# A NEW ERROR BOUND FOR LINEAR COMPLEMENTARITY PROBLEMS FOR $B$-MATRICES* 

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#### Abstract

A new error bound for the linear complementarity problem is given when the involved matrix is a $B$-matrix. It is shown that this bound improves the corresponding result in [M. GarcíaEsnaola and J.M. Peña. Error bounds for linear complementarity problems for B-matrices. Appl. Math. Lett., 22:1071-1075, 2009.] in some cases, and that it is sharper than that in [C.Q. Li and Y.T. Li. Note on error bounds for linear complementarity problems for B-matrices. Appl. Math. Lett., 57:108-113, 2016.].


Key words. Error bound, Linear complementarity problem, $B$-Matrix.

AMS subject classifications. 90C33, 65G50, 65F35.

1. Introduction. A linear complementarity problem $\operatorname{LCP}(M, q)$ tries to find a vector $x \in R^{n}$ such that

$$
\begin{equation*}
x \geq 0, \quad M x+q \geq 0, \quad(M x+q)^{T} x=0, \tag{1.1}
\end{equation*}
$$

where $M=\left[m_{i j}\right] \in R^{n \times n}$ and $q \in R^{n}$. The $L C P(M, q)$ has various applications in the Nash equilibrium point of a bimatrix game, the contact problem and the free boundary problem for journal bearing; for details, see 4, 5, 17.

It is well-known that the $\operatorname{LCP}(M, q)$ has a unique solution for any $q \in R^{n}$ if and only if $M$ is a $P$-matrix [5. Here, a matrix $M \in R^{n \times n}$ is called a $P$-matrix if all its principal minors are positive [6. In 3], Chen and Xiang gave the following error bound of the $\operatorname{LCP}(M, q)$ when $M$ is a $P$-matrix:

$$
\left\|x-x^{*}\right\|_{\infty} \leq \max _{d \in[0,1]^{n}}\left\|(I-D+D M)^{-1}\right\|_{\infty}\|r(x)\|_{\infty},
$$

where $x^{*}$ is the solution of the $\operatorname{LCP}(M, q), r(x)=\min \{x, M x+q\}, D=\operatorname{diag}\left(d_{i}\right)$ with $0 \leq d_{i} \leq 1$, and the min operator $r(x)$ denotes the componentwise minimum of two vectors. If $M$ satisfies certain structure, then some bounds of $\max _{d \in[0,1]^{n}} \|(I-D+$ $D M)^{-1} \|_{\infty}$ can be derived; for details, see [2, 7, 8, 10, 14] and references therein.

[^0]When $M$ is a $B$-matrix introduced by Peña in [6] as a subclass of $P$-matrices, García-Esnaola and Peña in [10] presented the following upper bound which is only related with the entries of $M$. Here a real matrix $M=\left[m_{i j}\right] \in R^{n \times n}$ is called a $B$-matrix [6] if for each $i \in N=\{1,2, \ldots, n\}$,

$$
\begin{equation*}
\sum_{k \in N} m_{i k}>0, \text { and } \frac{1}{n}\left(\sum_{k \in N} m_{i k}\right)>m_{i j} \text { for any } j \in N \text { and } j \neq i \tag{1.2}
\end{equation*}
$$

Theorem 1.1. [10, Theorem 2.2] Let $M=\left[m_{i j}\right] \in R^{n \times n}$ be a $B$-matrix with the form

$$
\begin{equation*}
M=B^{+}+C \tag{1.3}
\end{equation*}
$$

where

$$
B^{+}=\left[b_{i j}\right]=\left[\begin{array}{ccc}
m_{11}-r_{1}^{+} & \cdots & m_{1 n}-r_{1}^{+}  \tag{1.4}\\
\vdots & & \vdots \\
m_{n 1}-r_{n}^{+} & \cdots & m_{n n}-r_{n}^{+}
\end{array}\right]
$$

and $r_{i}^{+}=\max \left\{0, m_{i j} \mid j \neq i\right\}$. Then

$$
\begin{equation*}
\max _{d \in[0,1]^{n}}\left\|(I-D+D M)^{-1}\right\|_{\infty} \leq \frac{n-1}{\min \{\beta, 1\}} \tag{1.5}
\end{equation*}
$$

where $\beta=\min _{i \in N}\left\{\beta_{i}\right\}$ and $\beta_{i}=b_{i i}-\sum_{j \neq i}\left|b_{i j}\right|$.
As shown in [15], if the diagonal dominance of $B^{+}$is weak, i.e.,

$$
\beta=\min _{i \in N}\left\{\beta_{i}\right\}=\min _{i \in N}\left\{b_{i i}-\sum_{j \neq i}\left|b_{i j}\right|\right\}
$$

is small, then the bound (1.5) may be very large when $M$ is a $B$-matrix, which leads to that the estimate in (1.5) is always inaccurate, for details, see [15, 16. To improve the bound (1.5), Li and $\mathrm{Li}[15]$ gave the following bound for $\max _{d \in[0,1]^{n}}\left\|(I-D+D M)^{-1}\right\|_{\infty}$ when $M$ is a $B$-matrix.

Theorem 1.2. [15, Theorem 4] Let $M=\left[m_{i j}\right] \in R^{n \times n}$ be a $B$-matrix with the form $M=B^{+}+C$, where $B^{+}=\left[b_{i j}\right]$ is the matrix of (1.4). Then

$$
\begin{equation*}
\max _{d \in[0,1]^{n}}\left\|(I-D+D M)^{-1}\right\|_{\infty} \leq \sum_{i=1}^{n} \frac{n-1}{\min \left\{\bar{\beta}_{i}, 1\right\}} \prod_{j=1}^{i-1}\left(1+\frac{1}{\bar{\beta}_{j}} \sum_{k=j+1}^{n}\left|b_{j k}\right|\right) \tag{1.6}
\end{equation*}
$$

$$
\begin{gathered}
\text { where } \bar{\beta}_{i}=b_{i i}-\sum_{j=i+1}^{n}\left|b_{i j}\right| l_{i}\left(B^{+}\right), l_{k}\left(B^{+}\right)=\max _{k \leq i \leq n}\left\{\frac{1}{\left|b_{i i}\right|} \sum_{\substack{j=k, j \neq i}}^{n}\left|b_{i j}\right|\right\} \text { and } \\
\prod_{j=1}^{i-1}\left(1+\frac{1}{\bar{\beta}_{j}} \sum_{k=j+1}^{n}\left|b_{j k}\right|\right)=1 \text { if } i=1
\end{gathered}
$$

Very recently, when $M$ is a weakly chained diagonally dominant $B$-matrix, Li and Li [16] gave a bound for $\max _{d \in[0,1]^{n}}\left\|(I-D+D M)^{-1}\right\|_{\infty}$. This bound holds true for the case that $M$ is a $B$-matrix because a $B$-matrix is a weakly chained diagonally dominant $B$-matrix [16].

Theorem 1.3. [16, Corollary 1] Let $M=\left[m_{i j}\right] \in R^{n \times n}$ be a $B$-matrix with the form $M=B^{+}+C$, where $B^{+}=\left[b_{i j}\right]$ is the matrix of (1.4). Then

$$
\max _{d \in[0,1]^{n}}\left\|(I-D+D M)^{-1}\right\|_{\infty} \leq \sum_{i=1}^{n}\left(\frac{n-1}{\min \left\{\tilde{\beta}_{i}, 1\right\}} \prod_{j=1}^{i-1} \frac{b_{j j}}{\tilde{\beta}_{j}}\right)
$$

where $\tilde{\beta}_{i}=b_{i i}-\sum_{j=i+1}^{n}\left|b_{i j}\right|>0$ and $\prod_{j=1}^{i-1} \frac{b_{j j}}{\tilde{\beta}_{j}}=1$ if $i=1$.
In this paper, we also focus on the error bound for the $\operatorname{LCP}(M, q)$, and gave a new bound for $\max _{d \in[0,1]^{n}}\left\|(I-D+D M)^{-1}\right\|_{\infty}$ when $M$ is a $B$-matrix. It is shown that this bound is more effective to estimate $\max _{d \in[0,1]^{n}}\left\|(I-D+D M)^{-1}\right\|_{\infty}$ than that in Theorem 1.1 and sharper than those in Theorems 1.2 and 1.3
2. Main results. We first recall some definitions. A matrix $A=\left[a_{i j}\right] \in C^{n \times n}$ is called a strictly diagonally dominant $(S D D)$ matrix if for each $i \in N,\left|a_{i i}\right|>\sum_{\substack{j=1, j \neq i}}^{n}\left|a_{i j}\right|$. It is well-known that an $S D D$ matrix is nonsingular [1]. A matrix $A=\left[a_{i j}\right]$ is called a $Z$-matrix if $a_{i j} \leq 0$ for any $i \neq j$, and a nonsingular $M$-matrix if $A$ is a $Z$-matrix with $A^{-1}$ being nonnegative [1]. Next, several lemmas which will be used later are given.

Lemma 2.1. [18, Theorem 3.2] Let $A=\left[a_{i j}\right] \in R^{n \times n}$ be an SDD M-matrix. Then

$$
\left\|A^{-1}\right\|_{\infty} \leq \sum_{i=1}^{n}\left(\frac{1}{a_{i i}\left(1-u_{i}(A) l_{i}(A)\right)} \prod_{j=1}^{i-1} \frac{1}{1-u_{j}(A) l_{j}(A)}\right)
$$

where $u_{i}(A)=\frac{1}{\left|a_{i i}\right|} \sum_{j=i+1}^{n}\left|a_{i j}\right|, l_{k}(A)=\max _{k \leq i \leq n}\left\{\frac{1}{\left|a_{i i}\right|} \sum_{\substack{j=k, j \neq i}}^{n}\left|a_{i j}\right|\right\}$ and

$$
\prod_{j=1}^{i-1} \frac{1}{1-u_{j}(A) l_{j}(A)}=1 \quad \text { if } i=1
$$

Lemma 2.2. [15, Lemma 3] Let $\gamma>0$ and $\eta \geq 0$. Then for any $x \in[0,1]$,

$$
\begin{equation*}
\frac{1}{1-x+\gamma x} \leq \frac{1}{\min \{\gamma, 1\}} \tag{2.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\eta x}{1-x+\gamma x} \leq \frac{\eta}{\gamma} \tag{2.2}
\end{equation*}
$$

Lemma 2.3. [16, Lemma 5] Let $A=\left[a_{i j}\right] \in R^{n \times n}$ with

$$
a_{i i}>\sum_{j=i+1}^{n}\left|a_{i j}\right| \text { for each } i \in N
$$

Then for any $x_{i} \in[0,1], i \in N$,

$$
\frac{1-x_{i}+a_{i i} x_{i}}{1-x_{i}+a_{i i} x_{i}-\sum_{j=i+1}^{n}\left|a_{i j}\right| x_{i}} \leq \frac{a_{i i}}{a_{i i}-\sum_{j=i+1}^{n}\left|a_{i j}\right|}
$$

Theorem 2.4. Let $M=\left[m_{i j}\right] \in R^{n \times n}$ be a $B$-matrix with the form $M=B^{+}+C$, where $B^{+}=\left[b_{i j}\right]$ is the matrix of (1.4). Then

$$
\begin{equation*}
\max _{d \in[0,1]^{n}}\left\|(I-D+D M)^{-1}\right\|_{\infty} \leq \sum_{i=1}^{n} \frac{n-1}{\min \left\{\bar{\beta}_{i}, 1\right\}} \prod_{j=1}^{i-1} \frac{b_{j j}}{\bar{\beta}_{j}} \tag{2.3}
\end{equation*}
$$

where $\bar{\beta}_{i}$ is defined in Theorem [1.2 and $\prod_{j=1}^{i-1} \frac{b_{j j}}{\beta_{j}}=1$ if $i=1$.
Proof. Let $M_{D}=I-D+D M$. Then

$$
M_{D}=I-D+D M=I-D+D\left(B^{+}+C\right)=B_{D}^{+}+C_{D}
$$

where $B_{D}^{+}=I-D+D B^{+}=\left[b_{i j}\right]$. Similarly to the proof of Theorem 2.2 in [10, we can obtain that $B_{D}^{+}$is an $S D D M$-matrix with positive diagonal elements and $C_{D}=D C$, and that
(2.4) $\left\|M_{D}^{-1}\right\|_{\infty} \leq\left\|\left(I+\left(B_{D}^{+}\right)^{-1} C_{D}\right)^{-1}\right\|_{\infty}\left\|\left(B_{D}^{+}\right)^{-1}\right\|_{\infty} \leq(n-1)\left\|\left(B_{D}^{+}\right)^{-1}\right\|_{\infty}$.

By Lemma 2.1,

$$
\begin{align*}
\left\|\left(B_{D}^{+}\right)^{-1}\right\|_{\infty} \leq \sum_{i=1}^{n} & \left(\frac{1}{\left(1-d_{i}+b_{i i} d_{i}\right)\left(1-u_{i}\left(B_{D}^{+}\right) l_{i}\left(B_{D}^{+}\right)\right)}\right.  \tag{2.5}\\
& \left.\times \prod_{j=1}^{i-1} \frac{1}{1-u_{j}\left(B_{D}^{+}\right) l_{j}\left(B_{D}^{+}\right)}\right)
\end{align*}
$$

where

$$
u_{i}\left(B_{D}^{+}\right)=\frac{\sum_{j=i+1}^{n}\left|b_{i j}\right| d_{i}}{1-d_{i}+b_{i i} d_{i}}, \text { and } l_{k}\left(B_{D}^{+}\right)=\max _{k \leq i \leq n}\left\{\frac{\sum_{\substack{j=k, j \neq i}}^{n}\left|b_{i j}\right| d_{i}}{1-d_{i}+b_{i i} d_{i}}\right\} .
$$

By Lemma 2.2, we can easily get that for each $k \in N$,

$$
\begin{equation*}
l_{k}\left(B_{D}^{+}\right) \leq \max _{k \leq i \leq n}\left\{\frac{1}{b_{i i}} \sum_{\substack{j=k, j \neq i}}^{n}\left|b_{i j}\right|\right\}=l_{k}\left(B^{+}\right)<1 \tag{2.6}
\end{equation*}
$$

and that for each $i \in N$,

$$
\begin{align*}
\frac{1}{\left(1-d_{i}+b_{i i} d_{i}\right)\left(1-u_{i}\left(B_{D}^{+}\right) l_{i}\left(B_{D}^{+}\right)\right)} & =\frac{1}{1-d_{i}+b_{i i} d_{i}-\sum_{j=i+1}^{n}\left|b_{i j}\right| d_{i} l_{i}\left(B_{D}^{+}\right)} \\
& \leq \frac{1}{\min \left\{b_{i i}-\sum_{j=i+1}^{n}\left|b_{i j}\right| l_{i}\left(B^{+}\right), 1\right\}} \\
& =\frac{1}{\min \left\{\bar{\beta}_{i}, 1\right\}} . \tag{2.7}
\end{align*}
$$

Furthermore, by Lemma 2.3 ,

$$
\begin{equation*}
\frac{1}{1-u_{i}\left(B_{D}^{+}\right) l_{i}\left(B_{D}^{+}\right)}=\frac{1-d_{i}+b_{i i} d_{i}}{1-d_{i}+b_{i i} d_{i}-\sum_{j=i+1}^{n}\left|b_{i j}\right| d_{i} l_{i}\left(B_{D}^{+}\right)} \leq \frac{b_{i i}}{\bar{\beta}_{i}} \tag{2.8}
\end{equation*}
$$

By (2.5), (2.6), (2.7) and (2.8), we have

$$
\begin{equation*}
\left\|\left(B_{D}^{+}\right)^{-1}\right\|_{\infty} \leq \frac{1}{\min \left\{\bar{\beta}_{1}, 1\right\}}+\sum_{i=2}^{n} \frac{1}{\min \left\{\bar{\beta}_{i}, 1\right\}} \prod_{j=1}^{i-1} \frac{b_{j j}}{\bar{\beta}_{j}} \tag{2.9}
\end{equation*}
$$

The conclusion follows from (2.4) and (2.9).

The comparisons of the bounds in Theorems $1.2,1.3$ and 2.4 are established as follows.

THEOREM 2.5. Let $M=\left[m_{i j}\right] \in R^{n \times n}$ be a $B$-matrix with the form $M=B^{+}+C$, where $B^{+}=\left[b_{i j}\right]$ is the matrix of (1.4). Let $\bar{\beta}_{i}$ and $\tilde{\beta}_{i}$ be defined in Theorems 1.2 and 1.3 respectively. Then

$$
\begin{aligned}
\sum_{i=1}^{n} \frac{n-1}{\min \left\{\bar{\beta}_{i}, 1\right\}} \prod_{j=1}^{i-1} \frac{b_{j j}}{\bar{\beta}_{j}} & \leq \sum_{i=1}^{n} \frac{n-1}{\min \left\{\bar{\beta}_{i}, 1\right\}} \prod_{j=1}^{i-1}\left(1+\frac{1}{\bar{\beta}_{j}} \sum_{k=j+1}^{n}\left|b_{j k}\right|\right) \\
& \leq \sum_{i=1}^{n}\left(\frac{n-1}{\min \left\{\tilde{\beta}_{i}, 1\right\}} \prod_{j=1}^{i-1} \frac{b_{j j}}{\tilde{\beta}_{j}}\right)
\end{aligned}
$$

Proof. Note that

$$
\tilde{\beta}_{i}=b_{i i}-\sum_{j=i+1}^{n}\left|b_{i j}\right|, \quad \bar{\beta}_{i}=b_{i i}-\sum_{j=i+1}^{n}\left|b_{i j}\right| l_{i}\left(B^{+}\right)
$$

and $l_{k}\left(B^{+}\right)=\max _{k \leq i \leq n}\left\{\frac{1}{\left|b_{i i}\right|} \sum_{\substack{j=k, j \neq i}}^{n}\left|b_{i j}\right|\right\}<1$. Hence, for each $i \in N, \tilde{\beta}_{i} \leq \bar{\beta}_{i}$ and

$$
\begin{equation*}
\frac{1}{\min \left\{\tilde{\beta}_{i}, 1\right\}} \geq \frac{1}{\min \left\{\bar{\beta}_{i}, 1\right\}} \tag{2.10}
\end{equation*}
$$

Meantime, for $j=1,2, \ldots, n-1$,

$$
\begin{equation*}
1+\frac{1}{\bar{\beta}_{j}} \sum_{k=j+1}^{n}\left|b_{j k}\right| \leq 1+\frac{1}{\tilde{\beta}_{j}} \sum_{k=j+1}^{n}\left|b_{j k}\right|=\frac{1}{\tilde{\beta}_{j}}\left(\tilde{\beta}_{j}+\sum_{k=j+1}^{n}\left|b_{j k}\right|\right)=\frac{b_{j j}}{\tilde{\beta}_{j}} . \tag{2.11}
\end{equation*}
$$

By (2.10) and (2.11), we have
(2.12) $\sum_{i=1}^{n} \frac{n-1}{\min \left\{\bar{\beta}_{i}, 1\right\}} \prod_{j=1}^{i-1}\left(1+\frac{1}{\bar{\beta}_{j}} \sum_{k=j+1}^{n}\left|b_{j k}\right|\right) \leq \sum_{i=1}^{n}\left(\frac{n-1}{\min \left\{\tilde{\beta}_{i}, 1\right\}} \prod_{j=1}^{i-1} \frac{b_{j j}}{\tilde{\beta}_{j}}\right)$.

Moreover, for $j=1,2, \ldots, n-1$,

$$
\begin{aligned}
\frac{b_{j j}}{\bar{\beta}_{j}} & =\prod_{j=1}^{i-1} \frac{b_{j j}-\sum_{k=j+1}^{n}\left|b_{j k}\right| l_{j}\left(B^{+}\right)+\sum_{k=j+1}^{n}\left|b_{j k}\right| l_{j}\left(B^{+}\right)}{\bar{\beta}_{j}} \\
& =\frac{\bar{\beta}_{j}+\sum_{k=j+1}^{n}\left|b_{j k}\right| l_{j}\left(B^{+}\right)}{\bar{\beta}_{j}}
\end{aligned}
$$

$$
\begin{aligned}
& =1+\frac{\sum_{k=j+1}^{n}\left|b_{j k}\right| l_{j}\left(B^{+}\right)}{\bar{\beta}_{j}} \\
& \leq 1+\frac{\sum_{k=j+1}^{n}\left|b_{j k}\right|}{\bar{\beta}_{j}},
\end{aligned}
$$

this implies

$$
\begin{equation*}
\sum_{i=1}^{n} \frac{n-1}{\min \left\{\bar{\beta}_{i}, 1\right\}} \prod_{j=1}^{i-1} \frac{b_{j j}}{\bar{\beta}_{j}} \leq \sum_{i=1}^{n} \frac{n-1}{\min \left\{\bar{\beta}_{i}, 1\right\}} \prod_{j=1}^{i-1}\left(1+\frac{1}{\bar{\beta}_{j}} \sum_{k=j+1}^{n}\left|b_{j k}\right|\right) \tag{2.13}
\end{equation*}
$$

The conclusion follows from (2.12) and (2.13).
Example 2.6. Consider the family of $B$-matrices in 15:

$$
M_{k}=\left[\begin{array}{cccc}
1.5 & 0.5 & 0.4 & 0.5 \\
-0.1 & 1.7 & 0.7 & 0.6 \\
0.8 & -0.1 \frac{k}{k+1} & 1.8 & 0.7 \\
0 & 0.7 & 0.8 & 1.8
\end{array}\right]
$$

where $k \geq 1$. Then $M_{k}=B_{k}^{+}+C_{k}$, where

$$
B_{k}^{+}=\left[\begin{array}{cccc}
1 & 0 & -0.1 & 0 \\
-0.8 & 1 & 0 & -0.1 \\
0 & -0.1 \frac{k}{k+1}-0.8 & 1 & -0.1 \\
-0.8 & -0.1 & 0 & 1
\end{array}\right]
$$

By Theorem 1.1 (Theorem 2.2 in (10), we have

$$
\max _{d \in[0,1]^{4}}\left\|\left(I-D+D M_{k}\right)^{-1}\right\|_{\infty} \leq \frac{4-1}{\min \{\beta, 1\}}=30(k+1)
$$

It is obvious that

$$
30(k+1) \rightarrow+\infty \text { when } k \rightarrow+\infty
$$

By Theorem 1.3, we have

$$
\max _{d \in[0,1]^{4}}\left\|\left(I-D+D M_{k}\right)^{-1}\right\|_{\infty} \leq \sum_{i=1}^{4}\left(\frac{3}{\min \left\{\tilde{\beta}_{i}, 1\right\}} \prod_{j=1}^{i-1} \frac{b_{j j}}{\tilde{\beta}_{j}}\right) \approx 15.2675 .
$$

By Theorem 1.2 we have

$$
\max _{d \in[0,1]^{4}}\left\|\left(I-D+D M_{k}\right)^{-1}\right\|_{\infty} \leq \frac{2.97(90 k+91)(190 k+192)+6.24(100 k+101)^{2}}{0.99(90 k+91)^{2}},
$$

and

$$
\frac{2.97(90 k+91)(190 k+192)+6.24(100 k+101)^{2}}{0.99(90 k+91)^{2}}<15.2675, \text { for any } k \geq 1
$$

By Theorem [2.4 we have

$$
\max _{d \in[0,1]^{4}}\left\|\left(I-D+D M_{k}\right)^{-1}\right\|_{\infty} \leq \frac{2.97(90 k+91)(190 k+191)+5.97(100 k+100)^{2}}{0.99(90 k+91)^{2}}
$$

and

$$
\begin{aligned}
& \frac{2.97(90 k+91)(190 k+191)+5.97(100 k+100)^{2}}{0.99(90 k+91)^{2}} \\
< & \frac{2.97(90 k+91)(190 k+192)+6.24(100 k+101)^{2}}{0.99(90 k+91)^{2}}
\end{aligned}
$$

In particular, when $k=1$,

$$
\frac{2.97(90 k+91)(190 k+191)+5.97(100 k+100)^{2}}{0.99(90 k+91)^{2}} \approx 13.6777
$$

and

$$
\frac{2.97(90 k+91)(190 k+192)+6.24(100 k+101)^{2}}{0.99(90 k+91)^{2}} \approx 14.1044
$$

When $k=2$,

$$
\frac{2.97(90 k+91)(190 k+191)+5.97(100 k+100)^{2}}{0.99(90 k+91)^{2}} \approx 13.7110
$$

and

$$
\frac{2.97(90 k+91)(190 k+192)+6.24(100 k+101)^{2}}{0.99(90 k+91)^{2}} \approx 14.1079
$$

In these two cases, the bounds in (1.5) are equal to $60(k=1)$ and $90(k=2)$, respectively. This example shows that the bound in Theorem 2.4 is sharper than those in Theorems 1.1, 1.2 and 1.3
3. Conclusions. In this paper, we give a new bound for $\max _{d \in[0,1]^{n}} \|(I-D+$ $D M)^{-1} \|_{\infty}$ when $M$ is a $B$-matrix, and show that it improves the bound of Theorem 2.2 of 10 in some cases, and that it is always sharper than those of Theorem 4 of [15] and of Corollary 1 of [16].

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