilde{M} - \Omega$.

For $\rho=\rho_{v,R}$ let $\bar{B}_\rho(x,\varepsilon)$ $(x\in\partial\tilde{M})$ be the closure of $B_\rho(x,\varepsilon)$ in $\partial\tilde{M}$. Let $\beta=\beta^{v,R}$ be as above and $\xi=\varphi_v(-\infty)$.

COROLLARY 13. *There is a number $\kappa>0$ such that for all $x\in\partial\tilde{M}-\xi$ and all $\varepsilon>0$, $\kappa\varepsilon^h\le\beta\bar{B}_\rho(x,\varepsilon)\le\kappa^{-1}\varepsilon^h$.*

Proof. Use the notations of the proof of Lemma 12. Let $D\subset\tilde{M}$ be a compact fundamental domain for $\Gamma=\pi_1M$ and define $u:S\tilde{M}|_D\to\mathbb{R}$, $w\to u(w)=\beta^w\bar{B}_{\rho_w}(\varphi_w(\infty),1)$.

Let $\{w_j\}\subset S\tilde{M}|_D$ be a sequence such that $u(w_j)\to\sup\{u(w)|w\in S\tilde{M}|_D\}$. By the compactness of $S\tilde{M}|_D$ we may assume that $\{w_j\}$ converges to $w\in S\tilde{M}|_D$ as $j\to\infty$.

Since by Lemma 7 ρ_w depends continuously on $w\in S\tilde{M}$, there is a number $i_0>0$ such that the closed ball of radius 1 around $\varphi_{w_i}(\infty)$ with respect to ρ_{w_i} is contained in $B=\bar{B}_{\rho_w}(\varphi_w(\infty),2)$. Thus $u(w_i)\le\beta^{w_i}B$ for all $i\ge i_0$ and Lemma 12 shows $\limsup_{j\to\infty}u(w_j)\le\beta^wB<\infty$. A similar argument yields $\inf\{u(w)\,|\,w\in S\tilde{M}|_D\}>0$, i.e. there is a number $\kappa>0$ such that $\kappa\le u(w)\le 1/\kappa$ for all $w\in S\tilde{M}|_D$.

For $\rho=\rho_{v,R}$ and $x\in\partial\tilde{M}-\varphi_v(-\infty)$, $\bar{B}_\rho(x,\varepsilon)$ is the projection in $\partial\tilde{M}-\varphi_v(-\infty)$ of the set $\bar{B}_d(\pi_{\log 1/\varepsilon,v}x,R)\cap\theta_v^{-1}(\log 1/\varepsilon)$ along the geodesics which are tangent to $W^u(v)$. Choose $\Phi\in\Gamma$ such that $\Phi(\pi_{\log 1/\varepsilon,v}x)\in D$. If $w\in S\tilde{M}$ is the tangent at $\log 1/\varepsilon$ of the geodesic $t\to\Phi(\pi_{t,v}x)$, then $\Phi\bar{B}_\rho(x,\varepsilon)=\bar{B}_{\rho_w}(\Phi x,1)$ and $\Phi\bar{B}\rho(y,\delta)=\bar{B}_{\rho_w}(\Phi y,\varepsilon^{-1}\delta)$ for all $y\in\bar{B}_\rho(x,\varepsilon)$ and all $\delta>0$ (recall that Γ acts on $\partial\tilde{M}$ in a natural way). By the definition of β this means $\beta\bar{B}_\rho(x,\varepsilon)=\varepsilon^h\beta^w\bar{B}_{\rho_w}(\Phi x,1)$, hence $\kappa\varepsilon^h\le\beta\bar{B}_\rho(x,\varepsilon)\le\kappa^{-1}\varepsilon^h$. □

Recall the definition of the *h-dim. spherical measure* $\sigma=\sigma^{v,R}=\sigma_\rho$ on $\partial\tilde{M}-\xi=\partial\tilde{M}-\varphi_v(-\infty)$ associated to $\rho=\rho_{v,R}$ (see [3]). For $\Omega\subset\partial\tilde{M}-\xi$, $\sigma(\Omega)=\sup_{\varepsilon\to0}\sigma_\varepsilon(\Omega)$ where $\sigma_\varepsilon(\Omega)=\inf\{\sum_{j=1}^\infty\varepsilon_j^h\,|\,\varepsilon_j\le\varepsilon$ and $\Omega\subset\bigcup_{j=1}^\infty\bar{B}_\rho(x_j,\varepsilon_j)$ for some $x_j\in\Omega\}$. Corollary 5 implies that σ is a Borel regular measure, i.e. $\sigma(\Omega)=\sup\{\sigma(B)\,|\,B\subset\Omega$ compact$\}$ for every Borel-subset Ω of $\partial\tilde{M}-\xi$ (compare the argument in [3] for spherical measures associated to distances).

COROLLARY 14. *Let $c>0$ be as in Corollary 5 and $\kappa>0$ be as in Corollary 13. Then $c^h\beta(\Omega)\ge\sigma(\Omega)\ge\kappa\beta(\Omega)$ for every Borel set $\Omega\subset\partial\tilde{M}-\xi$.*

Proof. By the definition of β and the fact that σ is Borel-regular it suffices to show the claim for compact subsets Ω of $\partial\tilde{M}-\xi$.

Let $\Omega\subset\partial\tilde{M}-\xi$ be compact, let $\varepsilon>0$ and $\{x_1,\dots,x_q\}\subset\Omega$ be a set of maximal cardinality with the property that the balls $B_\rho(x_i,\varepsilon)$ are pairwise disjoint. Then for every $y\in\Omega$ there is $i\in\{1,\dots,q\}$ and $z\in B_\rho(x_i,\varepsilon)$ such that $\rho(y,z)<\epsilon$. Hence by Corollary 5 the balls $B_\rho(x_i,c\varepsilon)$ cover Ω, which shows $\sigma_{c\varepsilon}(\Omega)\le q\varepsilon^hc^h=c^h\beta_\varepsilon(\Omega)$. Thus $\sigma(\Omega)\le c^h\beta(\Omega)$. On the other hand, for each $\delta>0$ there is a covering of Ω by balls $\bar{B}_\rho(x_i,\varepsilon_i)$ $(i\ge1)$ such that $\sum_{i=1}^\infty\varepsilon_i^h\le\sigma(\Omega)+\delta$. Corollary 13 and property (2) of β implies

$$\beta(\Omega)\le\frac{1}{\kappa}\sum\varepsilon_i^h\le\frac{1}{\kappa}\sigma(\Omega)+\frac{\delta}{\kappa}.$$

Since $\delta>0$ was arbitrary, this is the claim. □

Now we are left with showing that the measures $\sigma^{v,R}$ indeed give rise to the Bowen–Margulis measure on $S\tilde{M}$.

For a fixed $R>0$ recall that $\sigma^v=\sigma^{v,R}$ depends on the choice of $v\in S\tilde{M}$ and $\sigma^v=\sigma^v$ if $w\in W^{su}(v)$. Now the strong unstable manifold $W^{su}(v)$ has a canonical identification with $\partial\tilde{M}-\varphi_v(-\infty)$ via the map $\pi: w\to\varphi_w(\infty)$. Thus σ^v can be viewed as a Borel measure on $W^{su}(v)$. In this way we obtain a Borel measure μ^{su} on the leaves of the foliation W^{su}. If $v\in S\tilde{M}$ and $t\in\mathbb{R}$, then $\theta_v=\theta_{g^tv}+t$, hence $\rho_{g^tv}=e^t\rho_v$ and $\mu^{su}\circ g^t=e^{ht}\mu^{su}$.

We have to construct a measure on $S\tilde{M}$ which is invariant under the geodesic flow and the isometry group of $\tilde{M}$ and restricts to the measures μ^{su} on the leaves of W^{su}. We first define a Borel measure μ^u on the leaves of the foliation W^u as follows: For $A\subset W^u$ let $\mu^u(A)$ be the infimum of all numbers $\sum_{j=1}^{\infty}\int_{T_j}\mu^{su}(g^tA_j)dt$ corresponding to all families of Borel sets $T_j\subset\mathbb{R}$, $A_j\subset W^{su}$ with $A\subset\bigcup_{j=1}^{\infty}(\bigcup_{t\in T_j}g^tA_j)$. μ^u can be viewed as a weighted product measure on $W^u(v)\approx W^{su}(v)\times\mathbb{R}(v\in A$; see [3] p. 114). If $A=\bigcup_{s\in T}g^s\tilde{A}$ for some Borel-set $\tilde{A}\subset W^{su}$ and a Borel-set $T\subset\mathbb{R}$, then $\mu^u(A)=\int_T\mu^{su}(g^t\tilde{A})dt=\mu^{su}(\tilde{A})\int_T e^{ht}dt$ (this follows as the analogous statement for product measures, see [3]). Furthermore $\mu^u(g^tA)=e^{ht}\mu^u(A)$ for all $t\in\mathbb{R}$.

For v, $w\in S\tilde{M}$ such that $\varphi_v(-\infty)\neq\varphi_w(\infty)$ there is a geodesic γ joining $\varphi_v(-\infty)=\gamma(-\infty)$ to $\varphi_w(\infty)=\gamma(\infty)$, and γ is unique up to reparametrization. Thus the intersection $W^u(v)\cap W^{ss}(w)$ consists of a unique point. Following Margulis ([6]) we call sets $A_1\subset W^u$, $A_2\subset W^u(w)$ *equivalent* if $A_2=\{W^u(w)\cap W^{ss}(v)\mid v\in A_1\}$. If $A_1\subset W^u$ and $w\in S\tilde{M}$ is such that $\varphi_w(-\infty)\notin\pi A_1$, then A_1 is equivalent to a subset of $W^u(w)$.

For equivalent sets A_1, $A_2\subset W^u$ there is a homeomorphism Ψ: $A_1\to A_2$ such that $\Psi(v)\in W^{ss}(v)$ for all $v\in A_1$. A_1 and A_2 are called *ε-equivalent* if A_1 and A_2 are equivalent and if furthermore the homeomorphism $\Psi: A_1\to A_2$ satisfies $d(Pw, P\Psi w)<\varepsilon$ for all $w\in A_1$. If A_1 and A_2 are ε-equivalent for some $\varepsilon>0$, then $\tilde{\mu}^u(A_1)=\tilde{\mu}^u(A_2)$ ([6]). This is also true for μ^u.

LEMMA 15. *If A_1, $A_2\subset W^u$ are relatively compact and equivalent, then $\mu^uA_1=\mu^uA_2$.*

Proof. We want to show $\mu^uA_1\geq\mu^uA_2$ if A_1, A_2 are as above. Since μ^u is Borel regular we may assume that A_1 is compact. Denote by W_i^u the leaft of the foliation W^u which contains A_i.

Let $\bar{v}\in A_1$, $w\in A_2$ and choose a compact subset Ω of $W^{su}(\bar{v})$ such that $\pi\Omega$ is a compact neighbourhood of πA_1 in $\partial\tilde{M}-\varphi_w(-\infty)$. Then there is a number $\tau>0$ such that $V=\bigcup_{-\tau\leq t\leq\tau}g^t\Omega$ is a compact neighbourhood of A_1 in W_1^u.

By the choice of Ω, V is equivalent to a subset of W_2^u. This means that there is a homeomorphism Ψ of V onto a compact neighbourhood ΨV of A_2 in W_2^u such that $\Psi A_1=A_2$ and $\Psi(v)\in W^{ss}(v)\cap W_2^u$ for all $v\in V$.

By the definition of μ^u, for every $\delta>0$ there are Borel sets $S_j\subset(-\tau,\tau)$, $\Omega_j\subset\Omega(j\geq1)$ such that $A_1\subset\bigcup_{j=1}^{\infty}(\bigcup_{s\in S_j}g^s\Omega_j)\subset V$ and $\mu^u(A_1)\geq\sum_{j=1}^{\infty}\int_{S_j}\mu^{su}(g^s\Omega_j)ds-\delta$. Since $\mu^u(A_2)\leq\sum_{j=1}^{\infty}\mu^u(\Psi(\bigcup_{s\in S_j}g^s\Omega_j))$ and $\mu^u(\bigcup_{s\in s_j}g^s\Omega_j)=\int_{S_j}\mu^{su}(g^s\Omega_j)ds$, it thus suffices to show $\mu^u(B_1)\geq\mu^u(\Psi B_1)$ for every subset B_1 of V of the form $B_1=\bigcup_{s\in S}g^sB$ with Borel sets $S\subset[-\tau,\tau]$, $B\subset\Omega$.

Let $\delta>0$, B_1 as above and $\lambda=\mu^{su}(g^{\tau}\Omega)<\infty$. Since the Lebesgue-measure on coincides with the 1-dim. spherical measure with respect to the Euclidean distanc there are countably many closed intervals $S_j\subset[-\tau,\tau](j\geq 1)$ such that $\int_S dt$ $\sum_{j=1}^{\infty}\int_{S_j}dt-\delta/\lambda$ and $S\subset\bigcup_{j=1}^{\infty}S_j$. Write $T_j=(S\cap S_j)\backslash\bigcup_{i=1}^{j-1}S_i$; then $S=\bigcup_{j=1}^{\infty}T_j$ ar

$$\int_S dt=\sum_{j=1}^{\infty}\int_{T_j}dt\geq\sum_{j=1}^{\infty}\int_{T_j}dt+\sum_{j=1}^{\infty}\int_{S_j-T_j}dt-\delta/\lambda.$$

Thus the choice of λ yields

$$\sum_{j=1}^{\infty}\int_{Sj}\mu^{su}(g^tB)dt\leq\int_S\mu^{su}(g^tB)dt+\delta.$$

Since the sets $\bigcup_{s\in S_j}g^sB(j\geq 1)$ cover B_1, it follows as above that we need on consider sets $B_1=\bigcup_{t\in T}g^tB$ where $B\subset\Omega$ is Borel and $T\subset[-\tau,\tau]$ is a closed interv

Assume without loss of generality that $B_1=\bigcup_{-\nu\leq s\leq\nu}g^sB$ for some $\nu>0$. I eventually enlarging Ω we may also suppose that the closure $\bar{B}$ of B is contain in the interior of Ω. Define $B_2=\Psi B_1$ and let $\varepsilon>0$. By continuity there is for eve $v\in\bar{B}$ an open neighbourhood $U(v)$ of v in Ω such that $d(P\Psi(w),P\Psi(v))<\varepsilon$ f all $w\in U(v)$. The compact set $\bar{B}$ admits a finite cover by open sets $U(v_i)$ $(v_i\in$ and $i=1,\ldots,k)$. In particular B has a Borel-partition $B=\sum_{i=1}^k C^i$ into pairwi disjoint sets $C^i\subset(U(v_i)\bigcap B)$.

Define $D^i=\bigcup_{-\nu\leq s\leq\nu}g^sC^i$; then $B_1=\bigcup_{i=1}^k D^i$ and $D^i\bigcap D^j=\varnothing$ if $i\neq j$, i. $\mu^u(B_1)=\sum_{i=1}^k\mu^u(D^i)$.

For fixed $i\in\{1,\ldots,k\}$ we want to compare the measures $\mu^u(D^i)$ and $\mu^u(\Psi D$

This is done by estimating the measure of a set $\tilde{E}^i\supset\Psi(D^i)$ which is defined l $\tilde{E}^i=\bigcup_{-\nu-\varepsilon<s<\nu+\varepsilon}g^sE^i$ where $E^i=\{w\in W^{su}(\Psi v_i)\mid\pi w\in\pi C^i\}$.

We have to show $\tilde{E}^i\supset\Psi(D^i)$: Indeed, for every $v\in C^i$ there is a number $s(v)\in$ such that $g^{s(v)}\Psi(v)\in E^i$. Then $s(v_i)=0$ and consequently $s(v)\leq d(P\Psi(v_i$ $P\Psi(v))<\varepsilon$ for all $v\in C^i$ by the choice of C^i. Since $\pi^{-1}(\pi v)\cap\Psi(D^i)$ $\bigcup_{-\nu\leq s\leq\nu}g^s\Psi(v)$ this implies $\Psi(D^i)\subset\tilde{E}^i$.

In order to estimate $\mu^u(\tilde{E}^i)$ we have to estimate $\mu^{su}(E^i)$. For this purpose l $v\in C^i$, $w\in E^i$ and $\rho=\rho_{v,R}$, $\tilde{\rho}=\rho_{w,R}$. Since $\bar{B}$ is compact and $\Psi(g^sv)=g^s\Psi(v)$ f all $s\in[-\nu,\nu]$, there is a number $t_0\in\mathbb{R}$ such that g^tB_1 and g^tB_2 are ε-equivalent f all $t\geq t_0$, i.e. $d(Pg^tv,Pg^t\Psi(v))<\varepsilon$ for all $v\in B_1$ (compare [6]).

Let $\delta<e^{-t_0}$. For every $x\in\pi C^i$ there are unique points $w_1(x)\in g^{\log 1/\delta}C^i$, $w_2(x)$ $g^{\log 1/\delta}E^i$ such that $\pi w_i(x)=x$ $(i=1,2)$. The choice of δ yields

$$\begin{aligned}d(Pw_1(x),Pw_2(x))&\leq d(Pw_1(x),P\Psi w_1(x))+d(P\Psi w_1(x),Pw_2(x))\\&<\varepsilon+|s(g^{-\log 1/\delta}w_1(x))|<2\epsilon.\end{aligned}$$

Thus $y\in\bar{B}_\rho(x,\delta)\bigcap\pi C^i$, i.e. $d(Pw_1(x),Pw_1(y))\leq R$, implies $d(Pw_2(x),Pw_2(y))$ $R+2\varepsilon$. If we define

$$\tau(\epsilon)=\left(\left(\sinh\frac{a}{2}R(1+2\varepsilon)\right)\Big/\left(\sinh\frac{a}{2}R\right)\right)^{1/a}$$

then Lemma 6 shows as before that $\bar{B}_\rho(x,\delta)\cap\pi C^i\subset\bar{B}_{\tilde{\rho}}(x,\tau(\varepsilon)\delta)\cap\pi C^i$ for a $x\in\pi C^i$, $\delta<\varepsilon^{-t_0}$.

Given $\delta \in (0, e^{-t_0})$ arbitrary, there is a covering of πC^i by balls $\bar{B}_\rho(x_j, \delta_j)$ ($x_j \in \pi C^i, j \geq 1$, $\delta_j \leq \delta$) such that $\sum_{j=1}^{\infty} \delta_j^h \leq \mu^{su}(C^i) + \delta$. By the above consideration the balls $\bar{B}_{\bar{\rho}}(x_j, \tau(\varepsilon)\delta_j)$ cover $\pi C^i = \pi E^i$ which implies $\mu^{su}(E^i) \leq (\tau(\varepsilon))^h \mu^{su}(C^i)$.

Using this inequality we obtain

$$\begin{aligned}\mu^u(\tilde{E}^i) &= \int_{-\nu-\varepsilon}^{\nu+\varepsilon} e^{ht}\mu^{su}(E^i)\,dt \\ &= \frac{1}{h}(e^{h(\nu+\varepsilon)} - e^{-h(\nu+\varepsilon)})\mu^{su}(E^i) \\ &\leq \frac{1}{h}\tau(\varepsilon)^h(e^{h\nu}e^{h\varepsilon} - e^{-h\nu}e^{-h\varepsilon})\mu^{su}(C^i)\end{aligned}$$

hence

$$\mu^u(B_2) \leq \sum_{i=1}^{k} \mu^u(\tilde{E}^i) \leq \frac{1}{h}\tau(\varepsilon)^h(e^{h\nu}e^{h\varepsilon} - e^{-h\nu}e^{-h\varepsilon})\mu^{su}(B).$$

On the other hand $\mu^u(B_1) = h^{-1}(e^{h\nu} - e^{-h\nu})\mu^{su}(B)$; since $\varepsilon > 0$ was arbitrary and $\tau(\varepsilon) \to 1$ as $\varepsilon \to 0$, this shows $\mu^u(B_2) \leq \mu^u(B_1)$ and finishes the proof of the lemma. □

For every $v \in S\tilde{M}$, the leaf $W^{ss}(v)$ of the strong stable foliation has a canonical identification with $W^{su}(-v)$. Thus μ^{su} induces a measure μ^{ss} on the leaves of W^{ss}. Clearly $\mu^{ss} \circ g^t = e^{-ht}\mu^{ss}$.

As in [**6**], Lemma 15 yields the existence of a g^t-invariant measure μ on $S\tilde{M}$ which restricts to μ^i on the leaves of W^i ($i = ss, u, su$). If $A \subset S\tilde{M}$ is compact and if $W^u(v) \cap A$ is equivalent to $W^u(w) \cap A$ for all v, $w \in A$, then we have

$$\mu(A) = \int_{W^u(v)\cap A} \mu^{ss}(W^{ss}(w) \cap A)\,d\mu^u$$

where $v \in A$ is arbitrary. Now μ^u and μ^{ss} are clearly invariant under the action of Γ on $S\tilde{M}$, hence the same is true for μ. Thus μ induces a finite Borel measure on SM which is positive on all open subsets of SM. The standard computation (see [**2**]) shows that the measure-theoretic entropy of this measure equals the topological entropy h of the geodesic flow on SM, so μ coincides indeed (up to a constant) with the Bowen–Margulis measure $\tilde{\mu}$. In particular the construction of μ and $\tilde{\mu}$ shows $\tilde{\mu}^{su} = \mu^{su}$ on the leaves of W^{su}.

Now let $\bar{\sigma} = \bar{\sigma}^{v,R}$ be the h-dim. spherical measure associated to $\eta = \eta_{v,R}$. Lemma 4 yields $\nu^h\bar{\sigma} \leq \sigma^{v,R} \leq \bar{\sigma}$; in particular $\bar{\sigma}$ is finite on compact subsets of $\partial\tilde{M} - \varphi_v(-\infty)$ and determines the same measure class as σ^v. The proof of Lemma 15 can easily be modified to be valid for the measure $\bar{\mu}^u$ on the leaves of W^u which is induced by the measures $\bar{\sigma}^{v,R}$ on $W^{su}(v) \approx \partial\tilde{M} - \varphi_v(-\infty)$. As above we obtain a measure $\bar{\mu}$ on $S\tilde{M}$ in the measure class of μ which is invariant under g^t and Γ and restricts to $\bar{\sigma}^{v,R}$ on $W^{su}(v)$. By the ergodicity of the geodesic flow on SM with respect to μ, $\bar{\mu}$ equals μ up to a constant. This finishes the proof of the theorem.

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