

## A NEW ERROR ANALYSIS FOR DISCONTINUOUS FINITE ELEMENT METHODS FOR LINEAR ELLIPTIC PROBLEMS

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ABSTRACT. The standard *a priori* error analysis of discontinuous Galerkin methods requires additional regularity on the solution of the elliptic boundary value problem in order to justify the Galerkin orthogonality and to handle the normal derivative on element interfaces that appear in the discrete energy norm. In this paper, a new error analysis of discontinuous Galerkin methods is developed using only the  $H^k$  weak formulation of a boundary value problem of order  $2k$ . This is accomplished by replacing the Galerkin orthogonality with estimates borrowed from *a posteriori* error analysis and by using a discrete energy norm that is well defined for functions in  $H^k$ .

### 1. INTRODUCTION

Let  $\Omega \subset \mathbb{R}^n$ , be a bounded polygonal domain and  $f \in L_2(\Omega)$ . For simplicity, we assume  $n = 2$ . However, our error analysis may be extended to any  $n \geq 1$ . To give the motivation, consider a model problem of finding  $u \in H_0^1(\Omega)$  such that

$$(1.1) \quad \int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx \quad \forall v \in H_0^1(\Omega).$$

Many discontinuous Galerkin (DG) methods [3] have been proposed for (1.1) based on a discrete space

$$V_h \subseteq \{v \in L_2(\Omega) : v|_T \in P_r(T) \, \forall T \in \mathcal{T}_h\},$$

where  $\mathcal{T}_h$  is a triangulation of the computational domain  $\Omega$ . For these methods, *a priori* error estimates are derived based on Galerkin orthogonality and Cea's Lemma assuming that the solution  $u$  of (1.1) has the following regularity:

$$u|_T \in H^s(T), \quad \forall T \in \mathcal{T}_h, \quad s > \frac{3}{2}.$$

To be more precise, consider the variational form of the symmetric interior penalty method [22, 39, 2] : Find  $u_h \in V_h$  such that

$$(1.2) \quad a_h(u_h, v_h) = \int_{\Omega} f v_h \, dx \quad \forall v_h \in V_h,$$

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Received by the editor January 5, 2009 and, in revised form, June 16, 2009.

2010 *Mathematics Subject Classification*. Primary 65N30, 65N15.

*Key words and phrases*. Optimal error estimates, elliptic regularity, finite element, discontinuous Galerkin, nonconforming.

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where

$$a_h(w, v) = \sum_{T \in \mathcal{T}_h} \int_T \nabla w \cdot \nabla v \, dx - \sum_{e \in \mathcal{E}_h} \int_e (\{\!\!\{ \nabla w \}\!\!\} [v] + \{\!\!\{ \nabla v \}\!\!\} [w]) \, ds \\ + \sum_{e \in \mathcal{E}_h} \int_e \frac{\sigma}{h_e} [w] [v] \, ds \quad w, v \in V_h,$$

the jumps and means are defined as in Section 3 and  $\sigma$  is a sufficiently large positive constant.

By writing the solution  $u$  of (1.1) as the sum of a regular part and a singular part [27], the following integration by parts formula [15, Lemma 2.1] can be proved under the regularity result that the solution  $u \in H^s(\Omega)$  for  $s > 3/2$  [27]:

$$\int_T \nabla u \cdot \nabla v \, dx = \int_{\partial T} \nabla u \cdot \nu \, ds + \int_T f v \, dx \quad \forall T \in \mathcal{T}_h, \quad \forall v \in V_h.$$

Hence the solution  $u$  of (1.1) satisfies

$$a_h(u, v_h) = \int_{\Omega} f v_h \, dx \quad \forall v_h \in V_h,$$

which implies the following Galerkin orthogonality:

$$(1.3) \quad a_h(u - u_h, v_h) = 0 \quad \forall v_h \in V_h.$$

Then the following *a priori* error estimate [35, 15, 13, 14, 34] is obtained from (1.3):

$$(1.4) \quad \|u - u_h\|_{1,h} \leq C \inf_{v_h \in V_h} \|u - v_h\|_{1,h},$$

where for  $w \in H^s(\Omega) + V_h$ ,  $s > 3/2$ ,

$$\|w\|_{1,h}^2 = \sum_{T \in \mathcal{T}_h} \|\nabla w\|_{L_2(T)}^2 + \sum_{e \in \mathcal{E}_h} h_e \|\{\!\!\{ \nabla w \}\!\!\}\|_{L_2(e)}^2 + \sum_{e \in \mathcal{E}_h} \frac{\sigma}{h_e} \|[w]\|_{L_2(e)}^2$$

and  $h_e$  is the length of  $e$ . Note that the regularity result  $u \in H^s(\Omega)$  for  $s > 3/2$  is needed to handle the term  $\|\{\!\!\{ \nabla(u - v_h) \}\!\!\}\|_{L_2(e)}$  in (1.4).

We see from the discussion above that the derivation of (1.3) for discontinuous Galerkin methods requires the nontrivial elliptic regularity theory in polygonal domains. The goal of this paper is to provide a new type of error estimates that does not require such regularity results. This new approach is particularly useful for more complicated problems, such as linear interface, where  $u$  has low regularity [31].

We will derive the new error estimates by an analog of the Berger–Scott–Strang lemma [5] that decomposes the error into two parts in which one measures the interpolation error and the other measures the nonconforming error and the consistency error together. Thereby the analysis does not require any regularity other than that the weak solution of a PDE of order  $2k$  belongs to  $H^k(\Omega)$ . We obtain the following error estimate for (1.2):

$$(1.5) \quad \|u - u_h\|_h \leq C \left( \inf_{v_h \in V_h} \|u - v_h\|_h + \text{Osc}_k(f) \right),$$

where

$$\|w\|_h^2 = \sum_{T \in \mathcal{T}_h} \|\nabla w\|_{L_2(T)}^2 + \sum_{e \in \mathcal{E}_h} \frac{\sigma}{h_e} \|[w]\|_{L_2(e)}^2 \quad w \in H^1(\Omega) + V_h$$

and  $Osc_k(f)$  which measures the oscillations of  $f$  is defined in (2.8). Note on the right-hand sides of (1.5) that the first term quantifies the interpolation error and the second term measures the oscillations of  $f$  which are of the same order with the first term (assuming that  $u$  has enough regularity) if  $f \in L_2(\Omega)$  and higher order if  $f$  is sufficiently smooth. Our error analysis is motivated by the recent *a posteriori* error analysis for discontinuous Galerkin methods [16, 18, 30, 29] and the results of [8]. The key ingredients are the discrete local efficiency arguments [37, 38] for *a posteriori* error estimators and an enriching map for piecewise smooth functions [6, 7, 8, 9, 10].

The rest of the article is organized as follows. In Section 2, we present the main result in an abstract lemma which enables us to decompose the error. Section 3 and Section 4 are devoted to the applications of the abstract lemma to various nonconforming and discontinuous Galerkin methods for second and fourth order elliptic problems, respectively. Finally, in Section 5, we present conclusions and possible extensions.

## 2. ABSTRACT RESULT

Recall the Sobolev-Hilbert space  $H^k(\Omega)$  which is the set of all  $L_2(\Omega)$  functions whose distributional derivatives up to order  $k$  are in  $L_2(\Omega)$ . Denote by  $V := H_0^k(\Omega)$ , the set of all functions in  $H^k(\Omega)$  whose traces up to order  $k - 1$  vanishes. Denote the norm on  $V$  by  $\|\cdot\|_V$ . The model problem is to find  $u \in V$  such that

$$(2.1) \quad a(u, v) = (f, v) \quad \forall v \in V,$$

where  $a(\cdot, \cdot)$  is the bilinear form for the underlying PDE of order  $2k$  and  $(\cdot, \cdot)$  denotes the  $L_2$  inner product. We assume that the bilinear form  $a$  is bounded and elliptic so that the model problem (2.1) has a unique solution  $u \in V$ . Denote by  $\mathcal{T}_h$  a regular (without hanging nodes) simplicial triangulation of  $\Omega$ . Let  $h_T = \text{diam } T$  and  $h = \max\{h_T : T \in \mathcal{T}_h\}$ . Let the discontinuous finite element space  $V_h$  be a subspace of

$$V_h^r = \{v_h \in L_2(\Omega) : v_h|_T \in P_r(T) \forall T \in \mathcal{T}_h\}$$

where  $P_r(D)$  is the space of polynomials of degree less than or equal to  $r$  restricted to the set  $D$ . Let  $\|\cdot\|_h$  be a mesh dependent norm on  $V + V_h$  and  $V_c$  a finite element subspace of  $V$  associated with  $\mathcal{T}_h$ .

The discontinuous finite element method is to find  $u_h \in V_h$  such that

$$(2.2) \quad a_h(u_h, v) = (f, v) \quad \forall v \in V_h,$$

where the bilinear form  $a_h(\cdot, \cdot)$  is defined on  $V_h \times (V_c + V_h)$ .

We make the following abstract assumptions:

(N1) There is a positive constant  $C$  independent of  $h$  such that

$$(2.3) \quad C\|v\|_h^2 \leq a_h(v, v) \quad \forall v \in V_h.$$

(N2) There is a positive constant  $C$  independent of  $h$  such that for all  $w \in V_c$ ,

$$(2.4) \quad |a(v, w) - a_h(v_h, w)| \leq C\|v - v_h\|_h\|w\|_V \quad \forall v \in V \text{ and } \forall v_h \in V_h.$$

(N3) There exists a linear map  $E_h : V_h \rightarrow V_c$  satisfying

$$(2.5) \quad \|E_h v\|_V \leq C\|v\|_h \quad \forall v \in V_h,$$

for some positive constant  $C$  independent of  $h$ .

It can be readily seen from (2.3) that there is a unique solution  $u_h \in V_h$  for (2.2).

We are now ready to prove an abstract lemma.

**Lemma 2.1.** *Let  $u \in V := H_0^k(\Omega)$  and  $u_h \in V_h$  be the solutions of (2.1) and (2.2), respectively. Assume that the assumptions (N1)–(N3) hold. Then there exists a positive constant  $C$  independent of  $h$  such that*

$$(2.6) \quad \|u - u_h\|_h \leq C \inf_{v \in V_h} \left[ \|u - v\|_h + \sup_{\phi \in V_h \setminus \{0\}} \frac{(f, \phi - E_h \phi) - a_h(v, \phi - E_h \phi)}{\|\phi\|_h} \right].$$

In addition, if there exists a positive constant  $C$  independent of  $h$  such that for any  $v \in V_h$ ,

$$(2.7) \quad \sup_{\phi \in V_h \setminus \{0\}} \frac{(f, \phi - E_h \phi) - a_h(v, \phi - E_h \phi)}{\|\phi\|_h} \leq C(\|u - v\|_h + \text{Osc}_k(f)),$$

where

$$(2.8) \quad \text{Osc}_k(f) = \left( \sum_{T \in \mathcal{T}_h} h_T^{2k} \left[ \inf_{\bar{f} \in P_{r-k}(T)} \|f - \bar{f}\|_{L_2(T)}^2 \right] \right)^{1/2},$$

then

$$(2.9) \quad \|u - u_h\|_h \leq \tilde{C} \left( \inf_{v \in V_h} \|u - v\|_h + \text{Osc}_k(f) \right).$$

Here  $\tilde{C}$  is a positive constant independent of  $h$ .

*Proof.* Let  $v \in V_h$  be such that  $v \neq u_h$ . Let  $\psi = u_h - v$ . From (2.3), (2.2) and (2.1), we get

$$\begin{aligned} C\|u_h - v\|_h^2 &\leq a_h(u_h - v, \psi) \\ &= (f, \psi) - a_h(v, \psi) \\ &= a(u, E_h \psi) - a_h(v, E_h \psi) + (f, \psi - E_h \psi) - a_h(v, \psi - E_h \psi). \end{aligned}$$

We obtain

$$\|u_h - v\|_h \leq C \left( \frac{a(u, E_h \psi) - a_h(v, E_h \psi)}{\|u_h - v\|_h} + \frac{(f, \psi - E_h \psi) - a_h(v, \psi - E_h \psi)}{\|u_h - v\|_h} \right).$$

Using (2.4) and (2.5), we find

$$\begin{aligned} |a(u, E_h \psi) - a_h(v, E_h \psi)| &\leq C\|u - v\|_h \|E_h \psi\|_V \\ &\leq C\|u - v\|_h \|\psi\|_h \\ &= C\|u - v\|_h \|u_h - v\|_h, \end{aligned}$$

which implies that

$$\frac{|a(u, E_h \psi) - a_h(v, E_h \psi)|}{\|u_h - v\|_h} \leq C\|u - v\|_h.$$

It is obvious that

$$\frac{(f, \psi - E_h \psi) - a_h(v, \psi - E_h \psi)}{\|\psi\|_h} \leq \sup_{\phi \in V_h \setminus \{0\}} \frac{(f, \phi - E_h \phi) - a_h(v, \phi - E_h \phi)}{\|\phi\|_h}.$$

Now a use of the triangle inequality yields the estimate (2.6). Finally, we use (2.7) in (2.6) to complete the proof.  $\square$

We have immediately the following corollary.

**Corollary 2.2.** *Assume that the hypothesis for Lemma 2.1 is true. Furthermore, assume that the oscillation  $Osc_k(f)$  is zero. Then, the error estimate (2.9) is quasi-optimal in the sense that there is a positive constant  $C$  independent of  $h$  such that*

$$\|u - u_h\|_h \leq C \inf_{v \in V_h} \|u - v\|_h.$$

3. SECOND ORDER PROBLEMS ( $k = 1$ )

Here we have  $V = H_0^1(\Omega)$  and  $\|v\|_V = \|\nabla v\|_{L_2(\Omega)}$ . For given  $f \in L_2(\Omega)$ , the model problem is to find  $u \in V$  such that

$$(3.1) \quad a(u, v) = (f, v) \quad \forall v \in V,$$

where

$$(3.2) \quad a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx.$$

We now introduce some notation. Denote the set of all interior edges of  $\mathcal{T}_h$  by  $\mathcal{E}_h^i$ , the set of boundary edges by  $\mathcal{E}_h^b$ , and define  $\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^b$ . The length of any edge  $e \in \mathcal{E}_h$  will be denoted by  $h_e$ . Define a broken Sobolev space

$$H^1(\Omega, \mathcal{T}_h) = \{v \in L_2(\Omega) : v_T = v|_T \in H^1(T) \quad \forall T \in \mathcal{T}_h\}.$$

For any  $e \in \mathcal{E}_h^i$ , there are two triangles  $T_+$  and  $T_-$  such that  $e = \partial T_+ \cap \partial T_-$ . Let  $n_-$  be the unit normal of  $e$  pointing from  $T_-$  to  $T_+$ , and  $n_+ = -n_-$ . (cf. Fig. 3.1). For any  $v \in H^1(\Omega, \mathcal{T}_h)$ , we define the jump and mean of  $v$  on  $e$  by

$$[[v]] = v_- n_- + v_+ n_+ \quad \text{and} \quad \{v\} = \frac{1}{2}(v_- + v_+),$$

where  $v_{\pm} = v|_{T_{\pm}}$ . Similarly, define for  $w \in H^1(\Omega, \mathcal{T}_h)^2$  the jump and mean of  $w$  on  $e \in \mathcal{E}_h^i$  by

$$[[w]] = w_- \cdot n_- + w_+ \cdot n_+, \quad \text{and} \quad \{w\} = \frac{1}{2}(w_- + w_+),$$

where  $w_{\pm} = w|_{T_{\pm}}$ .

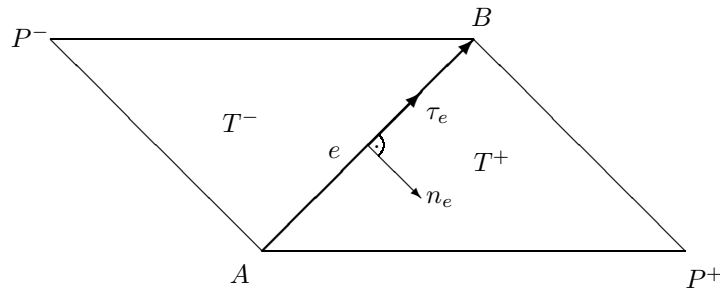


FIGURE 3.1. Two neighboring triangles  $T_-$  and  $T_+$  that share the edge  $e = \partial T_- \cap \partial T_+$  with initial node  $A$  and end node  $B$  and unit normal  $n_e$ . The orientation of  $n_e = n_- = -n_+$  equals the outer normal of  $T_-$ , and hence, points into  $T_+$ .

For any edge  $e \in \mathcal{E}_h^b$ , there is a triangle  $T \in \mathcal{T}_h$  such that  $e = \partial T \cap \partial\Omega$ . Let  $n_e$  be the unit normal of  $e$  that points outside  $T$ . For any  $v \in H^1(T)$ , we set on  $e \in \mathcal{E}_h^b$ ,

$$[[v]] = vn_e \text{ and } \{\{v\}\} = v,$$

and for  $w \in H^1(T)^2$ ,

$$[[w]] = w \cdot n_e, \text{ and } \{\{w\}\} = w.$$

We recall the following trace inequality on  $V_h$  [11, 19].

**Lemma 3.1.** *There exists a positive  $C$  independent of  $h$  such that for  $v_h \in V_h^T$ ,*

$$(3.3) \quad \|v_h\|_{L_2(e)} \leq Ch_e^{-1/2} \|v_h\|_{L_2(T)} \quad \forall T \in \mathcal{T}_h,$$

where  $e$  is an edge of  $T$ .

Throughout this section,  $\Delta$  denotes the Laplacian.

The proof of the following lemma is similar to well-known discrete local efficiency estimates [1, 38, 37, 30] in a *posteriori* error analysis of second order problems when  $v = u_h$  is the finite element solution. We state the result here and omit the proof.

**Lemma 3.2.** *Let  $v \in V_h$ . Then there is a positive constant  $C$  independent of  $h$  such that*

$$(3.4) \quad \sum_{T \in \mathcal{T}_h} h_T^2 \|f + \Delta v\|_{L_2(T)}^2 \leq C \left( \sum_{T \in \mathcal{T}_h} \|\nabla(u - v)\|_{L_2(T)}^2 + Osc_1(f)^2 \right)$$

and

$$(3.5) \quad \sum_{e \in \mathcal{E}_h^i} h_e \|[[\nabla v]]\|_{L_2(e)}^2 \leq C \left( \sum_{T \in \mathcal{T}_h} \|\nabla(u - v)\|_{L_2(T)}^2 + Osc_1(f)^2 \right),$$

where  $Osc_1(f)$  is defined in (2.8).

Let  $V_c = V_h \cap V$  be the conforming finite element space. The construction of an enriching map  $E_h : V_h \rightarrow V_c$  can be done by averaging [6, 7, 8]. Let  $p$  be any interior node of the Lagrange  $P_r$  finite element space associated with the triangulation  $\mathcal{T}_h$  and let  $\mathcal{T}_p$  be the set of all triangles sharing the node  $p$ . For  $v \in V_h$ , define  $E_h v \in V_c$  by

$$(3.6) \quad E_h v(p) = \frac{1}{|\mathcal{T}_p|} \sum_{T \in \mathcal{T}_p} v|_T(p)$$

where  $|\mathcal{T}_p|$  is the cardinality of  $\mathcal{T}_p$ . For all boundary nodes  $p$ , set  $E_h v(p) = 0$ . For the rest of the section, we use this  $E_h$ .

We now present the applications of Lemma 2.1 to a wide range of discontinuous finite element methods.

**3.1. Classical nonconforming method.** The discrete space is the Crouzeix-Raviart [21] nonconforming  $P_1$  finite element space defined by

$$(3.7) \quad V_h = \{v_h \in L_2(\Omega) : v_h|_T \in P_1(T) \quad \forall T \in \mathcal{T}_h, \int_e [[v]] ds = 0 \quad \forall e \in \mathcal{E}_h\}.$$

Define the norm  $\|\cdot\|_h$  by

$$(3.8) \quad \|v\|_h^2 = \sum_{T \in \mathcal{T}_h} \int_T |\nabla v|^2 dx.$$

The nonconforming approximate solution  $u_h \in V_h$  is obtained by solving

$$(3.9) \quad a_h(u_h, v) = (f, v) \quad \forall v \in V_h,$$

where

$$(3.10) \quad a_h(w, v) = \sum_{T \in \mathcal{T}_h} \int_T \nabla w \cdot \nabla v \, dx \quad \forall w, v \in V_h.$$

It is easy to check the assumptions (N1) and (N2). The enriching map  $E_h$  in (3.6) satisfies [6, 7, 8, 30]

$$(3.11) \quad \sum_{T \in \mathcal{T}_h} h_T^{-2} \|E_h v - v\|_{L_2(T)}^2 + \|E_h v\|_V^2 \leq C \|v\|_h^2 \quad \forall v \in V_h.$$

Therefore, the estimate (2.6) is valid for the nonconforming method (3.9).

We now verify the estimate (2.7). For this, let  $v, \phi \in V_h$  and denote  $\psi = \phi - E_h \phi$ . Using (3.3), (3.4), (3.5) and (3.11),

$$\begin{aligned} (f, \psi) - a_h(v, \psi) &= (f, \psi) - \sum_{T \in \mathcal{T}_h} \int_T \nabla v \cdot \nabla \psi \, dx = (f, \psi) - \sum_{e \in \mathcal{E}_h^i} \int_e \llbracket \nabla v \rrbracket \{\!\!\{ \psi \}\!\!\} \, ds \\ &\leq C \sum_{T \in \mathcal{T}_h} \|f\|_{L_2(T)} \|\phi - E_h \phi\|_{L_2(T)} \\ &\quad + C \sum_{e \in \mathcal{E}_h^i} \|\llbracket \nabla v \rrbracket\|_{L_2(e)} \|\{\!\!\{ \phi - E_h \phi \}\!\!\}\|_{L_2(e)} \\ &\leq C \left( \sum_{T \in \mathcal{T}_h} h_T^2 \|f\|_{L_2(T)}^2 + \sum_{e \in \mathcal{E}_h^i} h_e \|\llbracket \nabla v \rrbracket\|_{L_2(e)}^2 \right)^{1/2} \|\phi\|_h \\ &\leq C (\|u - v\|_h + \text{Osc}_1(f)) \|\phi\|_h. \end{aligned}$$

Therefore,

$$\sup_{\phi \in V_h \setminus \{0\}} \frac{(f, \phi - E_h \phi) - a_h(v, \phi - E_h \phi)}{\|\phi\|_h} \leq C (\|u - v\|_h + \text{Osc}_1(f)).$$

Hence, the estimate (2.9) holds for the nonconforming finite element method.

**3.2. Discontinuous Galerkin (DG) methods.** In this section, we present the application of Lemma 2.1 to discontinuous Galerkin methods [3] for second order elliptic problems.

The DG finite element space is defined by

$$V_h = \{v_h \in L_2(\Omega) : v_h|_T \in P_r(T) \, \forall T \in \mathcal{T}_h\}$$

for any  $r \geq 1$ . Define the norm  $\|\cdot\|_h$  on  $V_h$  by

$$\|v\|_h^2 = \sum_{T \in \mathcal{T}_h} \int_T |\nabla v|^2 \, dx + \sum_{e \in \mathcal{E}_h} \frac{\sigma}{h_e} \|\llbracket v \rrbracket\|_{L_2(e)}^2,$$

where  $\sigma > 0$  is the stabilizing parameter.

The enriching map  $E_h$  in (3.6) satisfies [6, 7, 8, 30]

$$(3.12) \quad \sum_{T \in \mathcal{T}_h} h_T^{-2} \|E_h v - v\|_{L_2(T)}^2 + \|E_h v\|_V^2 \leq C \|v\|_h^2 \quad \forall v \in V_h.$$

This validates the assumption (N3).

In order to define DG methods, we introduce the following. Let  $W_h = V_h \times V_h$ . Given  $e \in \mathcal{E}_h$ , define the local lifting operators  $r_e : L_2(e)^2 \rightarrow W_h$  and  $\ell_e : L_2(e) \rightarrow W_h$  by

$$\begin{aligned} \int_{\Omega} r_e(q) \cdot \tau_h \, dx &= \int_e q \cdot \{\{\tau_h\}\} \, ds \quad \text{for all } \tau_h \in W_h, \\ \int_{\Omega} \ell_e(v) \cdot \tau_h \, dx &= \int_e v \llbracket \tau_h \rrbracket \, ds \quad \text{for all } \tau_h \in W_h. \end{aligned}$$

The global lifting operators  $r : L_2(\mathcal{E}_h)^2 \rightarrow W_h$  and  $\ell : L_2(\mathcal{E}_h^i) \rightarrow W_h$ ,

$$r := \sum_{e \in \mathcal{E}_h} r_e \quad \text{and} \quad \ell := \sum_{e \in \mathcal{E}_h^i} \ell_e$$

satisfy

$$\begin{aligned} \int_{\Omega} r(q) \cdot \tau_h \, dx &= \sum_{e \in \mathcal{E}_h} \int_e q \cdot \{\{\tau_h\}\} \, ds \quad \text{for all } \tau_h \in W_h, \\ \int_{\Omega} \ell(v) \cdot \tau_h \, dx &= \sum_{e \in \mathcal{E}_h^i} \int_e v \cdot \llbracket r_h \rrbracket \, ds \quad \text{for all } \tau_h \in W_h. \end{aligned}$$

The variational form of the DG methods [17] is to find  $u_h \in V_h$  such that

$$(3.13) \quad a_h(u_h, v) = (f, v) \quad \forall v \in V_h,$$

where for  $w, v \in V_h$ ,

$$\begin{aligned} a_h(w, v) &= \sum_{T \in \mathcal{T}_h} \int_T \nabla w \cdot \nabla v \, dx - \sum_{e \in \mathcal{E}_h} \int_e (\{\{\nabla w\}\} \llbracket v \rrbracket + \theta \{\{\nabla v\}\} \llbracket w \rrbracket) \, ds \\ &\quad + \sum_{e \in \mathcal{E}_h^i} \int_e (\beta \cdot \llbracket w \rrbracket \llbracket \nabla v \rrbracket + \llbracket \nabla w \rrbracket \beta \cdot \llbracket v \rrbracket) \, ds \\ &\quad + \int_{\Omega} \alpha [r(\llbracket w \rrbracket) + \ell(\beta \cdot \llbracket w \rrbracket)] \cdot [r(\llbracket v \rrbracket) + \ell(\beta \cdot \llbracket v \rrbracket)] \, ds \\ &\quad + \sum_{e \in \mathcal{E}_h} \int_e \frac{\sigma}{h_e} \llbracket w \rrbracket \llbracket v \rrbracket \, ds \end{aligned}$$

for  $\theta = \pm$ ,  $\alpha \in \{0, 1\}$ ,  $\beta \in \mathbb{R}^2$  and  $\sigma > 0$ . Here  $\theta = 1$ ,  $\beta = (0, 0)$  and  $\alpha = 0$  give the symmetric interior penalty method (SIPG) [22, 39, 2],  $\theta = -1$ ,  $\beta = (0, 0)$  and  $\alpha = 0$  give the nonsymmetric interior penalty method (NIPG) [36], and  $\theta = 1$ ,  $\beta \in \mathbb{R}^2$  and  $\alpha = 1$  give the local discontinuous Galerkin method (LDG) [20, 17].

Under the assumption  $\sigma \geq \sigma_* > 0$  ( $\sigma_*$  is sufficiently large for SIPG and  $\sigma_* > 0$  for NIPG and LDG), it is known [3] that there is a positive constant  $C$  such that  $C \|v\|_h^2 \leq a_h(v, v)$  for all  $v \in V_h$ . This verifies the assumption (N1).

For  $v \in V$ ,  $w \in V_c$  and  $v_h \in V_h$ , note that

$$\begin{aligned} a(v, w) - a_h(v_h, w) &= \sum_{T \in \mathcal{T}_h} \int_T \nabla(v - v_h) \cdot \nabla w \, dx - \sum_{e \in \mathcal{E}_h} \int_e \{\{\nabla w\}\} \llbracket v - v_h \rrbracket \, ds \\ &\quad + \sum_{e \in \mathcal{E}_h^i} \int_e \beta \cdot \llbracket \nabla w \rrbracket \llbracket v - v_h \rrbracket \, ds \end{aligned}$$



where we have used the fact that for all  $e \in \mathcal{E}_h$ ,  $[[v]] = 0$ . A use of the trace inequality (3.3) implies that

$$\begin{aligned} |a(v, w) - a_h(v_h, w)| &\leq \sum_{T \in \mathcal{T}_h} \|\nabla(v - v_h)\|_{L^2(T)} \|\nabla w\|_{L^2(T)} \\ &\quad + \sum_{e \in \mathcal{E}_h} \|\{\{\nabla w\}\}\|_{L^2(e)} \|[[v - v_h]]\|_{L^2(e)} \\ &\quad + \sum_{e \in \mathcal{E}_h^i} \|\beta \cdot [[\nabla w]]\|_{L^2(e)} \|[[v - v_h]]\|_{L^2(e)} \\ &\leq \sum_{T \in \mathcal{T}_h} \|\nabla(v - v_h)\|_{L^2(T)} \|\nabla w\|_{L^2(T)} \\ &\quad + C_{\sigma^*, \beta} \left( \sum_{T \in \mathcal{T}_h} \|\nabla w\|_{L^2(T)}^2 \right)^{1/2} \left( \sum_{e \in \mathcal{E}_h} \frac{\sigma}{h_e} \|[[v - v_h]]\|_{L^2(e)}^2 \right)^{1/2}. \end{aligned}$$

Hence, the assumption (N2) holds.

We now verify the estimate (2.7). Let  $v, \phi \in V_h$  and denote  $\psi = \phi - E_h\phi$ . Then

$$\begin{aligned} (f, \psi) - a_h(v, \psi) &= \sum_{T \in \mathcal{T}_h} \int_T (f + \Delta v)\psi \, dx - \sum_{e \in \mathcal{E}_h^i} \int_e [[\nabla v]] \{\{\psi\}\} \, ds \\ &\quad + \theta \sum_{e \in \mathcal{E}_h} \int_e \{\{\nabla \psi\}\} [[v]] \, ds - \sum_{e \in \mathcal{E}_h} \int_e \frac{\sigma}{h_e} [[v]] [[\psi]] \, ds \\ &\quad - \sum_{e \in \mathcal{E}_h^i} \int_e \alpha(\beta \cdot [[v]] [[\nabla \psi]] + [[\nabla v]] \beta \cdot [[\psi]]) \, ds \\ &\quad - \int_{\Omega} [r([[v]]) + \ell(\beta \cdot [[v]])] \cdot [r([[ \psi]]) + \ell(\beta \cdot [[\psi]])] \, ds. \end{aligned}$$

Using the bounds for  $r$  and  $\ell$  in [3] that

$$\begin{aligned} \|r([[v]])\|_{L_2(\Omega)}^2 &\leq C \sum_{e \in \mathcal{E}_h} \frac{1}{h_e} \|[[v]]\|_{L_2(e)}^2 \quad \forall v \in V_h, \\ \|\ell(\beta \cdot [[v]])\|_{L_2(\Omega)}^2 &\leq C \sum_{e \in \mathcal{E}_h^i} \frac{1}{h_e} \|[[v]]\|_{L_2(e)}^2 \quad \forall v \in V_h, \end{aligned}$$

the trace inequality (3.3), (3.12) and Lemma 3.2, we obtain

$$\sup_{\phi \in V_h \setminus \{0\}} \frac{(f, \phi - E_h\phi) - a_h(v, \phi - E_h\phi)}{\|\phi\|_h} \leq C(\|u - v\|_h + Osc_1(f))$$

and hence we conclude the error estimate in (2.9) for the DG methods in (3.13).

**3.3. Weakly over-penalized symmetric interior penalty (WOPSIP) method.** Here we highlight that the above analysis is also applicable to over-penalized interior penalty methods. We consider the weakly over-penalized symmetric interior penalty method in [15]. For any  $v \in H^1(\Omega, \mathcal{T}_h)$ , define on  $e \in \mathcal{E}_h$ ,

$$\pi_e([[v]]) = \frac{1}{h_e} \int_e [[v]] \, ds.$$

Note that  $\pi_e(\llbracket u \rrbracket) = 0$  for all  $e \in \mathcal{E}_h$ . Define

$$V_h = \{v_h \in L_2(\Omega) : v_h|_T \in P_1(T) \forall T \in \mathcal{T}_h\},$$

and the norm  $\|\cdot\|_h$  on  $V_h$  by

$$\|v\|_h^2 = \sum_{T \in \mathcal{T}_h} \int_T |\nabla v|^2 dx + \sum_{e \in \mathcal{E}_h} \frac{1}{h_e^2} \pi_e(\llbracket v \rrbracket)^2.$$

The WOPSIP method is to find  $u_h \in V_h$  such that

$$a_h(u_h, v) = (f, v) \quad \forall v \in V_h,$$

where for  $w, v \in V_h$ ,

$$a_h(w, v) = \sum_{T \in \mathcal{T}_h} \int_T \nabla w \cdot \nabla v dx + \sum_{e \in \mathcal{E}_h} \frac{1}{h_e^2} \pi_e(\llbracket w \rrbracket) \pi_e(\llbracket v \rrbracket).$$

Let  $E_h : V_h \rightarrow V_c$  be defined as in (3.6), where  $V_c$  is the conforming  $P_1$  finite element space. Then it can be seen [6, 7, 8] that

$$\sum_{T \in \mathcal{T}_h} h_T^{-2} \|E_h v - v\|_{L_2(T)}^2 + \|E_h v\|_V^2 \leq C \|v\|_h^2 \quad \forall v \in V_h.$$

Note that

$$a_h(v, v) = \|v\|_h^2 \quad \forall v \in V_h$$

and for  $v \in V$ ,  $w \in V_c$  and  $v_h \in V_h$ ,

$$|a(v, w) - a_h(v_h, w)| = \sum_{T \in \mathcal{T}_h} \int_T \nabla(v - v_h) \cdot \nabla w dx \leq C \|v - v_h\|_h \|w\|_V.$$

We now verify a variant of (2.7). Let  $v, \phi \in V_h$  and  $\psi = \phi - E_h \phi$ . Then using (3.4) and (3.5), we get

$$\begin{aligned} (f, \psi) - a_h(v, \psi) &= (f, \psi) - \sum_{T \in \mathcal{T}_h} \int_T \nabla v \cdot \nabla \psi dx - \sum_{e \in \mathcal{E}_h} \frac{1}{h_e^2} \pi_e(\llbracket v \rrbracket) \pi_e(\llbracket \psi \rrbracket) \\ &= \sum_{T \in \mathcal{T}_h} \int_T f \psi dx - \sum_{e \in \mathcal{E}_I} \int_e \llbracket \nabla v \rrbracket \llbracket \psi \rrbracket ds \\ &\quad - \sum_{e \in \mathcal{E}_h^i} \int_e \llbracket \nabla v \rrbracket \pi_e(\llbracket \psi \rrbracket) ds + \sum_{e \in \mathcal{E}_h} \frac{1}{h_e^2} \pi_e(\llbracket u - v \rrbracket) \pi_e(\llbracket \psi \rrbracket) \\ &\leq C(\|u - v\|_h + \text{Osc}_1(f)) \|\phi\|_h + C \left( \sum_{T \in \mathcal{T}_h} h_T^2 \|\nabla u\|_{L_2(T)}^2 \right)^{1/2} \|\phi\|_h. \end{aligned}$$

Therefore,

$$\begin{aligned} \sup_{\phi \in V_h \setminus \{0\}} \frac{(f, \phi - E_h \phi) - a_h(v, \phi - E_h \phi)}{\|\phi\|_h} &\leq C(\|u - v\|_h + \text{Osc}_1(f)) \\ &\quad + C \left( \sum_{T \in \mathcal{T}_h} h_T^2 \|\nabla u\|_{L_2(T)}^2 \right)^{1/2}. \end{aligned}$$

Hence,

$$\|u - u_h\|_h \leq C \left[ \inf_{v \in V_h} \|u - v\|_h + Osc_1(f) + \left( \sum_{T \in \mathcal{T}_h} h_T^2 \|\nabla u\|_{L_2(T)}^2 \right)^{1/2} \right].$$

The error estimate in (3.3) is slightly different compared to the other DG methods. However, the estimate is still optimal up to the regularity of  $u$ .

*Remark 3.3.* We note that analogous error estimates hold for other DG formulations in [3, Table 3.2] and in [13].

4. FOURTH ORDER PROBLEMS ( $k = 2$ )

The model problem is to find  $u \in H_0^2(\Omega)$  such that

$$(4.1) \quad a(u, v) = (f, v) \quad \forall v \in H_0^2(\Omega),$$

where

$$a(w, v) = \int_{\Omega} D^2 w : D^2 v \, dx \quad \forall w, v \in H_0^2(\Omega),$$

$$D^2 w : D^2 v = \sum_{i,j=1}^2 \frac{\partial^2 w}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j}.$$

Therefore, for this section  $V = H_0^2(\Omega)$  and the norm  $\|\cdot\|_V = |\cdot|_{H^2(\Omega)}$ .

We define the Sobolev space  $H^s(\Omega, \mathcal{T}_h)$  associated with the triangulation  $\mathcal{T}_h$  as follows:

$$H^s(\Omega, \mathcal{T}_h) = \{v \in L_2(\Omega) : v_T = v|_T \in H^s(T) \quad \forall T \in \mathcal{T}_h\}.$$

For this section, we slightly alter the definition of the jumps and means.

For any  $e \in \mathcal{E}_h^i$ , there are two triangles  $T_+$  and  $T_-$  such that  $e = \partial T_+ \cap \partial T_-$ . Let  $n_e$  be the unit normal of  $e$  pointing from  $T_-$  to  $T_+$  (cf. Fig. 3.1). For any  $v \in H^2(\Omega, \mathcal{T}_h)$ , we define the jump and mean of the normal derivative of  $v$  on  $e$  by

$$\left[ \left[ \frac{\partial v}{\partial n} \right] \right] = \frac{\partial v_+}{\partial n_e} \Big|_e - \frac{\partial v_-}{\partial n_e} \Big|_e \quad \text{and} \quad \left\{ \left\{ \frac{\partial v}{\partial n} \right\} \right\} = \frac{1}{2} \left( \frac{\partial v_+}{\partial n_e} \Big|_e + \frac{\partial v_-}{\partial n_e} \Big|_e \right),$$

where  $v_{\pm} = v|_{T_{\pm}}$ .

Similarly, for any  $v \in H^3(\Omega, \mathcal{T}_h)$ , we define the jump and mean of the second order normal derivative across  $e$  by

$$\left[ \left[ \frac{\partial^2 v}{\partial n^2} \right] \right] = \frac{\partial^2 v_+}{\partial n_e^2} \Big|_e - \frac{\partial^2 v_-}{\partial n_e^2} \Big|_e \quad \text{and} \quad \left\{ \left\{ \frac{\partial^2 v}{\partial n^2} \right\} \right\} = \frac{1}{2} \left( \frac{\partial^2 v_+}{\partial n_e^2} \Big|_e + \frac{\partial^2 v_-}{\partial n_e^2} \Big|_e \right).$$

For any edge  $e \in \mathcal{E}_h^b$ , there is a triangle  $T \in \mathcal{T}_h$  such that  $e = \partial T \cap \partial \Omega$ . Let  $n_e$  be the unit normal of  $e$  that points outside  $T$ . For any  $v \in H^2(T)$ , we set

$$\left[ \left[ \frac{\partial v}{\partial n} \right] \right] = -\frac{\partial v_T}{\partial n_e},$$

and for any  $v \in H^3(T)$ , we set

$$\left\{ \left\{ \frac{\partial^2 v}{\partial n^2} \right\} \right\} = \frac{\partial^2 v_T}{\partial n_e^2}.$$

Let  $V_c \subset H_0^2(\Omega)$  be the Hsieh-Clough-Tocher finite element space associated with  $\mathcal{T}_h$  [19, 11]. As in the previous section, the linear map  $E_h : V_h \rightarrow V_c$  is constructed

by averaging [12, 16]. Let  $N$  be any (global) degree of freedom of  $V_c$ , i.e.,  $N$  is either the evaluation of a shape function or its first order derivatives at an interior node of  $\mathcal{T}_h$ , or the evaluation of the normal derivative of a shape function at a node on an interior edge. For  $v_h \in V_h$ , we define

$$(4.2) \quad N(E_h v_h) = \frac{1}{|\mathcal{T}_N|} \sum_{T \in \mathcal{T}_N} N(v_T),$$

where  $\mathcal{T}_N$  is the set of triangles in  $\mathcal{T}_h$  that share the degree of freedom,  $N$  and  $|\mathcal{T}_N|$  is the number of elements of  $\mathcal{T}_N$ .

We now derive the following discrete local efficiency estimates.

**Lemma 4.1.** *Let  $v \in V_h$ . Then there is a positive constant  $C$  independent of  $h$  such that*

$$(4.3) \quad \sum_{T \in \mathcal{T}_h} h_T^4 \|f - \Delta^2 v\|_{L_2(T)}^2 \leq C \left( \sum_{T \in \mathcal{T}_h} \|D^2(u - v)\|_{L_2(T)}^2 + \text{Osc}_2(f)^2 \right),$$

$$(4.4) \quad \sum_{e \in \mathcal{E}_h^i} h_e \|\llbracket \partial^2 v / \partial n^2 \rrbracket\|_{L_2(e)}^2 \leq C \left( \sum_{T \in \mathcal{T}_h} \|D^2(u - v)\|_{L_2(T)}^2 + \text{Osc}_2(f)^2 \right),$$

where  $\text{Osc}_2(f)$  is defined in (2.8).

*Proof.* The proof is again based on bubble function techniques [16, 38].

Let  $v \in V_h$  and  $\bar{f} \in P_{r-2}(T)$  for an arbitrary  $T \in \mathcal{T}_h$ . Let  $b_T \in P_6(T) \cap H_0^2(T)$  be the bubble function defined on  $T$  such that  $b_T(x_T) = 1$ , where  $x_T$  is the barycenter of  $T$ . Let  $\phi = b_T(\bar{f} - \Delta^2 v)$  on  $T$  and extend it to be zero on  $\Omega \setminus T$ . We have for some mesh independent constants  $C_1$  and  $C_2$  that

$$C_1 \|\bar{f} - \Delta^2 v\|_{L_2(T)} \leq \|\phi\|_{L_2(T)} \leq C_2 \|\bar{f} - \Delta^2 v\|_{L_2(T)}.$$

It follows from (4.1) and integration by parts that

$$\begin{aligned} (f - \Delta^2 v, \phi) &= \int_{\Omega} f \phi \, dx - \int_T \Delta^2 v \phi \, dx \\ &= \int_{\Omega} D^2 u : D^2 \phi \, dx - \int_T D^2 v : D^2 \phi \, dx = \int_T D^2(u - v) : D^2 \phi \, dx. \end{aligned}$$

Using a standard inverse estimate [11, 19], we find

$$\begin{aligned} C_1 \|\bar{f} - \Delta^2 v\|_{L_2(T)}^2 &\leq \int_T (\bar{f} - \Delta^2 v) \phi \, dx = \int_T (\bar{f} - f) \phi \, dx + \int_T (f - \Delta^2 v) \phi \, dx \\ &= \int_T (\bar{f} - f) \phi \, dx + \int_T D^2(u - v) : D^2 \phi \, dx \\ &\leq \|f - \bar{f}\|_{L_2(T)} \|\phi\|_{L_2(T)} + |u - v|_{H^2(T)} |\phi|_{H^2(T)} \\ &\leq \left( \|f - \bar{f}\|_{L_2(T)} + Ch_T^{-2} |u - v|_{H^2(T)} \right) \|\phi\|_{L_2(T)} \\ &\leq C \left( \|f - \bar{f}\|_{L_2(T)} + h_T^{-2} |u - v|_{H^2(T)} \right) \|\bar{f} - \Delta^2 v\|_{L_2(T)}, \end{aligned}$$

which implies

$$h_T^2 \|\bar{f} - \Delta^2 v\|_{L_2(T)} \leq C \left( |u - v|_{H^2(T)} + h_T^2 \|f - \bar{f}\|_{L_2(T)} \right).$$

Using the triangle inequality, we obtain

$$h_T^2 \|f - \Delta^2 v\|_{L_2(T)} \leq C \left( |u - v|_{H^2(T)} + h_T^2 \|f - \bar{f}\|_{L_2(T)} \right)$$

which implies (4.3).

We now prove the second estimate (4.4). Let  $e \in \mathcal{E}_h^i$  and  $T_\pm$  be two triangles sharing the edge  $e$ . Let  $\mathcal{T}_e$  be the set of the two triangles  $T_\pm$  and let  $n_e$  denote the unit normal of  $e$  pointing from  $T_-$  to  $T_+$ . (cf. Fig. 3.1).

Let  $\beta$  be the jump  $[[\partial^2 v / \partial n^2]]$  across  $e$  and extend it outside  $e$  so that it is constant along the lines perpendicular to  $e$ . Let  $\zeta_1 \in \mathbb{P}_{r-1}(T_+ \cup T_-)$  be defined by

$$(4.5) \quad \zeta_1 = 0 \quad \text{on the edge } e \quad \text{and} \quad \frac{\partial \zeta_1}{\partial n_e} = \beta.$$

A simple scaling argument shows that

$$(4.6) \quad |\zeta_1|_{H^1(T_\pm)} \approx \left( \int_e h_e \left[ \left[ \frac{\partial^2 v}{\partial n^2} \right] \right]^2 ds \right)^{1/2},$$

$$(4.7) \quad \|\zeta_1\|_{L^\infty(T_\pm)} \approx \left( \int_e h_e \left[ \left[ \frac{\partial^2 v}{\partial n^2} \right] \right]^2 ds \right)^{1/2}.$$

Next we define  $\zeta_2 \in \mathbb{P}_8(T_+ \cup T_-)$  by the following properties:

- (i)  $\zeta_2$  vanishes to the first order on  $(\partial T_+ \cup \partial T_-) \setminus e$  (i.e., the union of the closed line segments  $AP^+$ ,  $AP^-$ ,  $BP^+$  and  $BP^-$  in Fig. 3.1).
- (ii)  $\zeta_2$  is positive on the (open) edge  $e$ .
- (iii)  $\int_{T_+ \cup T_-} \zeta_2 dx = |T_+| + |T_-|$ .

By scaling we have

$$(4.8) \quad \int_e \zeta_2 ds \approx |e|,$$

$$(4.9) \quad |\zeta_2|_{H^1(T_\pm)} \approx 1 \approx \|\zeta_2\|_{L^\infty(T_\pm)}.$$

It follows from (4.1), property (i) in the definition of  $\zeta_2$ , (4.5), (4.8), and integration by parts that

$$(4.10) \quad \begin{aligned} C \int_e \left[ \left[ \frac{\partial^2 v}{\partial n^2} \right] \right]^2 ds &\leq \int_e \beta^2 \zeta_2 ds = \int_e \left[ \left[ \frac{\partial^2 v}{\partial n^2} \right] \right] \frac{\partial \zeta_1}{\partial n_e} \zeta_2 ds = \int_e \left[ \left[ \frac{\partial^2 v}{\partial n^2} \right] \right] \frac{\partial(\zeta_1 \zeta_2)}{\partial n_e} ds \\ &= - \sum_{T \in \mathcal{T}_e} \left( \int_T D^2 v : D^2(\zeta_1 \zeta_2) dx + \int_T (\nabla \cdot D^2 v) \cdot \nabla(\zeta_1 \zeta_2) dx \right) \\ &= - \sum_{T \in \mathcal{T}_e} \left( \int_T D^2 v : D^2(\zeta_1 \zeta_2) dx - \int_T \Delta^2 v (\zeta_1 \zeta_2) dx \right) \\ &= \sum_{T \in \mathcal{T}_e} \int_T D^2(u - v) : D^2(\zeta_1 \zeta_2) dx - \sum_{T \in \mathcal{T}_e} \int_T D^2 u : D^2(\zeta_1 \zeta_2) dx \\ &\quad + \sum_{T \in \mathcal{T}_e} \int_T \Delta^2 v (\zeta_1 \zeta_2) dx \\ &= \sum_{T \in \mathcal{T}_e} \int_T D^2(u - v) : D^2(\zeta_1 \zeta_2) dx - \sum_{T \in \mathcal{T}_e} \int_T (f - \Delta^2 v)(\zeta_1 \zeta_2) dx. \end{aligned}$$

In view of the Poincaré inequality

$$\|\zeta_1 \zeta_2\|_{L_2(T_\pm)} \leq Ch_T^2 |\zeta_1 \zeta_2|_{H^2(T_\pm)}$$

and the inverse inequality

$$|\zeta_1 \zeta_2|_{H^2(T_{\pm})} \leq Ch_T^{-1} |\zeta_1 \zeta_2|_{H^1(T_{\pm})} \leq Ch_e^{-1} |\zeta_1 \zeta_2|_{H^1(T_{\pm})},$$

we deduce from (4.10),

$$(4.11) \quad \int_e \left[ \left[ \frac{\partial^2 v}{\partial n^2} \right] \right]^2 ds \leq C \sum_{T \in \mathcal{T}_e} (|u - v|_{H^2(T)} + h_T^2 \|f - \Delta^2 v\|_{L_2(T)}) |\zeta_1 \zeta_2|_{H^2(T)} \\ \leq C \sum_{T \in \mathcal{T}_e} (|u - v|_{H^2(T)} + h_T^2 \|f - \Delta^2 v\|_{L_2(T)}) h_e^{-1} |\zeta_1 \zeta_2|_{H^1(T_{\pm})}.$$

From (4.6), (4.7) and (4.9) we have

$$|\zeta_1 \zeta|_{H^1(T_{\pm})} \leq |\zeta_1|_{L_{\infty}(T_{\pm})} |\zeta_2|_{H^1(T_{\pm})} + |\zeta_1|_{H^1(T_{\pm})} |\zeta_2|_{L_{\infty}(T_{\pm})} \\ \leq C \left( \int_e h_e \left[ \left[ \frac{\partial^2 v}{\partial n^2} \right] \right]^2 ds \right)^{1/2},$$

which together with (4.11) implies

$$(4.12) \quad \left( \int_e h_e \left[ \left[ \frac{\partial^2 v}{\partial n^2} \right] \right]^2 ds \right)^{1/2} \leq C \sum_{T \in \mathcal{T}_e} (|u - v|_{H^2(T)} + h_T^2 \|f - \Delta^2 v\|_{L_2(T)}).$$

The estimate (4.4) now follows from (4.3) and (4.12). □

We will now present the applications of Lemma 2.1 to a few discontinuous finite element methods for fourth order problems.

**4.1. Morley nonconforming method.** The discrete space [32] is

$$V_h = \{v \in L_2(\Omega) : v|_T \in P_2(T) \ \forall T \in \mathcal{T}_h, \\ v \text{ is continuous at all the vertices of } \mathcal{T}_h, \\ \partial v / \partial n \text{ is continuous at the midpoints of the edges of } \mathcal{T}_h, \\ v = 0 \text{ at all the vertices on } \partial\Omega, \\ \partial v / \partial n = 0 \text{ at all the midpoints of the edges on } \partial\Omega\}.$$

The nonconforming method is to find  $u_h \in V_h$  such that

$$a_h(u_h, v) = (f, v) \quad \forall v \in V_h,$$

where for  $w, v \in V_h$ ,

$$a_h(w, v) = \sum_{T \in \mathcal{T}_h} \int_T D^2 w : D^2 v \, dx.$$

Note that  $a_h$  is also well defined on  $V_c$ . Define the norm  $\|v\|_h$  by  $\|v\|_h^2 = a_h(v, v)$ . To verify (N2), let  $v \in V, w \in V_c$  and  $v_h \in V_h$ . We then note that

$$|a(v, w) - a_h(v_h, w)| \leq \sum_{T \in \mathcal{T}_h} \int_T |D^2(v - v_h)| |D^2 w| \, dx \leq \|v - v_h\|_h \|w\|_V.$$

The map  $E_h$  defined in (4.2) satisfies the estimate [12, 16]

$$(4.13) \quad \sum_{T \in \mathcal{T}_h} \left( h_T^{-4} \|v - E_h v\|_{L_2(T)}^2 + h_T^{-2} |v - E_h v|_{H^1(T)}^2 \right) + \|E_h v\|_V^2 \leq C \|v\|_h^2 \quad \forall v_h \in V_h.$$

Let  $v, \phi \in V_h$  and  $\psi = \phi - E_h \phi$ . For any  $v \in V_h$  and for all vertices  $p$  in  $\mathcal{T}_h$ , we have

$$(4.14) \quad E_h v(p) = v(p).$$

We find that

$$\begin{aligned} a_h(v, \phi - E_h \phi) &= \sum_{T \in \mathcal{T}_h} \int_T D^2 v : D^2(\phi - E_h \phi) \, dx \\ &= \sum_{T \in \mathcal{T}_h} \int_{\partial T} \left( \frac{\partial^2 v}{\partial n^2} \frac{\partial(\phi - E_h \phi)}{\partial n} + \frac{\partial^2 v}{\partial \tau \partial n} \frac{\partial(\phi - E_h \phi)}{\partial \tau} \right) ds \\ &= - \sum_{e \in \mathcal{E}_h^i} \int_e \left[ \frac{\partial^2 v}{\partial n^2} \right] \left\{ \left\{ \frac{\partial(\phi - E_h \phi)}{\partial n} \right\} \right\} ds. \end{aligned}$$

Here we have used the following consequence of (4.14):

$$(4.15) \quad \sum_{T \in \mathcal{T}_h} \int_{\partial T} \frac{\partial^2 v}{\partial \tau \partial n} \frac{\partial(\phi - E_h \phi)}{\partial \tau} ds = \sum_{T \in \mathcal{T}_h} \sum_{e \in \partial T} \frac{\partial^2 v}{\partial \tau \partial n} \int_e \frac{\partial(\phi - E_h \phi)}{\partial \tau} ds = 0,$$

where  $\partial/\partial\tau$  denotes the tangential derivative along  $\partial T$ .

Using (4.3) and (4.4), we find that

$$\begin{aligned} (f, \psi) - a_h(v, \psi) &= (f, \psi) + \sum_{e \in \mathcal{E}_h^i} \int_e \left[ \frac{\partial^2 v}{\partial n^2} \right] \left\{ \left\{ \frac{\partial \psi}{\partial n} \right\} \right\} ds \\ &\leq C \left( \sum_{T \in \mathcal{T}_h} h_T^4 \|f\|_{L_2(T)}^2 + \sum_{e \in \mathcal{E}_h^i} h_e \left\| \left[ \frac{\partial^2 v}{\partial n^2} \right] \right\|_{L_2(e)}^2 \right)^{1/2} \|\phi\|_h \\ &\leq C (\|u - v\|_h + \text{Osc}_2(f)) \|\phi\|_h. \end{aligned}$$

Therefore,

$$\sup_{\phi \in V_h \setminus \{0\}} \frac{(f, \phi - E_h \phi) - a_h(v, \phi - E_h \phi)}{\|\phi\|_h} \leq C (\|u - v\|_h + \text{Osc}_2(f)).$$

Hence, the estimate (2.9) holds true.

**4.2.  $C^0$  interior penalty method.** Define the space  $V_h$  as

$$V_h = \{v \in H_0^1(\Omega) : v|_T \in P_2(T) \quad \forall T \in \mathcal{T}_h\}$$

and the norm  $\|\cdot\|_h$  on  $V_h$  by

$$\|v\|_h^2 = \sum_{T \in \mathcal{T}_h} \int_T D^2 v : D^2 v \, dx + \sum_{e \in \mathcal{E}_h} \int_e \frac{\sigma}{h_e} \left[ \frac{\partial v}{\partial n} \right]^2 ds \quad \forall v \in V_h,$$

where  $\sigma > 0$  is the stabilizing parameter.

The  $C^0$  interior penalty method [12, 23] for (4.1) is to find  $u_h \in V_h$  such that

$$a_h(u_h, v) = (f, v) \quad \forall v \in V_h,$$

where for  $w_h, v_h \in V_h$ ,

$$(4.16) \quad \begin{aligned} a_h(w_h, v_h) = & \sum_{T \in \mathcal{T}_h} \int_T D^2 w_h : D^2 v_h \, dx + \sum_{e \in \mathcal{E}_h} \int_e \left\{ \left\{ \frac{\partial^2 w_h}{\partial n^2} \right\} \right\} \left[ \left[ \frac{\partial v_h}{\partial n} \right] \right] ds \\ & + \sum_{e \in \mathcal{E}_h} \int_e \left\{ \left\{ \frac{\partial^2 v_h}{\partial n^2} \right\} \right\} \left[ \left[ \frac{\partial w_h}{\partial n} \right] \right] ds + \sum_{e \in \mathcal{E}_h} \frac{\sigma}{h_e} \int_e \left[ \left[ \frac{\partial w_h}{\partial n} \right] \right] \left[ \left[ \frac{\partial v_h}{\partial n} \right] \right] ds. \end{aligned}$$

Note that  $a_h$  is well defined on  $V_c$ . The method is stable when  $\sigma$  is sufficiently large. More precisely,

$$(4.17) \quad a_h(v, v) \geq C \|v\|_h^2 \quad \forall v \in V_h$$

for  $\sigma \geq \sigma_* > 0$  and  $\sigma_*$  sufficiently large.

The map  $E_h$  defined in (4.2) satisfies the estimate [12, 16]

$$(4.18) \quad \begin{aligned} & \sum_{T \in \mathcal{T}_h} \left( h_T^{-4} \|v - E_h v\|_{L_2(T)}^2 + h_T^{-2} |v - E_h v|_{H^1(T)}^2 \right) \\ & + \|E_h v\|_V^2 \leq C \|v\|_h^2 \quad \forall v_h \in V_h. \end{aligned}$$

Let  $v \in V$ ,  $w \in V_c$  and  $v_h \in V_h$ . Then

$$\begin{aligned} a(v, w) - a_h(v_h, w) &= \sum_{T \in \mathcal{T}_h} \int_T D^2(v - v_h) : D^2 w \, dx + \sum_{e \in \mathcal{E}_h} \int_e \left\{ \left\{ \frac{\partial^2 w}{\partial n^2} \right\} \right\} \left[ \left[ \frac{\partial v_h}{\partial n} \right] \right] ds \\ &\leq C_{\sigma_*} \|v - v_h\|_h \|w\|_V. \end{aligned}$$

Again, using the fact that for any  $v \in V_h$ ,  $E_h v(p) = v(p)$  for all  $v \in V_h$  and vertices  $p$  in  $\mathcal{T}_h$ , we find the following analog of (4.15) for  $v, \phi \in V_h$  and  $\psi = \phi - E_h \phi$ :

$$(4.19) \quad \begin{aligned} & \sum_{T \in \mathcal{T}_h} \int_T D^2 v : D^2 \psi \, dx = \sum_{T \in \mathcal{T}_h} \int_{\partial T} \left( \frac{\partial^2 v}{\partial n^2} \frac{\partial \psi}{\partial n} + \frac{\partial^2 v}{\partial \tau \partial n} \frac{\partial \psi}{\partial \tau} \right) ds \\ &= - \sum_{e \in \mathcal{E}_h} \int_e \left\{ \left\{ \frac{\partial^2 v}{\partial n^2} \right\} \right\} \left[ \left[ \frac{\partial \psi}{\partial n} \right] \right] ds - \sum_{e \in \mathcal{E}_h^i} \int_e \left\{ \left\{ \frac{\partial \psi}{\partial n} \right\} \right\} \left[ \left[ \frac{\partial^2 v}{\partial n^2} \right] \right] ds. \end{aligned}$$

Then using (4.18) and Lemma 4.1, we find

$$\begin{aligned} (f, \psi) - a_h(v, \psi) &= (f, \psi) + \sum_{e \in \mathcal{E}_h^i} \int_e \left[ \left[ \frac{\partial^2 v}{\partial n^2} \right] \right] \left\{ \left\{ \frac{\partial \psi}{\partial n} \right\} \right\} ds \\ &\quad - \sum_{e \in \mathcal{E}_h} \int_e \left\{ \left\{ \frac{\partial^2 \psi}{\partial n^2} \right\} \right\} \left[ \left[ \frac{\partial v}{\partial n} \right] \right] ds - \sum_{e \in \mathcal{E}_h} \frac{\sigma}{h_e} \int_e \left[ \left[ \frac{\partial v}{\partial n} \right] \right] \left[ \left[ \frac{\partial \psi}{\partial n} \right] \right] ds \\ &\leq C_{\sigma_*} (\|u - v\|_h + \text{Osc}_2(f)) \|\phi\|_h. \end{aligned}$$

Hence, the assumption (2.7) is valid and the estimate (2.9) holds.



**4.3. Discontinuous Galerkin methods.** The model problem (4.1) is rewritten in the following form. Find  $u \in H_0^2(\Omega)$  such that

$$a(u, v) = (f, v) \quad \forall v \in H_0^2(\Omega),$$

where

$$a(w, v) = \int_{\Omega} \Delta w \Delta v \, dx \quad \forall w, v \in H_0^2(\Omega),$$

and  $\Delta$  denotes the Laplacian. Set  $V = H_0^2(\Omega)$  and  $\|v\|_V = |\Delta v|_{L^2(\Omega)}$  for all  $v \in V$ . For this last section, we switch back to the definitions of jump and mean in Section 3.

We now prove the following lemma.

**Lemma 4.2.** *Let  $v \in V_h$ . Then there is a positive constant  $C$  independent of  $h$  such that*

$$(4.20) \quad \sum_{T \in \mathcal{T}_h} h_T^4 \|f - \Delta^2 v\|_{L_2(T)}^2 \leq C \left( \sum_{T \in \mathcal{T}_h} \|\Delta(u - v)\|_{L_2(T)}^2 + Osc_2(f)^2 \right),$$

$$(4.21) \quad \sum_{e \in \mathcal{E}_h^i} h_e \|\llbracket \Delta v \rrbracket\|_{L_2(e)}^2 \leq C \left( \sum_{T \in \mathcal{T}_h} \|\Delta(u - v)\|_{L_2(T)}^2 + Osc_2(f)^2 \right),$$

and

$$(4.22) \quad \sum_{e \in \mathcal{E}_h^i} h_e^3 \|\llbracket \nabla \Delta v \rrbracket\|_{L_2(e)}^2 \leq C \left( \sum_{T \in \mathcal{T}_h} \|\Delta(u - v)\|_{L_2(T)}^2 + Osc_2(f)^2 \right),$$

where  $Osc_2(f)$  is defined in (2.8).

*Proof.* The proofs of (4.20) and (4.21) are similar to the proofs of (4.3) and (4.4), respectively, and hence we omit the proofs.

We will prove (4.22) using the bubble function techniques [38, 16, 26]. Let  $e \in \mathcal{E}_h^i$  and  $T_{\pm}$  be the triangles sharing this edge  $e$ . Denote by  $\mathcal{T}_e$  the patch of the two triangles  $T_{\pm}$  (cf. Fig. 3.1). Consider  $\llbracket \nabla \Delta v \rrbracket$  on  $e$  and extend it to  $T_{\pm}$  by constants along the lines orthogonal to  $e$ . Denote the resulting function  $\zeta_1 \in P_{r-3}(\mathcal{T}_e)$ . It is then obvious that  $\zeta_1 = \llbracket \nabla \Delta v \rrbracket$  on  $e$ . Construct a piecewise polynomial bubble function  $\zeta_2 \in H_0^2(\mathcal{T}_e)$  such that  $\zeta_2(x_e) = 1$ , where  $x_e$  is the midpoint of  $e$ . Denote  $\phi = \zeta_1 \zeta_2$  and extend it to be zero on  $\Omega \setminus \mathcal{T}_e$ . We have by scaling [37],

$$(4.23) \quad C \|\phi\|_{L^2(\mathcal{T}_e)} \leq \left( \int_e h_e \llbracket \nabla \Delta v \rrbracket^2 ds \right)^{1/2},$$

for some mesh independent constant  $C$ . Then, using (4.1), Poincaré's inequality and a standard inverse inequality, we have

$$\begin{aligned}
\int_e \llbracket \nabla \Delta v \rrbracket^2 ds &\leq C \int_e \llbracket \nabla \Delta v \rrbracket \zeta_1 \zeta_2 ds = C \left( \sum_{T \in \mathcal{T}_e} \int_T (\nabla \Delta v \cdot \nabla \phi + \Delta^2 v \phi) dx \right) \\
&= C \left( \sum_{T \in \mathcal{T}_e} \int_T (-\Delta v \Delta \phi + \Delta^2 v \phi) dx \right) + C \sum_{T \in \mathcal{T}_e} \int_{\partial T} \llbracket \Delta v \rrbracket \{\{\nabla \phi\}\} ds \\
&= C \left( \sum_{T \in \mathcal{T}_e} \int_T \Delta(u-v) \Delta \phi dx - \sum_{T \in \mathcal{T}_e} \int_T (f - \Delta^2 v) \phi dx \right) \\
&\quad + C \sum_{T \in \mathcal{T}_e} \int_{\partial T} \llbracket \Delta v \rrbracket \{\{\nabla \phi\}\} ds \\
&\leq C \left( \sum_{T \in \mathcal{T}_e} \left( h_T^4 \|f - \Delta^2 v\|_{L^2(T)}^2 + |\Delta(u-v)|_{L^2(T)}^2 \right) \right)^{1/2} h_e^{-2} \|\phi\|_{L^2(\mathcal{T}_e)} \\
&\quad + C \left( \sum_{T \in \mathcal{T}_e} \int_{\partial T} h_e \llbracket \Delta v \rrbracket^2 ds \right)^{1/2} h_e^{-2} \|\phi\|_{L^2(\mathcal{T}_e)}
\end{aligned}$$

and then we use (4.20) and (4.21) to complete the proof of (4.22).  $\square$

4.3.1. *Symmetric interior penalty galerkin (SIPG) method.* Let

$$V_h = \{v \in L_2(\Omega) : v|_T \in P_r(T) \quad \forall T \in \mathcal{T}_h\}$$

and define the norm  $\|\cdot\|_h$  on  $V_h$  by

$$\|v\|_h^2 = \sum_{T \in \mathcal{T}_h} \int_T |\Delta v|^2 dx + \sum_{e \in \mathcal{E}_h} \int_e \frac{1}{h_e} \llbracket \nabla v \rrbracket^2 ds + \sum_{e \in \mathcal{E}_h} \int_e \frac{1}{h_e^3} \llbracket v \rrbracket^2 ds \quad \forall v \in V_h.$$

The symmetric interior penalty method [4, 24, 33] is to find  $u_h \in V_h$  such that

$$a_h(u_h, v) = (f, v) \quad \forall v \in V_h,$$

where for  $w, v \in V_h$ ,

$$\begin{aligned}
a_h(w, v) &= \sum_{T \in \mathcal{T}_h} \int_T \Delta w \Delta v dx + \sum_{e \in \mathcal{E}_h} \int_e \{\{\nabla \Delta w\}\} \llbracket v \rrbracket ds - \sum_{e \in \mathcal{E}_h} \int_e \{\{\Delta w\}\} \llbracket \nabla v \rrbracket ds \\
&\quad + \sum_{e \in \mathcal{E}_h} \int_e \{\{\nabla \Delta v\}\} \llbracket w \rrbracket ds - \sum_{e \in \mathcal{E}_h} \int_e \{\{\Delta v\}\} \llbracket \nabla w \rrbracket ds \\
&\quad + \sum_{e \in \mathcal{E}_h} \frac{\sigma_1}{h_e} \int_e \llbracket \nabla w \rrbracket \llbracket \nabla v \rrbracket ds + \sum_{e \in \mathcal{E}_h} \frac{\sigma_2}{h_e^3} \int_e \llbracket w \rrbracket \llbracket v \rrbracket ds
\end{aligned}$$

and  $\sigma_1 > 0$ ,  $\sigma_2 > 0$  are penalty parameters.

For sufficiently large  $\sigma_1$  and  $\sigma_2$ , it holds that

$$a_h(v, v) \geq C \|v\|_h^2 \quad \forall v \in V_h,$$

where  $C$  is a mesh independent constant. Note for  $v \in V$ ,  $w \in V_c$  and  $v_h \in V_h$  that

$$\begin{aligned} a(v, w) - a_h(v_h, w) &= \sum_{T \in \mathcal{T}_h} \int_T \Delta(v - v_h) \Delta w \, dx + \sum_{e \in \mathcal{E}_h} \int_e \{\{\nabla \Delta w\}\} \llbracket v - v_h \rrbracket ds \\ &\quad - \sum_{e \in \mathcal{E}_h} \int_e \{\{\Delta w\}\} \llbracket \nabla(v - v_h) \rrbracket ds \\ &\leq C \|v - v_h\|_h \|w\|_V. \end{aligned}$$

The map  $E_h$  defined in (4.2) satisfies the estimate [12, 16, 26]

$$(4.24) \quad \begin{aligned} &\sum_{T \in \mathcal{T}_h} \left( h_T^{-4} \|v - E_h v\|_{L_2(T)}^2 + h_T^{-2} |v - E_h v|_{H^1(T)}^2 \right) \\ &\quad + \|E_h v\|_V^2 \leq C \|v\|_h^2 \quad \forall v_h \in V_h. \end{aligned}$$

Let  $v, \phi \in V_h$  and  $\psi = \phi - E_h \phi$ . Then after two integration by parts and using earlier arguments with (4.20)–(4.22), we find that

$$\begin{aligned} (f, \psi) - a_h(v, \psi) &= \sum_{T \in \mathcal{T}_h} \int_T (f - \Delta^2 v) \psi \, dx + \sum_{e \in \mathcal{E}_h} \int_e \llbracket \nabla \Delta v \rrbracket \{\{\psi\}\} ds \\ &\quad - \sum_{e \in \mathcal{E}_h} \int_e \llbracket \Delta v \rrbracket \{\{\nabla \psi\}\} ds - \sum_{e \in \mathcal{E}_h} \int_e \{\{\nabla \Delta \psi\}\} \llbracket v \rrbracket ds \\ &\quad + \sum_{e \in \mathcal{E}_h} \int_e \{\{\Delta \psi\}\} \llbracket \nabla v \rrbracket ds - \sum_{e \in \mathcal{E}_h} \frac{\sigma_1}{h_e} \int_e \llbracket \nabla v \rrbracket \llbracket \nabla \psi \rrbracket ds \\ &\quad - \sum_{e \in \mathcal{E}_h} \frac{\sigma_2}{h_e^3} \int_e \llbracket v \rrbracket \llbracket \psi \rrbracket ds \\ &\leq C (\|u - v\|_h + \text{Osc}_2(f)) \|\phi\|_h. \end{aligned}$$

Therefore, the assumption (2.7) is valid and hence the estimate (2.9) holds.

### 5. CONCLUSIONS

We have developed a new approach to discontinuous finite element methods and proved that all the well-known classical nonconforming methods and discontinuous Galerkin methods for second and fourth order elliptic problems are quasi-optimal up to higher order data oscillations, using only the weak formulations of the boundary value problems. Since the analysis involves techniques from *a priori* error analysis and *a posteriori* error analysis, it may be referred to as a *medius* error analysis.

This approach puts discontinuous finite element methods on an equal footing with conforming finite element methods in the sense that the first stage of the *a priori* error analysis does not require the elliptic regularity theory. It is particularly useful for more complicated problems (such as interface problems) where the exact solution  $u$  has low regularity, i.e.,  $u \in H^s(\Omega)$ , where  $s \leq 3/2$  for second order problems and  $s \leq 7/2$  for fourth order problems.

To keep the technicalities to a minimum, the results in this paper are presented for simple model problems. With appropriate modifications these results can be extended to second order mixed boundary value problems, interface problems, non-homogeneous Dirichlet problems and fourth order problems with different boundary conditions.

The application of this new approach to  $hp$  error estimates and nonconforming meshes will be investigated in the near future.

#### ACKNOWLEDGMENTS

The author would like to thank Susanne C. Brenner and Li-yeng Sung for many helpful suggestions.

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