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A New Error Analysis for Brun's Constant

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ABSTRACT

Enumeration of the twin primes, and the sum of their reciprocals, is extended to 3×10^{15}, yielding the count $\pi_2(3 \times 10^{15}) = 3310517800844$. A more accurate estimate is obtained for Brun's constant, $B_2 = 1.90216\,05823 \pm 0.00000\,00008$. Error analysis is presented to support the contention that this estimate produces a 95 % confidence interval for B_2. In addition, published values of the count $\pi(x)$ of primes, obtained previously by indirect means, are verified by direct count to $x = 3 \times 10^{15}$.

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INTRODUCTION

The set $K_2 = \{(3,5),(5,7),(11,13),(17,19),\ldots\}$ of twin prime pairs $(q, q+2)$ has been studied by many investigators, including Glaisher (1878), Brun (1919), Hardy and Littlewood (1923), Sutton (1937), Selmer (1942), Sexton (1954), Lehmer (1957), Fröberg (1961), Gruenberger and Armerding (1965), Weintraub (1973), Bohman (1973), Shanks and Wrench (1974), Brent (1975), and Nicely (1995).

The present study results from the continuation of a project initiated in 1993, with results to 10^{14} previously published in (Nicely, 1995). A detailed description of the general problem, the computational methods employed, and the incidental discovery of the Pentium® FDIV flaw may be found there, with additional details given in (Nicely, 1999); only a brief summary will be included here.

The prime numbers themselves continue to retain most of their secrets, but still less is known about the twin primes. A matter as fundamental as the infinitude of K_2 remains undecided—the famous "twin prime conjecture." Nonetheless, Brun (1919) proved that in any event the sum of the reciprocals,

$$(1) \quad B_2 = \left(\frac{1}{3}+\frac{1}{5}\right)+\left(\frac{1}{5}+\frac{1}{7}\right)+\left(\frac{1}{11}+\frac{1}{13}\right)+\left(\frac{1}{17}+\frac{1}{19}\right)+\cdots,$$

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is convergent, in contrast to the known divergence of the sum of the reciprocals of all the primes (Brun actually omitted the first term in parentheses, which of course does not affect the convergence). The limit of this sum, styled Brun's sum or Brun's constant, is often denoted as simply B, but henceforth the author will use B_2. In this instance, as in a number of others noted in this paper, identifiers have been changed from those in (Brent, 1975) and (Nicely, 1995), in anticipation of the need for analogous symbols to be used in the study of prime constellations other than the twins.

The twin prime conjecture is a consequence of a much stronger result, an asymptotic relationship conjectured by Hardy and Littlewood (1923, pp. 42-44):

$$\pi_2(x) \sim L_2(x) = 2c_2 \int_2^x \frac{dt}{\ln^2 t} \quad , \tag{2}$$

where $\pi_2(x)$ represents the count of twin prime pairs $(q, q+2)$ such that $q \leq x$, and c_2 denotes the "twin-prime constant," computed to 42D by Wrench (1961),

$$c_2 = 0.66016\ 18158\ 46869\ 57392\ 78121\ 10014\ 55577\ 84326\ 23\ldots \quad . \tag{3}$$

The validity of the conjecture (2), often titled the Hardy-Littlewood approximation, is central to the estimation of Brun's constant and the error bounds in this paper. The Hardy-Littlewood approximation is itself a consequence of the yet more general "prime k-tuples conjecture," also set forth in their 1923 work. See Riesel (1994, pp. 60–83) for an illuminating exposition of these concepts.

Although (1) is convergent, the monotonically increasing partial sums approach the limit with agonizing slowness; summing the first thousand million reciprocals is still insufficient to bring us within five percent of the estimated value of the limit. However, assuming the validity of the Hardy-Littlewood approximation (2), a first-order extrapolation was derived by Fröberg (1961) and further studied by Brent (1975),

$$B_2 = S_2(x) + \frac{4c_2}{\ln x} + O\left(\frac{1}{\sqrt{x}\ \ln x}\right) \quad , \tag{4}$$

with an accelerated rate of convergence $O(\sqrt{x})$ faster than (1). Here $S_2(x)$ is the partial sum

$$S_2(x) = \sum_{q \leq x} \left(\frac{1}{q} + \frac{1}{q+2}\right) \ . \tag{5}$$

of the reciprocals of all the twin prime pairs $(q, q+2)$ for which $q \leq x$. Note that $S_2(x)$ is written as $B(x)$ in (Brent, 1975) and (Nicely, 1995). The first-order extrapolation of $S_2(x)$ to approximate B_2 consists of the first two terms of the right hand side of (4); this was indicated as $B^*(x)$ in (Brent, 1975) and (Nicely, 1995), but we write it here as $F_2(x)$:

$$F_2(x) = S_2(x) + \frac{4c_2}{\ln x} \quad . \tag{6}$$

The final term in (4) is the author's conjectured error or remainder term, inspired by Brent's (1975) probabilistic analysis. As discussed by Shanks and Wrench (1974, p. 298), no effective second-order extrapolation is known.

COMPUTATIONAL TECHNIQUE

These calculations were carried out as part of a more comprehensive project, including in addition the tabulation of prime gaps and other prime constellations. Computations began in 1993 and have since proceeded almost without interruption, although several months' work was lost early on due to the Pentium® FDIV flaw. The calculations were distributed asynchronously across several (varying from a few to more than two dozen) personal computers, using Intel® processors (mostly classic Pentiums®), extended DOS and Windows™ operating systems, and code written in C. The algorithm employed the classic sieve of Eratosthenes to carry out an exhaustive generation and enumeration of the primes. To guard against errors, all calculations were performed in duplicate on separate systems; in addition, the count $\pi(x)$ of primes was maintained and checked periodically against known values, such as those published by Riesel (1994, pp. 380–383). The values obtained for the count $\pi_2(x)$ of twin prime pairs agreed to 10^{11} with those of Brent (1975, with addendum), and to 10^{14} with those of Kutrib and Richstein (1996). Excluding software bugs and the Pentium® FDIV flaw, approximately forty-nine instances of machine errors were detected and corrected, most apparently the result of transient bit errors in memory (DRAM) chips. One of these instances contained at least 364 individual errors.

As mentioned previously, additional details regarding the computational technique, and the Pentium® FDIV affair, are available in (Nicely, 1995, 1999), and also at the author's URL.

COMPUTATIONAL RESULTS

Table 1 contains a brief summary of the computational results, including the counts $\pi_2(x)$ of twin prime pairs; the values of the discrepancy, denoted here by $\delta_2(x)$, between $\pi_2(x)$ and the Hardy-Littlewood approximation:

$$\delta_2(x) = L_2(x) - \pi_2(x) \quad ; \tag{7}$$

the partial sums $S_2(x)$ of the reciprocals of the twins; and the first-order extrapolations $F_2(x)$ of $S_2(x)$ to the limit, according to (6), members of a sequence believed to be converging to Brun's constant B_2. Note that the discrepancy $\delta_2(x)$ was written in (Brent, 1975) and (Nicely, 1995) as $r_3(x)$; Brent also rounded this value to the nearest integer.

Table 1 includes results previously published for powers of ten to 10^{14}, in addition to new results from the present study at additional increments of 10^{14}, ending with the results for the present upper bound of computation, $x_0 = 3 \times 10^{15}$. Updated and more extensive versions of Table 1 are being maintained at the author's URL. Also available there are equally extensive

TABLE 1. Counts of twin prime pairs and estimates of Brun's constant.

x	$\pi_2(x)$	$\delta_2(x)$	$S_2(x)$	$F_2(x)$
10^1	2	2.84	0.8761904761904761905	2.0230090113326
10^2	8	5.54	1.3309903657190867570	1.9043996332901
10^3	35	10.80	1.5180324635595909885	1.9003053086070
10^4	205	9.21	1.6168935574322006462	1.9035981912177
10^5	1224	24.71	1.6727995848277415480	1.9021632918562
10^6	8169	79.03	1.7107769308042211063	1.9019133533279
10^7	58980	-226.18	1.7383570439172709388	1.9021882632233
10^8	440312	55.79	1.7588156210679749679	1.9021679379607
10^9	3424506	802.16	1.7747359576385368007	1.9021602393210
10^{10}	27412679	-1262.47	1.7874785027192415475	1.9021603562335
10^{11}	224376048	-7183.32	1.7979043109551191615	1.9021605414226
10^{12}	1870585220	-25353.18	1.8065924191758825917	1.9021606304377
10^{13}	15834664872	-66566.94	1.8139437606846070596	1.9021605710802
10^{14}	135780321665	-56770.51	1.8202449681302705289	1.9021605777833
2.0×10^{14}	259858400254	-286596.19	1.8219692563019236634	1.9021605806674
3.0×10^{14}	380041003032	-386165.49	1.8229446574498899187	1.9021605813179
4.0×10^{14}	497794845572	-687458.42	1.8236224494488219106	1.9021605828234
5.0×10^{14}	613790177314	-495402.94	1.8241402488570614635	1.9021605819011
6.0×10^{14}	728412916123	-399030.90	1.8245582810368460212	1.9021605816028
7.0×10^{14}	841912734248	-330271.47	1.8249082431039834264	1.9021605813540
8.0×10^{14}	954464283498	-207253.20	1.8252088524969516994	1.9021605810407
9.0×10^{14}	1066196920739	-459168.78	1.8254720744000806297	1.9021605816527
10^{15}	1177209242304	-750443.32	1.8257060132402797152	1.9021605822498
1.1×10^{15}	1287579137984	-732612.87	1.8259164099409972759	1.9021605822159
1.2×10^{15}	1397370335220	-761338.54	1.8261074785718993129	1.9021605822802
1.3×10^{15}	1506635099560	-762644.45	1.8262824008978027694	1.9021605822837
1.4×10^{15}	1615417411648	-785068.05	1.8264436378766369280	1.9021605823288
1.5×10^{15}	1723754585354	-761213.67	1.8265931311402050729	1.9021605823084
1.6×10^{15}	1831678961614	-851925.37	1.8267324395006005931	1.9021605824283
1.7×10^{15}	1939218595600	-1129122.83	1.8268628327687977085	1.9021605827604
1.8×10^{15}	2046397121805	-678331.73	1.8269853577548725890	1.9021605822393
1.9×10^{15}	2153237307407	-562823.58	1.8271008903959923363	1.9021605821153
2.0×10^{15}	2259758303674	-612652.24	1.8272101680098151140	1.9021605821628
2.1×10^{15}	2365977242191	-653062.89	1.8273138179643056714	1.9021605822014
2.2×10^{15}	2471909670028	-643465.53	1.8274123785364204712	1.9021605821937
2.3×10^{15}	2577569863563	-750111.35	1.8275063150448871463	1.9021605822851
2.4×10^{15}	2682970233099	-552427.29	1.8275960317894826243	1.9021605821145
2.5×10^{15}	2788122612616	-168258.89	1.8276818830618905359	1.9021605818032
2.6×10^{15}	2893038573759	-430246.96	1.8277641812367275962	1.9021605820124
2.7×10^{15}	2997726948096	-292107.29	1.8278432012390461693	1.9021605819106
2.8×10^{15}	3102197972961	-876051.32	1.8279191890118998763	1.9021605823359
2.9×10^{15}	3206458423771	-521046.38	1.8279923621701145073	1.9021605820865
3.0×10^{15}	3310517800844	-897422.15	1.8280629180352850193	1.9021605823404

tables of the values of $\pi(x)$ recorded in this project, at much finer granularity than those commonly available for arguments exceeding 10^{11}. Indeed, the direct enumeration of the primes has been extended to a new upper bound by this project, culminating in the value $\pi(3 \times 10^{15}) = 86688602810119$. This result, as well as a number of other previously published values (Riesel, 1994, pp. 380–382) which were known only through indirect calculations, is now confirmed by direct count in the present study.

BRUN'S CONSTANT AND THE ERROR ANALYSIS

The first-order extrapolation $F_2(x_0) = F_2(3 \times 10^{15})$ is believed to yield the most accurate value known to date for Brun's constant,

$$B_2 = 1.90216\,05823 \pm 0.00000\,00008 \tag{8}$$

The error estimate is believed to define a 95 % confidence interval for the value of B_2. I have no rigorous analytical proof of this assertion regarding the error estimate; rather, it is an inference from the analysis (presented below) of the available numerical data. The notion of a "95 % confidence interval" is to be interpreted as follows. Based on the available numerical data, the author believes that whenever the technique used for this error analysis is applied to a sufficiently numerous sample of distinct integers $x > 1$, Brun's constant B_2 will lie between $F_2(x) - E_2(x)$ and $F_2(x) + E_2(x)$ for at least 95 % of the integers in the sample. Here $E_2(x)$ is the error bound function stated in (11) below; the error estimate given in (8) is a special case of this error bound function, namely $E_2(x_0)$. More precisely, given any set Z_1 of distinct integers $x > 1$, there will always exist a superset Z_2 of distinct integers $x > 1$, $Z_1 \subseteq Z_2$, such that $F_2(x) - E_2(x) \leq B_2 \leq F_2(x) + E_2(x)$ for at least 95 % of the integers in Z_2.

The algorithm for obtaining and validating this error bound function will now be explained. Discussion and justification of certain details of the procedure will be deferred until a later point in this paper.

(A) A set S of sample test points is chosen from the available numerical data; this set should be a reasonably large subset of all the available data points, avoiding any known bias in the associated values of S_2 or F_2. Indeed, S might be chosen as the entire set T of all recorded data points, up to and including the current upper bound $x_0 = 3 \times 10^{15}$ of computation; there are 300081 points in T, consisting of the lattice $(10^{10})(10^{10})(x_0)$ together with the "decade values" $x = k \cdot 10^n$ $(k = 1 \ldots 9,\ n = 1 \ldots 9)$. However, the calculations to be carried out in the error analysis then become excessive. We choose instead for S the lattice $(10^{12})(10^{12})(x_0)$, consisting of 3000 equally spaced data points, extending to the current upper limit of computation, the increment being one (U. S.) trillion.

(B) For each $x \in S$, we obtain an error bound on $F_2(x)$, presumably representing a 95 % confidence interval, by determining the value of a parameter

$K_{95}(x)$ such that, for at least 95 % of the points in the set $U = \{t : t \in T, t \leq x/2\}$,

$$|F_2(x) - F_2(t)| < \frac{K_{95}(x)}{\sqrt{t} \cdot \ln t} . \tag{9}$$

Here the form of the "scaling factor" in the denominator is inferred from the remainder term conjectured in (4). The data points $t > x/2$ are excluded from U to minimize any artificial reduction in the error estimate resulting from the implicit bias of $F_2(t)$ toward $F_2(x)$ as $t \to x$.

(C) We now reason as follows. Since for each $x \in S$, at least 95 % of the (relevant) preceding extrapolations $F_2(t)$, $t \in U$, agree with $F_2(x)$ within the bound in (9), we assume that this property will remain valid for arbitrarily large values of x as well. We now interchange x and t in (9), as well as the order of the resulting terms on the left hand side, and take the limit as $t \to +\infty$.

$$|F_2(x) - \lim_{t \to +\infty} F_2(t)| \leq \frac{\lim_{t \to +\infty} K_{95}(t)}{\sqrt{x} \cdot \ln x} . \tag{10}$$

The numerical evidence indicates that the positive function $K_{95}(x)$ is either roughly constant, or exhibits an overall decreasing trend masked by small scale variations (see Table 2). Thus we can obtain an approximate upper bound on the error by using $K_{95}(x)$ in place of the (unknown) limit of $K_{95}(t)$ in (10). This produces the desired error bound function $E_2(x)$:

$$|F_2(x) - B_2| \leq E_2(x) = \frac{K_{95}(x)}{\sqrt{x} \cdot \ln x} . \tag{11}$$

Determination of the error bound at any specific x then becomes a matter of calculating $K_{95}(x)$ and substituting into (11).

Analysis of the data yields the value $K_{95}(x_0) = 1.380$. Substitution into (11) then gives

$$E_2(x_0) = \frac{1.380}{\sqrt{x_0} \cdot \ln x_0} \approx 0.00000\ 00007\ 06989 . \tag{12}$$

Rounding up produces the error estimate stated in (8).

VALIDATION OF THE ERROR ANALYSIS

Since the *ad hoc* error analysis algorithm described and employed clearly lacks a rigorous analytical foundation, additional examination of the empirical evidence was undertaken, in an effort to find supporting evidence, or lack thereof.

The validation process consisted of comparing the confidence intervals obtained for B_2 at each x in the "lower half" $S' = \{x : x \in S, x \leq x_0/2\}$ of S (the values near x_0 being excluded for reasons similar to those given for set U) with the (presumably) best value obtained at x_0. Simply put, the issue is this: what percentage of the confidence intervals obtained for each $x \in S'$ actually contain the best known point estimate for B_2, given in (8) (and to

TABLE 2. Performance data for the error analysis algorithm.

$x/10^{12}$	$K_{95}(x)$	$E_2(x) \times 10^{10}$	Success %
1	2.074	750.61	100.00
10	2.218	234.32	80.00
100	1.758	54.53	94.00
200	1.602	34.40	83.50
300	1.582	27.40	89.00
400	2.047	30.44	91.75
500	1.688	22.30	93.40
600	1.564	18.76	94.50
700	1.451	16.04	94.86
800	1.320	13.60	95.25
900	1.487	14.39	94.89
1000	1.658	15.18	95.40
1100	1.623	14.13	95.82
1200	1.626	13.52	96.17
1300	1.608	12.82	96.46
1400	1.606	12.31	96.71
1500	1.584	11.70	96.93
1600	1.612	11.51	97.12
1700	1.747	12.08	97.29
1800	1.511	10.14	97.44
1900	1.455	9.49	97.58
2000	1.454	9.23	97.70
2100	1.450	8.97	97.81
2200	1.434	8.65	97.91
2300	1.449	8.54	98.00
2400	1.381	7.96	98.08
2500	1.269	7.16	98.16
2600	1.321	7.30	98.23
2700	1.277	6.92	98.30
2800	1.399	7.43	98.36
2900	1.307	6.82	98.41
3000	1.380	7.07	98.47

greater precision, if not accuracy, in the last entry of Table 1)? For example, applying our error analysis technique to the data for $x \leq 10^{14}$ yields $K_{95}(10^{14}) = 1.758$, and substitution into (11) then produces the confidence interval $B_2 = 1.90216\,05777\,83 \pm 0.00000\,00054\,53$. Since our best estimate for B_2 lies within this interval, we consider the error estimate algorithm to be a success at $x = 10^{14}$. On the other hand, applying the algorithm to the data for $x \le x_1 = 8.13 \times 10^{14}$, we obtain $K_{95}(x_1) = 1.306$, with the resulting confidence interval $B_2 = 1.90216\ 05809\ 53 \ \pm\ 0.00000\ 00013\ 34$, which constitutes a failure.

A survey of all the points $x \in S'$ reveals that 96.93 % (1454 of 1500) produce confidence intervals containing our current best point estimate for B_2. These calculations, briefly summarized in Table 2, also show the trends in the values of $K_{95}(x)$, $E_2(x)$, and the cumulative percentage of successful (in the sense described above) error estimates generated by the algorithm. The available data thus indicates that our algorithm has been successful (actually performing beyond expectation) in producing valid 95 % confidence intervals for the estimates of B_2. Therefore we anticipate that the error bounds thus obtained for larger values of x, including our current upper bound of computation $x_0 = 3 \times 10^{15}$, will also yield valid 95 % confidence intervals for the value of B_2.

CRITIQUE AND FURTHER REMARKS

The results of additional analysis of the data, conducted in order to address various weaknesses of the error analysis algorithm described, are now summarized.

- Further reduction of the "cutoff" fraction for the selection of sample points in sets U and S' (for example, restricting these sets to the smallest quarter, rather than the smaller half of the eligible values) had no significant effect on the results. Of course, if the restriction is relaxed or eliminated, the effect is to artificially inflate the success percentage of the algorithm. This may be observed in the entries of the last column of Table 2, for values of $x > 1.5 \times 10^{15}$.
- Increasing the density of the sample sets S and S' in T (for example, reducing the increment to 10^{11} rather than 10^{12}) had no significant effect on the results.
- Replacing the presumed best estimate $F_2(x_0)$ for B_2 by another value within the specified confidence interval (8) (both endpoint values were tested) had no significant effect on the conclusions.
- Replacing Brent's (1975) scaling factor $\sqrt{x} \cdot \ln x$ (corresponding to the denominator of the remainder term in (4)) with other plausible possibilities had no significant effect on the results. Among the candidates checked were $\sqrt{x}\cdot\ln x\cdot\ln\ln x$, $\sqrt{x}\cdot(\ln x)^2$, $\sqrt{x}\cdot\ln x\cdot(\ln\ln x)^2$, $\sqrt{x}\cdot\ln x\cdot\ln\ln x\cdot\ln\ln\ln x$, and $\sqrt{x}$. Results produced by each of these scaling factors are summarized in Table 3; note that the values for K_{95} and the error are calculated at $x = x_0 = 3 \times 10^{15}$, while the success percentages are evaluated at $x = x_0/2 = 1.5 \times 10^{15}$, as in our principal error analysis; furthermore, the values of the error are in units of 10^{-10}. The available numerical data is seen to be insufficient to either confirm or reject the error term conjectured by the author in (4), or any of the alternatives. On the other hand, since the use of these alternatives had little impact on the final results of the error analysis, the validity of the algorithm

TABLE 3. Impact of various scaling factors on the error analysis.

Scaling factor	K_{95}	Error	Success %
$\sqrt{x} \cdot \ln x$	1.380	7.07	96.93
$\sqrt{x} \cdot \ln x \cdot \ln\ln x$	4.809	6.89	96.27
$\sqrt{x} \cdot (\ln x)^2$	45.40	6.53	94.80
$\sqrt{x} \cdot \ln x \cdot (\ln\ln x)^2$	16.80	6.74	95.67
$\sqrt{x} \cdot \ln x \cdot \ln\ln x \cdot \ln\ln\ln x$	6.016	6.77	95.87
$\sqrt{x}$	0.043	7.83	99.00

appears to be relatively insensitive to the precise nature of the scaling factor (remainder term). Let it be noted that one could make a case, based on the results in Table 3, for a more agressive error estimate of 0.00000 00006 58 in (8); the author prefers the more conservative value previously stated.

• Other analysis techniques were investigated as well, but none was found superior to the one described. Efforts to use weighted or unweighted data averaging or smoothing, or linear regression techniques, in an attempt to obtain a more accurate value of Brun's constant, have not met with success. Harmonic analysis and fast Fourier transforms have been suggested by various colleagues as promising techniques for analysis of the data, but I have not pursued this avenue. I will attempt to post enough of the raw data at my URL so that other investigators may experiment with their own techniques; perhaps some other method will indeed be more successful than my own in producing a more accurate extrapolation, or a superior error bound.

• The error bound formula $E_2(x)$ in (11) is a generalization of that obtained by Brent (1975). As a consequence of a quite different line of reasoning, Brent arrived at the constant 3.5 in place of $K_{95}(x)$, and believed this to produce an 88 % confidence interval for his estimate for B_2. It now appears that Brent's error estimate was quite conservative. On the other hand, the error bound obtained in (Nicely, 1995) was specifically designed to represent one computed standard deviation at 10^{14}, and the present estimate for B_2 differs from that value by more than two of those standard deviations. As pointed out above, the present technique, when applied to the portion of the data for $x \leq 10^{14}$, produces a 95 % confidence interval containing the current best estimate for B_2 (and even containing the entire current best confidence interval); since a 95 % confidence interval corresponds to about ± 1.96 standard deviations (for a normal distribution), that result implies $\sigma(10^{14}) = 0.00000\ 00028$, a more conservative value than the estimate of 0.00000 00021 arrived at (using a different approach) in (Nicely, 1995).

• As the upper bound x_2 of computation for $\pi_2(x)$ and $S_2(x)$ is extended, corresponding error estimates can be obtained by analyzing the new totality of data to determine $K_{95}(x_2)$, according to part (B) of the error analysis algorithm, then substituting into (11). Note that there is no need to recompute $K_{95}(x)$ for any x other than the value for which a new error bound is desired; computation of $K_{95}(x)$ over the entire sample set was carried out only to explain and validate the algorithm. Indeed, based on the variation exhibited by $K_{95}(x)$ in Table 2, a rough error estimate could be obtained by simply using $K_{95}(x_2) = K_{95}(x_0) = 1.380$; or if a quite conservative value is desired, use $K_{95}(x_2) = 2$.

• Finally, it must be emphasized that both the value of B_2 and the associated error estimate obtained in this paper are entirely dependent on the validity of the Hardy-Littlewood approximation (2). All the numerical evidence to date strongly supports this conjecture, but one must maintain some informed skepticism; after all, the numerical evidence to the current level of computations also supports the famous conjecture that $\mathrm{Li}(x) > \pi(x)$, eventually disproved by Littlewood (1914) himself. Absent a major theoretical breakthrough, it will be difficult indeed to improve significantly on either the estimate or error bound herein presented for Brun's constant. As Shanks and Wrench (1974, p. 299) noted, the calculation of B_2 to eight or nine decimals is (was) extremely difficult—or at least computationally intensive—and twenty decimals of precision remains as remote now as it was then. Equation (11) indicates that computations may have to be extended to 10^{17} just to settle the tenth decimal place, and twenty decimals would require calculations out to perhaps 10^{36} —a figure far exceeding the total number of machine cycles available in the cumulative projected lifetimes of all the CPUs currently on our planet.

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