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A New *F* Approximation for the Pillai–Bartlett Trace under H_0

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Abstract

Pillai suggested two approximations for the Pillai–Bartlett trace statistic in the null case. The first one matches one moment of a β_1 random variable, and corresponds to an *F* random variable, and the second matches four moments in the Pearson system. Although intuitively appealing and widely used in current statistical packages, the first lacks accuracy even with moderate sample size. The second matches two moment ratios in the Pearson system and provides much greater accuracy. Two new approximations match two moments of a β_1 random variable, and hence correspond to an *F* random variable, yet achieve most of the accuracy of Pillai's second approximation. The second of the two new approximations provides the best combination of logical properties and numerical accuracy.

Keywords

Multivariate linear models; Repeated measures

1. INTRODUCTION

1.1 Motivation

Consider $\boldsymbol{H}$ and $\boldsymbol{E}$, independent, $b \times b$, central Wishart matrices, with common covariance matrix, Σ_*, and respective degrees of freedom a and ν_E. In turn $\boldsymbol{T} = \boldsymbol{H} + \boldsymbol{E}$ represents a $b \times b$ central Wishart matrix, with covariance Σ_* and degrees of freedom $a + \nu_E$. Such matrices arise under the null in testing the general linear hypothesis, in the context of the general linear multivariate model (GLMM). The Pillai-Bartlett trace, $V = \text{tr}(\boldsymbol{H}\boldsymbol{T}^{-1})$, provides a common test statistic. See Muller, LaVange, Ramey, and Ramey (1992) for a detailed statement of the underlying problem in the context of power analysis. See Pillai (1976, 1977) for a detailed survey of distributional results for GLMM tests.

Pillai (1954, 1955) suggested approximating V by a β_1, which corresponds to an *F* random variable. For $a = 2$, $b = 3$, and $\nu_E = 24$, the approximation yields around two digits of accuracy for quantiles in small samples (Pillai 1954, tab. 5.5.1). Exact probabilities exceed the corresponding approximate ones by as much as .01 in the same setting. The approach (1) matches the first moment of V with the β_1; (2) uses the intuitively appealing value of ab for the numerator degrees of freedom of the *F*; (3) reduces to the exact answer if $s = \min(a, b) = 1$; and (4) provides asymptotically correct performance. However, Pillai recommended avoiding the approximation in small samples due to limited accuracy. Itô (1956) suggested a series approximation. Pillai (1957) used the Pearson system to provide an approximation that uses four moments to match two moment ratios. Davis (1970) described a method for computing exact values, based on solving a differential equation. He also examined the accuracy of the Itô (1956) and Pillai (1957) approximations in providing percentiles. Davis reported that Pillai's Pearson curve approximation provides four digits of accuracy, except when the degrees of freedom for both $\boldsymbol{H}$ and $\boldsymbol{T}$ are small, and that Itô's approximation does not perform as well.

A parallel problem arises for the Hotelling–Lawley trace. McKeon (1974) used the moments of $U=\text{tr}(\boldsymbol{HE}^{-1})$ to chose $\{\gamma_m, \nu_{1,m}, \nu_{2,m}\}$ to define approximating random variables of the form $U_{*m} = \gamma_m \cdot F(\nu_{1,m}, \nu_{2,m})$, a scaled central F random variable. Matching the first three moments of U with U_{*3} yields one approximation. Another form matches only the first two moments, but adds the constraint $\nu_{1,2} = ab$ to define the third equation needed to uniquely determine U_{*2}. The constraint describes the limiting value. Both forms reduce to the exact answer if $s = 1$. Either has much better accuracy than previous approximations, including a one-moment method of Pillai and Samson (1959). McKeon (1974) recommended U_{*2} due to slightly better average accuracy for the conditions he studied and the simplicity of the numerator degrees of freedom. The success of McKeon's approximation encourages examining a similar strategy for the Pillai-Bartlett trace.

1.2 Moments of *V*

Pillai (1954, p. 95) derived the moments of V in the null case. With $\varepsilon V = \mu_1$ and $\varepsilon(V - \varepsilon V)^2 = \mu_2$,

$$\mu_1 = \frac{ab}{\nu_E + a} \quad (1.1)$$

and

$$\mu_2 = \frac{2ab\nu_E(\nu_E + a - b)}{(\nu_E + a)^2(\nu_E + a + 2)(\nu_E + a - 1)}. \quad (1.2)$$

2. Approximations

2.1 The Form of the Approximations

The desire to reduce to the exact result if $s = \min(a, b) = 1$ leads to approximating V by $V_{*m} = \gamma_m \cdot \beta_1(\nu_{1,m}/2, \nu_{2,m}/2)$, a β_1 random variable (Johnson and Kotz 1970, chap. 24). Observe that

$$\varepsilon V_{*m} = \gamma_m \frac{\nu_{1,m}}{\nu_{1,m} + \nu_{2,m}}, \quad (2.1)$$

and

$$\varepsilon(V_{*m} - \varepsilon V_{*m})^2 = \gamma_m^2 \frac{2\nu_{1,m}\nu_{2,m}}{(\nu_{1,m} + \nu_{2,m})^2(\nu_{1,m} + \nu_{2,m} + 2)}. \quad (2.2)$$

With $V \approx V_{*m} = \gamma_m \cdot \beta_1(\nu_{1,m}/2, \nu_{2,m}/2)$, consider $F_{*m} = F(\nu_{1,m}, \nu_{2,m})$, a central F random variable. The fact that $V_{*m} = \nu_{1,m}F_{*m}/(\nu_{1,m}F_{*1} + \nu_{2,m})$ allows writing

$$V \approx \gamma_m \cdot \frac{\nu_{1,m}F_{*m}}{\nu_{1,m}F_{*m} + \nu_{2,m}}, \quad (2.3)$$

and

$$\frac{(V/\gamma_m)/\nu_{1,m}}{(1 - V/\gamma_m)/\nu_{2,m}} \approx F(\nu_{1,m}, \nu_{2,m}). \quad (2.4)$$

2.2 Pillai'S Approximation

Assume $\nu_{1,0} = ab$ and $\gamma_0 = s$. The Pillai (1954) approximation results from assuming $\gamma_0 = s$ and $\nu_{1,0} = ab$, then solving

$$s \cdot \frac{ab}{ab+\nu_{2,0}} = \mu_1 \quad (2.5)$$

for $\nu_{2,0}$. This yields $\nu_{2,0} = s(\nu_E + a) - ab = s(\nu_E + s - b)$.

2.3 Method 1: A New Approximation

Assume $\nu_{1,1} = ab$ and γ_1 free to vary. With the assumption $\nu_{1,1} = ab$, the asymptotic value, write the equations of interest as

$$\gamma_1 \frac{ab}{ab+\nu_{2,1}} = \mu_1, \quad (2.6)$$

and

$$\gamma_1^2 \frac{2ab\nu_{2,1}}{(ab+\nu_{2,1})^2(ab+\nu_{2,1}+2)} = \mu_2. \quad (2.7)$$

Solving for $\nu_{2,1}$ and γ_1 yields

$$\nu_{2,1} = \frac{(ab+2)\nu_E(\nu_E+a-b)}{\nu_E(a+1+b)+a^2+a-2}, \quad (2.8)$$

$$\gamma_1 = \frac{ab+\nu_{2,1}}{(\nu_E+a)}. \quad (2.9)$$

Note that Method 1 implies $\Pr\{V \leq \upsilon\} = 1.0$ if $\upsilon > \gamma_1$.

2.4 Method 2: A New Approximation

Assume $\nu_{1,2}$ free to vary and $\gamma_2 = s$. With the assumption $\gamma_2 = s$, the upper bound on V, write the system of equations of interest as

$$\frac{\nu_{1,2}}{\nu_{1,2}+\nu_{2,2}} = \frac{\mu_1}{s}, \quad (2.10)$$

$$\frac{\nu_{1,2}\nu_{2,2}}{(\nu_{1,2}+\nu_{2,2})^2(\nu_{1,2}+\nu_{2,2}+1)} = \frac{\mu_2}{s^2}. \quad (2.11)$$

Solving for $\nu_{1,2}$ and $\nu_{2,2}$ yields

$$\nu_{1,2} = \frac{ab}{s(\nu_E+a)}\left[\frac{s(\nu_E+s-b)(\nu_E+a+2)(\nu_E+a-1)}{\nu_E(\nu_E+a-b)} - 2\right] \quad (2.12)$$

and

$$\nu_{2,2} = \frac{(\nu_E+s-b)}{(\nu_E+a)}\left[\frac{s(\nu_E+s-b)(\nu_E+a+2)(\nu_E+a-1)}{\nu_E(\nu_E+a-b)} - 2\right]. \quad (2.13)$$

Observe that, with

$$K=\frac{1}{s(\nu_E+a)}\left[\frac{s(\nu_E+s-b)(\nu_E+a+2)(\nu_E+a-1)}{\nu_E(\nu_E+a-b)}-2\right], \quad (2.14)$$

$\nu_{1,2} = K\nu_{1,0}$ and $\nu_{2,2} = K\nu_{2,0}$. If $s = a$ then

$$K=\frac{a\nu_E+(a-1)(a+2)}{a\nu_E}\geq 1, \quad (2.15)$$

while if $s = b$ then

$$K=\frac{b(\nu_E+a)+(b-2)}{b[(\nu_E+a)-b]}\geq 1. \quad (2.16)$$

Montonicity properties of the F distribution and $K \geq 1$ ensure that the probability for Method 2 will never be smaller than for Method 0, and hence the p value for a test will never be larger.

3. NUMERICAL COMPARISONS

Pillai (1954, p. 111) reported some $\alpha = .05$ critical values for $m = (|a - b| - 1)/2 = 0$, and a range of $n = (\nu_E - b - 1)/2$. Table 1 contains the critical values, as well as approximate probabilities computed with Pillai's approximation, as well as the two new methods (Method 1 in Section 2.3 and Method 2 in Section 2.4). Both of the new methods provide substantially better accuracy than does Pillai's approximation, with Method 1 slightly better than Method 2. Values in Table 1 greater than .05 imply that the approximation provides a conservative test in data analysis, while values less than .05 imply that the approximation provides a liberal test.

Pillai (1954) also reported some exact probabilities for certain other quantiles. Table 2 contains exact probabilities of the exact quantiles, as well as approximate quantiles from Pillai's approximation and the two new methods. The pattern parallels that in Table 1, except for the most extreme quantile. Large quantiles create difficulty for Method 1, which uses $\gamma_1 \leq s$ and implies $\Pr\{V \leq \upsilon\} = 1$ for $\upsilon > \gamma_1$ (which occurs with small but nonzero probability).

A simulation was conducted in SAS IML® to examine performance for extreme quantiles. A total of 500,000 values of V were tabulated for $a = 2$, $b = 3$, and $\nu_E = 10$ (and hence $n = 3$ and $m = 0$). For each replication the SAS function NORMAL created two matrices, $\mathbf{Z}_H (a \times b)$ and $\mathbf{Z}_E (\nu_E \times b)$, of pseudo-random, i.i.d. Gaussian data having mean zero and unit variance. In turn $\mathbf{H} = \mathbf{Z}'_H \mathbf{Z}_H$, $\mathbf{E} = \mathbf{Z}'_E \mathbf{Z}_E$, and $V = \mathrm{tr}[\mathbf{H}(\mathbf{H} + \mathbf{E})^{-1}]$. No other covariance structure needs to be considered due to the invariance of V under full rank linear transformation of the data.

Table 3 contains empirical quantiles and associated approximate probabilities. The results parallel those in Tables 1 and 2: overall, both new methods performed noticeably better than Pillai's approximation. Method 1 performs the best for probabilities no more than .99, with Method 2 superior in the extreme right tail. As suspected, the need to map all probabilities for values greater than γ_1 into 1.0 hurts the accuracy of Method 1 near the boundary.

Table 2 in Anderson (1984, appendix B) contains approximate critical values for a simple multiple of the Pillai–Bartlett trace, computed with the Pearson system method of Pillai. Extensive comparison of those values to ones computed with the new methods provides

results consistent with the results presented here. Some related additional simulations were also conducted, and also supported the conclusions presented here.

4. CONCLUSIONS

1. Pillai's F and the two new F approximations have logical and practical appeal because they
    a. correctly reduce to exact answer if $s = 1$;
    b. always match at least one moment exactly;
    c. have appropriate asymptotic behavior; and
    d. allow simple and convenient computations.
2. The new approximations provide substantially greater accuracy than the widely used Pillai F, which tends to be too conservative in small samples.
3. Method 1 provides the best average performance among the F approximations, but can sometimes be liberal.
4. The possibility of liberality, difficulties with $V > \gamma_1$, and the importance of Bonferroni corrections in multivariate data analysis, combine to substantially reduce the appeal of Method 1.
5. Pillai's Pearson system approximation provides some additional accuracy, but lacks most of the appeal of the F approximations.
6. Method 2 (Sec. 2.4) provides the best combination of accuracy of type I error control and logical properties, and deserves to replace Pillai's F approximation in data analysis.
7. Method 2 merits consideration in future research on power approximations.

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Table 1

Test Size for Exact Critical Values α =.05, $m = 0$)

			Test size		
s	**n**	**$V_{.95}$**	**Pillai**	**Method 1**	**Method 2**
2	10	.451	.0567	.0503	.0508
	15	.332	.0549	.0506	.0508
	20	.264	.0526	.0494	.0495
	25	.218	.0524	.0498	.0499
	30	.186	.0519	.0497	.0497
3	10	.697	.0615	.0501	.0515

Table 2

Approximate Quantiles for Exact Probabilities ($s = 2$, $m = 0$)

		Exact quantile	Approximate quantile		
n	**Pr{V ≤v}**	**v**	**Pillai**	**Method 1**	**Method 2**
10	.1192	.1000	.0971	.0998	.1016
10	.4534	.2000	.1972	.1999	.1992
10	.7471	.3000	.3004	.3003	.2983
10	.9084	.4000	.4067	.4006	.3997
30	.6353	.1000	.0998	.1000	.0998
30	.9661	.2000	.2022	.2001	.2003
30	.9984	.3000	3071	.2995	.3029
30	.9999	.4000	.3913	.3771	.3853

Table 3

Empirical Quantiles and Approximate Probabilities ($a = 2,\ b = 3,\ \nu_E = 10$)

Empirical quantile			Approximate probability, $F_{V*}(v)$		
v	$\hat{F}_V(v)$	±st. dev.	Pillai	Method 1	Method 2
.093	.01	± .00014	.01257	.01024	.00653
.170	.05	± .00031	.06073	.05158	.04238
.225	.10	± .00042	.11748	.10250	.09225
.336	.25	± .00061	.27669	.25375	.25050
.483	.50	± .00071	.51792	.50071	.51125
.646	.75	± .00061	.74265	.74548	.75658
.798	.90	± .00042	.87949	.89612	.89765
.890	.95	± .00031	.93064	.94921	.94599
1.071	.99	± .00014	.98118	.99289	.98824
1.284	.999	± .00004	.99755	.99993	.98894
1.453	.9999	± .00001	.99973	1.0	.99992
1.558	.99999	± .000004	.99996	1.0	.99999