A new family of distributions: the Kumaraswamy odd log-logistic, properties and applications

Morad Alizadeh*, † Mahdi Emadi‡, Mahdi Doostparast§, Gauss M.Cordeiro¶, Edwin M.M.Ortega‖, and Rodrigo R.Pescim **

Abstract

In this paper, a new family of distributions, called the Kumaraswamy odd log-logistic, is proposed and studied. Some mathematical properties are presented and special models are discussed. The asymptotes and shapes are investigated. The family density function is given by a linear combination of exponentiated densities following the same baseline model. We derive a power series for the quantile function, explicit expressions for the moments, quantile and generating functions and order statistics. We provide a bivariate extension of the new family. Its performance is illustrated by means of two real data sets.

Keywords: Generated function; Log-logistic distribution; Maximum likelihood; Moment; Order statistic; Quantile function.

2000 AMS Classification: AMS

1. Introduction

Recently, some attempts have been made to define new families of distributions that extend well-known distributions and at the same time provide great flexibility in modelling data in practice. So, several classes by adding one or more parameters to generate new distributions have been proposed in the statistical literature. Some well-known generators are the Marshall-Olkin generated family (MO-G) by Marshall and Olkin (1997), the beta-G by Eugene *et al.* (2002), the Kumaraswamy-G (Kw-G for short) by Cordeiro and de Castro (2011), the McDonald-G (Mc-G) by Alexander *et al.* (2012), the gamma-G by Zografos and Balakrishanan (2012), the

*Department of Statistics, Ferdowsi University of Mashhad, Iran., Email: moradalizadeh78@gmail.com

†Corresponding Author.

‡Department of Statistics, Ferdowsi University of Mashhad, Iran., Email: emadi@math.um.ac.ir

§Department of Statistics, Ferdowsi University of Mashhad, Iran., Email: doustparast@math.um.ac.ir

¶Departamento de Estatística, Universidade Federal de Pernambuco, Brazil., Email: gauss@de.ufpe.br

‖Departamento de Ciências Exatas, Universidade de São Paulo, ESALQ, Brazil., Email: edwin@usp.br

**Departamento de Ciências Exatas, Universidade de São Paulo, ESALQ, Brazil., Email: rrpescim@gmail.com

transformer (T-X) by Alzaatreh *et al.* (2013), the Weibull-G by Bourguignon *et al.* (2014) and the exponentiated half-logistic family by Cordeiro *et al.* (2014).

Let $G(x;\boldsymbol{\xi})$ be a baseline cumulative distribution function (cdf) and $\boldsymbol{\xi}$ the $p \times 1$ vector of associated parameters. Recently, da Cruz *et al.* (2014) introduced a class of distributions named the *odd log-logistic family* with one extra shape parameter $\alpha > 0$ defined by the cdf

$$H(x) = \frac{G(x;\boldsymbol{\xi})^{\alpha}}{G(x;\boldsymbol{\xi})^{\alpha} + \bar{G}(x;\boldsymbol{\xi})^{\alpha}}, \tag{1.1}$$

where $\bar{G}(x;\boldsymbol{\xi}) = 1 - G(x;\boldsymbol{\xi})$.

Let $r(t)$ be the probability density function (pdf) of a random variable $T \in [a, b]$ for $-\infty \leq a < b < \infty$ and let $W[G(x)]$ be a function of the cdf of a random variable X such that $W[G(x)]$ satisfies the following conditions:

$$\begin{cases} (i) & W\left[G(x)\right] \in [a, b], \\ (ii) & W\left[G(x)\right] \text{ is differentiable and monotonically non-decreasing, and} \\ (iii) & W\left[G(x)\right] \to a \ \text{ as } \ x \to -\infty \text{ and } W\left[G(x)\right] \to b \ \text{ as } \ x \to \infty. \end{cases} \tag{1.2}$$

Alzaatreh *et al.* (2013) defined the *T-X* family of distributions by

$$F(x) = \int_{a}^{W[G(x)]} r(t)\, dt, \tag{1.3}$$

where $W[G(x)]$ satisfies the conditions (1.2). The pdf corresponding to (1.3) is given by

$$f(x) = \left\{ \frac{d}{dx}\, W[G(x)] \right\} r\left\{ W[G(x)] \right\}. \tag{1.4}$$

In this paper, we propose a new wider class of continuous distributions called the *Kumaraswamy odd log-logistic-G* ("KwOLL-G" for short) family by taking $W[G(x)] = \frac{G(x;\boldsymbol{\xi})^{\alpha}}{G(x;\boldsymbol{\xi})^{\alpha} + \bar{G}(x;\boldsymbol{\xi})^{\alpha}}$ and $r(t) = a\, b\, t^{a-1}\,(1 - t^{a})^{b-1},\ 0 < t < 1$. Its cdf is given by

$$\begin{aligned} F(x) &= \int_{0}^{\frac{G(x;\boldsymbol{\xi})^{\alpha}}{G(x;\boldsymbol{\xi})^{\alpha} + \bar{G}(x;\boldsymbol{\xi})^{\alpha}}} a\, b\, t^{a-1}(1 - t^{a})^{b-1} dt \\ &= 1 - \left\{ 1 - \left[\frac{G(x,\boldsymbol{\xi})^{\alpha}}{G(x,\boldsymbol{\xi})^{\alpha} + \bar{G}(x,\boldsymbol{\xi})^{\alpha}} \right]^{a} \right\}^{b}, \end{aligned} \tag{1.5}$$

where $\alpha > 0$, $a > 0$ and $b > 0$ are three extra shape parameters to the baseline cdf $G(x,\boldsymbol{\xi})$. The KwOLL-G family (1.5) includes the *Kumaraswamy generalized* family (Cordeiro and de Castro, 2011), the proportional and reversed hazard rate models, the *odd log-logistic* family (da Cruz *et al.,* 2014), the *exponentiated OLL-G* family (Cordeiro *et al.,* 2014a), the *odd Burr* family (Alizadeh *et al.,* 2014), among others. Some special models of (1.5) are listed in Table 1.

The paper is organized as follows. In Section 2, we provide a physical interpretation of the KwOLL-G family. Four special cases are described in Section 3 with some details. In Section 4, the asymptotes and shapes of the density and hazard rate functions are investigated analytically. Some useful expansions are obtained in Section 5. In Section 6, we derive a power series for the quantile function (qf).

Table 1. Some special models.

a	b	α	Reduced distribution
-	-	1	Kumaraswamy generalized family of distributions (Cordeiro and de Castro, 2011)
1	1	-	Odd log-logistic family (da Cruz *et al.,* 2014)
-	1	-	exp OLL-G family of distributions (Cordeiro *et al.,* 2014a)
1	-	0	Odd Burr family of distributions (Alizadeh *et al.,* 2014)
1	-	1	Proportional hazard rate model (Gupta *et al.,* 1998)
-	1	1	Proportional reversed hazard rate model (Gupta and Gupta, 2007)
1	1	1	$G(x)$

In Sections 7 and 8, we obtain the ordinary and incomplete moments and the generating function, respectively. The order statistics are derived in Section 9. In Section 10, we introduce a bivariate extension of the new family. The estimation of the model parameters by maximum likelihood is performed in Section 11. Two applications to real data illustrate the potentiality of the proposed family in Section 12. Section 13 provides some conclusions.

2. The new family

The pdf corresponding to (1.5) is

$$f(x;a,b,\alpha,\boldsymbol{\xi})=\frac{ab\alpha g(x,\boldsymbol{\xi})G(x,\boldsymbol{\xi})^{\alpha a-1}\bar{G}(x,\boldsymbol{\xi})^{\alpha-1}}{\left[G(x,\boldsymbol{\xi})^{\alpha}+\bar{G}(x,\boldsymbol{\xi})^{\alpha}\right]^{a+1}}\left\{1-\left[\frac{G(x,\boldsymbol{\xi})^{\alpha}}{G(x,\boldsymbol{\xi})^{\alpha}+\bar{G}(x,\boldsymbol{\xi})^{\alpha}}\right]^{a}\right\}^{b-1}, \tag{2.1}$$

where $g(x;\boldsymbol{\xi})=dG(x;\boldsymbol{\xi})/dx$. Hereafter, $X\sim$ KwOLL-G$(a,b,\alpha,\boldsymbol{\xi})$ denotes a random variable having the density function (2.1). Further, we sometimes omit the dependence on the vector $\boldsymbol{\xi}$ and write simply $G(x)=G(x;\boldsymbol{\xi})$.

A physical interpretation of the KwOLL-G cdf (for a and b positive integers) is as follows. Equation (1.5) denotes the cdf of the lifetime of a series-parallel system consisting of independent components with the common cdf $H(x)$ given by (1.1). Consider that a system is formed by b independent series subsystems and that each of the subsystems is made up of a independent parallel components. Let $X_{ij}\sim H(x)$, for $1\le i\le a$ and $1\le j\le b$, denote the lifetime of the ith component in the jth subsystem and X denotes the lifetime of the entire system. We have

$$Pr(X\le x)=1-\left\{1-Pr(X_{11}\le x,\cdots,X_{1a}\le x)\right\}^{b}=1-\left\{1-Pr^{a}(X_{11}\le x)\right\}^{b},$$

and then X has pdf (2.1).

The hazard rate function (hrf) of X is given by

$$h(x;a,b,\alpha,\boldsymbol{\xi})=\frac{a\,b\,\alpha\,g(x,\boldsymbol{\xi})\,G(x,\boldsymbol{\xi})^{\alpha a-1}\bar{G}(x,\boldsymbol{\xi})^{\alpha-1}}{\left[G(x,\boldsymbol{\xi})^{\alpha}+\bar{G}(x,\boldsymbol{\xi})^{\alpha}\right]\left\{\left[G(x,\boldsymbol{\xi})^{\alpha}+\bar{G}(x,\boldsymbol{\xi})^{\alpha}\right]^{a}-G(x,\boldsymbol{\xi})^{a\alpha}\right\}}. \tag{2.2}$$

The KwOLL-G family is simulated by inverting $F(x) = u$ in (1.5) as follows: if u has a uniform $U(0,1)$ distribution, the solution of the nonlinear equation

$$x_u = G^{-1}\left(\frac{\left[1-(1-u)^{\frac{1}{b}}\right]^{\frac{1}{a\alpha}}}{\left[1-(1-u)^{\frac{1}{b}}\right]^{\frac{1}{a\alpha}} + \left\{1-\left[1-(1-u)^{\frac{1}{b}}\right]^{\frac{1}{a}}\right\}^{\frac{1}{\alpha}}} \right) \qquad (2.3)$$

has the pdf (2.1).

3. Four special cases of the KwOLL-G family

Equation (2.1) will be most tractable when $G(x;\boldsymbol{\xi})$ and $g(x;\boldsymbol{\xi})$ have closed-forms. Now, we provide only four cases of so many distributions which can be special models of the KwOLL-G family.

3.1. The Kumaraswamy odd log-logistic-normal (KwOLLN) distribution. By taking $G(x;\boldsymbol{\xi})$ and $g(x;\boldsymbol{\xi})$ in (2.1) to be the cdf and pdf of the normal $N(\mu,\sigma^2)$ distribution, where $\boldsymbol{\xi} = (\mu,\sigma)^T$, the KwOLLN pdf follows as

$$\begin{aligned} f(x;a,b,\alpha,\mu,\sigma) \quad &= \quad \frac{a\,b\,\alpha\,\phi(\frac{x-\mu}{\sigma})\,\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\alpha\,a-1}\,\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\alpha-1}}{\sigma\left\{\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\alpha}+\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\alpha}\right\}^{a+1}} \\ &\quad \times\left\{1-\left[\frac{\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\alpha}}{\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\alpha}+\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\alpha}}\right]^{a}\right\}^{b-1}, \qquad (3.1) \end{aligned}$$

where $x \in \mathbb{R}$, $\mu \in \mathbb{R}$ is a location parameter, $\sigma > 0$ is a scale parameter, and $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and cdf of the standard normal distribution, respectively. We denote by $X \sim \text{KwOLLN}(a,b,\alpha,\mu,\sigma)$ a random variable with pdf (3.1). For $\mu = 0$ and $\sigma = 1$, we obtain the standard KwOLLN distribution, and for $a = b = \alpha = 1$, it reduces to the normal distribution. For $\alpha = 1$, we have the Kumaraswamy normal (KwN) (Cordeiro and de Castro, 2011) distribution. Further, if $\alpha = 1$ in addition to $b = 1$, it gives the exponentiated-normal (EN) distribution. Plots of the KwOLLN pdf for selected parameter values are displayed in Figure 1.

3.2. The Kumaraswamy odd log-logistic-Weibull (KwOLLW) distribution. By taking $G(x;\boldsymbol{\xi}) = 1 - \mathrm{e}^{-(\beta \mathrm{x})^{\lambda}}$ to be the Weibull distribution with scale parameter $\beta > 0$ and shape parameter $\lambda > 0$, where $\boldsymbol{\xi} = (\lambda,\beta)^T$, we obtain the KwOLLW pdf (for $x > 0$) as

$$\begin{aligned} f(x) = f(x;a,b,\alpha,\lambda,\beta) \quad &= \quad \frac{a\,b\,\alpha\,\lambda\,\beta^{\lambda}\,x^{\lambda-1}\left\{1-\exp\left[-(\beta\,x)^{\lambda}\right]\right\}^{\alpha\,a-1}\left\{\exp\left[-(\beta\,x)^{\lambda}\right]\right\}^{\alpha}}{\left\{\left[1-\exp\left[-(\beta\,x)^{\lambda}\right]\right]^{\alpha}+\left[\exp\left[-(\beta\,x)^{\lambda}\right]\right]^{\alpha}\right\}^{a+1}} \\ &\quad \times\left\{1-\left[\frac{\left\{1-\exp\left[-(\beta\,x)^{\lambda}\right]\right\}^{\alpha}}{\left\{1-\exp\left[-(\beta\,x)^{\lambda}\right]\right\}^{\alpha}+\left\{\exp\left[-(\beta\,x)^{\lambda}\right]\right\}^{\alpha}}\right]^{a}\right\}^{b-1}. \qquad (3.2) \end{aligned}$$

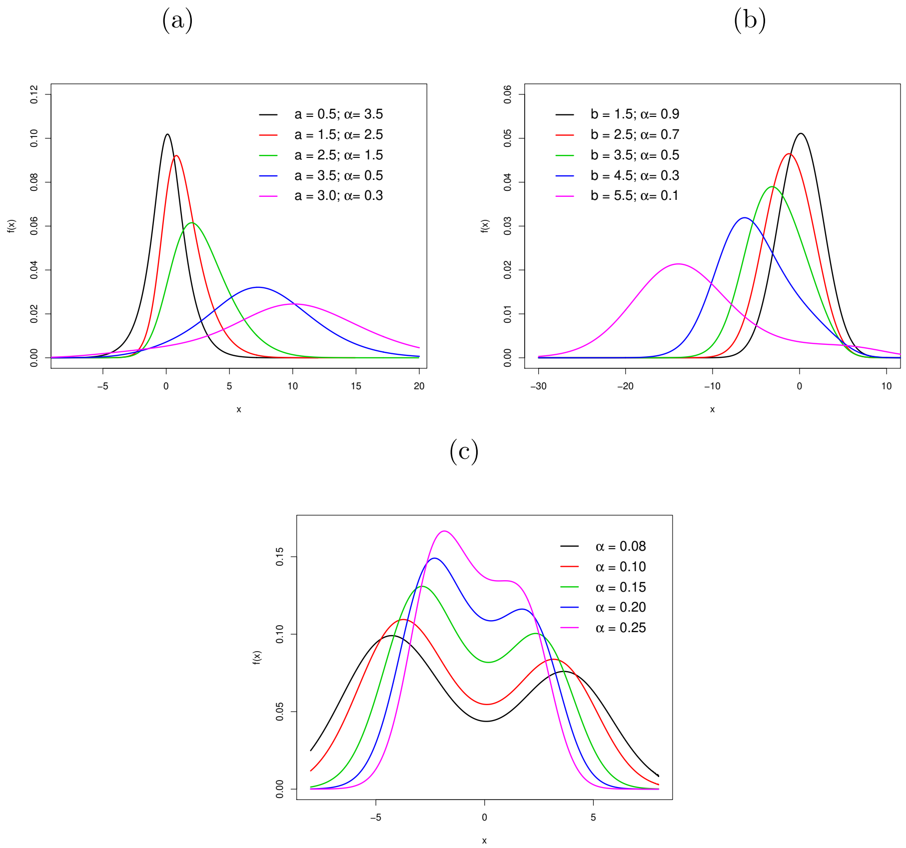

Figure 1. The KwOLLN pdf: (a) For $b = 0.5$, $\mu = 0$ and $\sigma = 3$. (b) For $a = 1.5$, $\mu = 0$ and $\sigma = 3$. (c) For $a = 1.5$, $b = 2.0$, $\mu = 0$ and $\sigma = 1$.

The Weibull distribution (with parameters λ and β) is a basic exemplar for $a = b = \alpha = 1$. Other special models include the Kumaraswamy Weibull (KwW) (Cordeiro *et al.*, 2010) for $\alpha = 1$ and the exponentiated Weibull (EW) (Mudholkar *et al.*, 1995; Mudholkar *et al.*, 1996; Nassar and Eissa, 2003; Nadarajah *et al.*, 2013) and exponentiated exponential (EE) (Gupta and Kundu, 2001) distributions for $b = \alpha = 1$ and $b = \alpha = \beta = 1$, respectively. Plots of the pdf and hrf of the KwOLLW distribution for selected parameter values are displayed in Figures 2 and 3, respectively. Further, it allows for five major hazard shapes: constant, increasing, decreasing, bathtub and unimodal hazard rates .

3.3. The Kumaraswamy odd log-logistic-Gumbel (KwOLLGu) distribution. Let $G(x; \boldsymbol{\xi})$ for $x \in \mathbb{R}$ be the Gumbel distribution with parameters (μ, σ), where $\mu \in \mathbb{R}$ is the location parameter and $\sigma > 0$ is the scale parameter, and cdf

(a) (b)

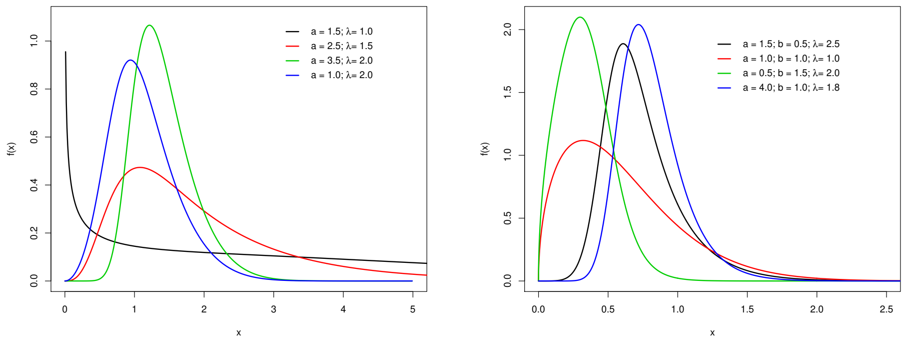

Figure 2. The KwOLLW pdf: (a) For $b = 0.5$, $\alpha = 0.5$ and $\beta = 1$. (b) For $\lambda = 1.5$ and $\beta = 1.5$.

given by

$$G(x;\boldsymbol{\xi}) = \exp\left[-\exp\left(-\frac{x-\mu}{\sigma}\right)\right], \quad x \in \mathbb{R}.$$

Inserting these expressions in equation (2.1) yields the KwOLLGu pdf

$$\begin{aligned} f(x;a,b,\alpha,\mu,\sigma) &= \frac{ab\alpha\exp\{-\frac{x-\mu}{\sigma}-\exp(-\frac{x-\mu}{\sigma})\}\left(\exp\left\{-\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}\right)^{\alpha\,a-1}}{\sigma\left\{\left[\exp\{-\exp(-\frac{x-\mu}{\sigma})\}\right]^{\alpha}+\left[1-\exp\{-\exp(-\frac{x-\mu}{\sigma})\}\right]^{\alpha}\right\}^{a+1}} \\ &\quad\times\left\{1-\left[\frac{\left[\exp\left\{-\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}\right]^{\alpha}}{\left[\exp\left\{-\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}\right]^{\alpha}+\left[1-\exp\left\{-\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}\right]^{\alpha}}\right]^{a}\right\}^{b-1} \\ &\quad\times(1-\exp\{-\exp(-\frac{x-\mu}{\sigma})\})^{\alpha-1}, \end{aligned} \tag{3.3}$$

where $x \in \mathbb{R}$. The Kumaraswamy Gumbel (KwGu) (Cordeiro *et al.*, 2010) model corresponds to $\alpha = 1$. The Lehmann type I Gumbel distribution refers to $b = \alpha = 1$. This case is usually called the exponentiated Gumbel (EGu) model. Indeed, the EGu cdf is defined by (for $\lambda > 0$)

$$F(x;\lambda,\boldsymbol{\xi}) = 1-[1-G(x;\boldsymbol{\xi})]^{\lambda}.$$

Plots of the KwOLLGu pdf for some parameter values are displayed in Figure 4.

4. Asymptotes and Shapes

4.1. Proposition. *The asymptotics of equations (1.5), (2.1) and (2.2) as* $x \to 0$ *are given by*

$$\begin{aligned} F(x) &\sim b\,G(x)^{a\,\alpha} \quad \text{as } \mathrm{G(x)} \to 0, \\ f(x) &\sim a\,b\,\alpha\,g(x)\,G(x)^{a\,\alpha-1} \quad \text{as } \mathrm{G(x)} \to 0, \\ h(x) &\sim a\,b\,\alpha\,g(x)\,G(x)^{a\,\alpha-1} \quad \text{as } \mathrm{G(x)} \to 0. \end{aligned}$$

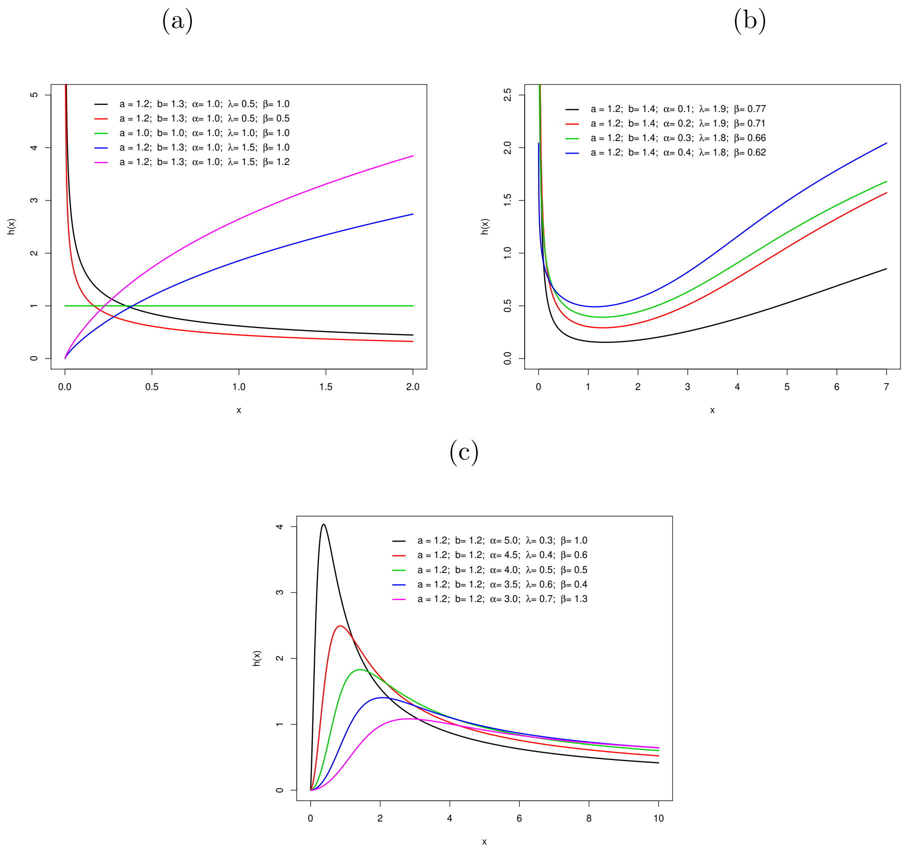

Figure 3. The KwOLLW hrf: (a) Constant, increasing and decreasing hrf. (b) Bathtub hrf. (c) Unimodal hrf.

4.2. Proposition. *The asymptotics of equations (1.5), (2.1) and (2.2) as $x \to \infty$ are given by*

$$\begin{aligned}
&1 - F(x) \sim \left[a\,\alpha\,\bar{G}(x)\right]^b \quad \text{as } \mathrm{x} \to \infty,\\
&f(x) \sim b\,(a\,\alpha)^b\,g(x)\,\bar{G}(x)^{b-1} \quad \text{as } \mathrm{x} \to \infty,\\
&h(x) \sim \frac{b\,g(x)}{\bar{G}(x)} \quad \text{as } \mathrm{x} \to \infty.
\end{aligned}$$

The shapes of the density and hazard rate functions can be described analytically. The critical points of the KwOLL-G pdf are the roots of the equation:

$$\begin{aligned}
&\frac{g'(x)}{g(x)} + (a\alpha - 1)\frac{g(x)}{G(x)} + (1-\alpha)\frac{g(x)}{\bar{G}(x)} - \alpha(a+1)g(x)\frac{G(x)^{\alpha-1} - \bar{G}(x)^{\alpha-1}}{G(x)^{\alpha} + \bar{G}(x)^{\alpha}}\\
= \quad & a(b-1)\alpha g(x)\frac{G(x)^{a\alpha-1}\bar{G}(x)^{\alpha-1}}{\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]\left\{\left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]^a - G(x)^{a\alpha}\right\}}.
\end{aligned}$$

(4.1)

(a) (b)

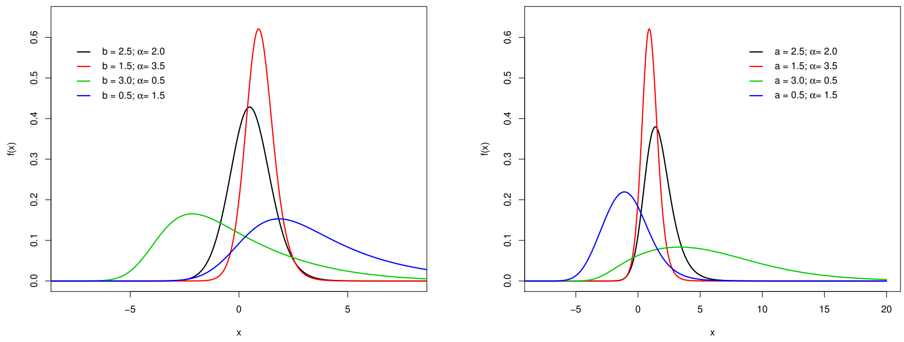

Figure 4. The KwOLLGu pdf: (a) For $a = 1.5$, $\mu = 0$ and $\sigma = 2.5$; (b) For $b = 1.5$, $\mu = 0$ and $\sigma = 2.5$.

There may be more than one root to (4.1). Let $\lambda(x) = \frac{d^2 \log[f(x)]}{dx^2}$. Then,

$$\begin{aligned}
\lambda(x) &= \frac{g''(x)g(x) - g'(x)^2}{g(x)^2} + (a\alpha - 1)\frac{g'(x)G(x) - g(x)^2}{G(x)^2} + (1-\alpha)\frac{g'(x)\bar{G}(x) + g(x)^2}{\bar{G}(x)^2} \\
&- \alpha(a+1)g'(x)\frac{G(x)^{\alpha-1} - \bar{G}(x)^{\alpha-1}}{G(x)^\alpha + \bar{G}(x)^\alpha} - \alpha(\alpha-1)(a+1)g(x)^2 \, \frac{G(x)^{\alpha-2} + \bar{G}(x)^{\alpha-2}}{G(x)^\alpha + \bar{G}(x)^\alpha} \\
&+ (a+1)\left\{\alpha g(x)\frac{G(x)^{\alpha-1} - \bar{G}(x)^{\alpha-1}}{G(x)^\alpha + \bar{G}(x)^\alpha}\right\}^2 \\
&- a(b-1)\alpha g'(x)\frac{G(x)^{a\alpha-1}\bar{G}(x)^{\alpha-1}}{\left[G(x)^\alpha + \bar{G}(x)^\alpha\right]\left\{\left[G(x)^\alpha + \bar{G}(x)^\alpha\right]^a - G(x)^{a\alpha}\right\}} \\
&- a(a\alpha-1)(b-1)\alpha\frac{g(x)^2 G(x)^{a\alpha-2}\bar{G}(x)^{\alpha-1}}{\left[G(x)^\alpha + \bar{G}(x)^\alpha\right]\left\{\left[G(x)^\alpha + \bar{G}(x)^\alpha\right]^a - G(x)^{a\alpha}\right\}} \\
&+ a(b-1)(\alpha-1)\alpha\frac{g(x)^2 G(x)^{a\alpha-1}\bar{G}(x)^{\alpha-2}}{\left[G(x)^\alpha + \bar{G}(x)^\alpha\right]\left\{\left[G(x)^\alpha + \bar{G}(x)^\alpha\right]^a - G(x)^{a\alpha}\right\}} \\
&+ a\alpha^2(b-1)\frac{g(x)^2\left[G(x)^{\alpha-1} - \bar{G}(x)^{\alpha-1}\right]G(x)^{a\alpha-1}\bar{G}(x)^{\alpha-1}}{\left[G(x)^\alpha + \bar{G}(x)^\alpha\right]^2\left\{\left[G(x)^\alpha + \bar{G}(x)^\alpha\right]^a - G(x)^{a\alpha}\right\}} \\
&+ \frac{a^2\alpha^2(b-1)\, g(x)G(x)^{a\alpha-1}\bar{G}(x)^{\alpha-1}\left\{\left[G(x)^{\alpha-1} - \bar{G}(x)^{\alpha-1}\right]\left[G(x)^\alpha + \bar{G}(x)^\alpha\right]^{a-1}\right\}}{\left[G(x)^\alpha + \bar{G}(x)^\alpha\right]\left\{\left[G(x)^\alpha + \bar{G}(x)^\alpha\right]^a - G(x)^{a\alpha}\right\}^2} \\
&- \frac{a^2\alpha^2(b-1)\, g(x)G(x)^{2a\alpha-2}\bar{G}(x)^{\alpha-1}}{\left[G(x)^\alpha + \bar{G}(x)^\alpha\right]\left\{\left[G(x)^\alpha + \bar{G}(x)^\alpha\right]^a - G(x)^{a\alpha}\right\}^2}.
\end{aligned}$$

If $x = x_0$ is a root of (4.1) then it corresponds to a local maximum (minimum) if $\lambda(x) > 0(< 0)$ for all $x < x_0$ and $\lambda(x) < 0(> 0)$ for all $x > x_0$. It yields points of inflexion if either $\lambda(x) > 0$ for all $x \neq x_0$ or $\lambda(x) < 0$ for all $x \neq x_0$.

The critical points of the hrf $h(x)$ are obtained from the equation:

$$
\begin{aligned}
& \frac{g'(x)}{g(x)} + (a\alpha-1)\frac{g(x)}{G(x)} + (1-\alpha)\frac{g(x)}{\bar{G}(x)} - \alpha g(x)\frac{\left[G(x)^{\alpha-1}-\bar{G}(x)^{\alpha-1}\right]}{G(x)^{\alpha}+\bar{G}(x)^{\alpha}} \\
= \; & a\alpha g(x)\frac{\left[G(x)^{\alpha-1}-\bar{G}(x)^{\alpha-1}\right]\left[G(x)^{\alpha}+\bar{G}(x)^{\alpha}\right]^{a-1}-G(x)^{a\alpha-1}}{\left[G(x)^{\alpha}+\bar{G}(x)^{\alpha}\right]^{a}-G(x)^{a\alpha}}.
\end{aligned}
$$

(4.2)

There may be more than one root to (4.2). Let $\tau(x) = d^2\log[h(x)]/dx^2$. We have

$$
\begin{aligned}
\tau(x) \; = \; & \frac{g''(x)g(x)-g'(x)^2}{g(x)^2} + (a\alpha-1)\frac{g'(x)G(x)-g(x)^2}{G(x)^2} + (1-\alpha)\frac{g'(x)\bar{G}(x)+g(x)^2}{\bar{G}(x)^2} \\
- \; & \alpha g'(x)\frac{G(x)^{\alpha-1}-\bar{G}(x)^{\alpha-1}}{G(x)^{\alpha}+\bar{G}(x)^{\alpha}} - \alpha(\alpha-1)g(x)^2\;\frac{G(x)^{\alpha-2}+\bar{G}(x)^{\alpha-2}}{G(x)^{\alpha}+\bar{G}(x)^{\alpha}} \\
+ \; & \left\{\alpha g(x)\frac{G(x)^{\alpha-1}-\bar{G}(x)^{\alpha-1}}{G(x)^{\alpha}+\bar{G}(x)^{\alpha}}\right\}^2 \\
- \; & a\alpha g'(x)\frac{\left[G(x)^{\alpha-1}-\bar{G}(x)^{\alpha-1}\right]\left[G(x)^{\alpha}+\bar{G}(x)^{\alpha}\right]^{a-1}-G(x)^{a\,\alpha-1}}{\left[G(x)^{\alpha}+\bar{G}(x)^{\alpha}\right]^{a}-G(x)^{a\alpha}} \\
- \; & a\,\alpha(\alpha-1)g(x)^2\frac{\left[G(x)^{\alpha-2}+\bar{G}(x)^{\alpha-2}\right]\left[G(x)^{\alpha}+\bar{G}(x)^{\alpha}\right]^{a-1}}{\left[G(x)^{\alpha}+\bar{G}(x)^{\alpha}\right]^{a}-G(x)^{a\alpha}} \\
- \; & a\,\alpha^2(a-1)g(x)^2\frac{\left[G(x)^{\alpha-1}-\bar{G}(x)^{\alpha-1}\right]^2\left[G(x)^{\alpha}+\bar{G}(x)^{\alpha}\right]^{a-2}}{\left[G(x)^{\alpha}+\bar{G}(x)^{\alpha}\right]^{a}-G(x)^{a\alpha}} \\
+ \; & a\,\alpha(a\,\alpha-1)g(x)^2\frac{G(x)^{a\,\alpha-2}}{\left[G(x)^{\alpha}+\bar{G}(x)^{\alpha}\right]^{a}-G(x)^{a\alpha}} \\
+ \; & \left\{a\,\alpha\,g(x)\frac{\left[G(x)^{\alpha-1}-\bar{G}(x)^{\alpha-1}\right]\left[G(x)^{\alpha}+\bar{G}(x)^{\alpha}\right]^{a-1}-G(x)^{a\,\alpha-1}}{\left[G(x)^{\alpha}+\bar{G}(x)^{\alpha}\right]^{a}-G(x)^{a\alpha}}\right\}^2.
\end{aligned}
$$

If $x = x_0$ is a root of (4.2) then it refers to a local maximum (minimum) if $\tau(x) > 0 (< 0)$ for all $x < x_0$ and $\tau(x) < 0 (< 0)$ for all $x > x_0$. It gives an inflexion point if either $\tau(x) > 0$ for all $x \neq x_0$ or $\tau(x) < 0$ for all $x \neq x_0$.

5. Some useful expansions

The cdf (1.5) of X admits the expansion

$$
\begin{aligned}
F(x) \; & = \; 1 - \sum_{m=0}^{\infty}(-1)^m\binom{b}{m}\;\frac{G(x)^{a\alpha m}}{\left[G(x)^{\alpha}+\bar{G}(x)^{\alpha}\right]^{am}} \\
& = \; 1 - \sum_{m=0}^{\infty}(-1)^m\binom{b}{m}\;\frac{\sum_{k=0}^{\infty}\delta_{1,k}^{(m)}\,G(x)^k}{\sum_{k=0}^{\infty}\delta_{2,k}^{(m)}\,G(x)^k} \\
& = \; 1 - \sum_{m=0}^{\infty}(-1)^m\binom{b}{m}\;\sum_{k=0}^{\infty}\beta_k^{(m)}\,G(x)^k,
\end{aligned}
$$

where (for $k \geq 0$)

$$\beta_k^{(m)} = \frac{1}{\delta_{2,0}^{(m)}} \left(\delta_{1,k}^{(m)} - \frac{1}{\delta_{2,0}^{(m)}} \sum_{r=1}^{k} \delta_{2,k}^{(m)}\, \beta_{k-r}^{(m)} \right), \quad \delta_{1,k}^{(m)} = \sum_{i=k}^{\infty} (-1)^{i+k} \binom{a\alpha m}{i} \binom{i}{k}$$

and $\delta_{2,k}^{(m)} = h_k(\alpha, a\, m)$ is defined in the Appendix. Then, we can write

$$F(x) = \sum_{k=0}^{\infty} b_k\, G(x)^k, \tag{5.1}$$

where

$$b_0 = 1 - \sum_{m=0}^{\infty} (-1)^m \binom{b}{m} \beta_0^{(m)}, \text{ and for } k \geq 1,\ b_k = \sum_{m=0}^{\infty} (-1)^{m+1} \binom{b}{m} \beta_k^{(m)}.$$

So, the pdf of X can be expressed as an infinite mixture of exponentiated-G ("exp-G") densities

$$f(x) = f(x; a, b, \alpha, \boldsymbol{\xi}) = \sum_{k=0}^{\infty} b_{k+1}\, h_{k+1}(x), \tag{5.2}$$

where $h_{k+1}(x) = (k+1)\, G(x)^k\, g(x)$ denotes the exp-G pdf with power parameter $k+1$. Structural properties of some exp-G distributions were studied by Mudholkar *et al.* (1996), Gupta and Kundu (2001), Nadarajah and Kotz (2006), Nadarajah and Gupta (2007) and Nadarajah *et al.* (2013), among others. So, some mathematical quantities of X can be derived from (5.2) and those exp-G properties. For example, the ordinary and incomplete moments and moment generating function (mgf) of X can be easily obtained from those of the exp-G quantities.

The formulae derived in the next sections can be easily handled in most symbolic computation software platforms such as MAPLE, MATHEMATICA and MATLAB. These platforms have currently the ability to deal with complex expressions. Established closed-form statistical measures can be more efficient than calculating them by numerical integration. The infinity limit in these sums can be substituted by a large positive integer such as twenty or thirty for most applications.

6. Quantile power series

Here, we derive a power series for the qf $x = Q(u) = F^{-1}(u)$ of X by expanding (2.3). First, if $Q_G(u)$ (the baseline qf) does not have an explicit expression, it can usually be expressed as a power series given by

$$Q_G(u) = \sum_{i=0}^{\infty} a_i\, u^i, \tag{6.1}$$

where the coefficients a_i's are suitably chosen real numbers which depend on the parameters of the G distribution. For several important distributions, such as the normal, Student t, gamma and beta distributions, $Q_G(u)$ does not have explicit expressions but it can be expanded as in equation (6.1).

Here and from now on, we use a result by Gradshteyn and Ryzhik (2007, Section 0.314) for a power series raised to a positive integer n (for $n \geq 1$)

$$Q_G(u)^n = \left(\sum_{i=0}^{\infty} a_i\, u^i\right)^n = \sum_{i=0}^{\infty} c_{n,i}\, u^i, \tag{6.2}$$

where the coefficients $c_{n,i}$ (for $i = 1, 2, \ldots$) can be obtained from the recurrence equation

$$c_{n,i} = (i\, a_0)^{-1} \sum_{m=1}^{i} [m(n+1) - i]\, a_m\, c_{n,i-m}, \tag{6.3}$$

and $c_{n,0} = a_0^n$. Clearly, the quantity $c_{n,i}$ can be determined numerically in any algebraic or numerical software from the quantities $a_0, \ldots, a_i$.

Second, we derive an expansion for the argument A of $Q_G(\cdot)$ in equation (2.3)

$$A = \frac{\left[1 - (1-u)^{\frac{1}{b}}\right]^{\frac{1}{a\alpha}}}{\left[1 - (1-u)^{\frac{1}{b}}\right]^{\frac{1}{a\alpha}} + \left\{1 - \left[1 - (1-u)^{\frac{1}{b}}\right]^{\frac{1}{a}}\right\}^{\frac{1}{\alpha}}} = \frac{\sum_{k=0}^{\infty} a_k^*\, u^k}{\sum_{k=0}^{\infty} b_k^*\, u^k},$$

where $a_k^* = \sum_{i=0}^{\infty} (-1)^{i+k} \binom{\frac{1}{\alpha a}}{i} \binom{\frac{i}{b}}{k}$ and $b_k^* = a_k^* + \sum_{i,j=0}^{\infty} (-1)^{i+j+k} \binom{\frac{1}{\alpha}}{i} \binom{\frac{i}{a}}{j} \binom{\frac{j}{b}}{k}$.

The quotient of the two power series is given by

$$A = \sum_{k=0}^{\infty} c_k^*\, u^k, \tag{6.4}$$

where the coefficients c_k^*'s ($k \geq 0$) are determined from the recurrence equation

$$c_k^* = \frac{1}{b_0^*}\left(a_k^* - \frac{1}{b_0^*} \sum_{r=1}^{k} b_r^*\, c_{k-r}^*\right).$$

Then, the qf of the KwOLL-G family can be reduced to

$$Q(u) = Q_G\left(\sum_{k=0}^{\infty} c_k^*\, u^k\right). \tag{6.5}$$

By combining (6.1) and (6.5) gives

$$Q(u) = \sum_{i=0}^{\infty} a_i \left(\sum_{k=0}^{\infty} c_k^*\, u^k\right)^i,$$

and then using (6.2) and (6.3), we have

$$Q(u) = \sum_{k=0}^{\infty} e_k\, u^k, \tag{6.6}$$

where $e_k = \sum_{i=0}^{\infty} a_i\, d_{i,k}$, $d_{i,0} = c_0^{*i}$ and (for $k > 1$)

$$d_{i,k} = (k\, c_0^*)^{-1} \sum_{m=1}^{k} [m(i+1) - k]\, c_m^*\, d_{i,k-m}.$$

Hence, equation (6.6) reveals that the qf of the KwOLL-G family can be expressed as a power series. So, several mathematical quantities of X can be reduced to integrals over $(0,1)$ based on this power series. For the great majority of these quantities, we can adopt twenty terms in this power series.

Let $W(\cdot)$ be any integrable function in the real line. We can write

$$\int_{-\infty}^{\infty} W(x)\, f(x) dx = \int_0^1 W\left(\sum_{k=0}^{\infty} e_k\, u^k\right) du. \tag{6.7}$$

Equations (6.6) and (6.7) are the main results of this section since we can obtain from them various KwOLL-G mathematical properties. In fact, they can follow by using the integral on the hight-hand side for special $W(\cdot)$ functions, which are usually more simple than if they are based on the left-hand integral. For example, a formula for the nth moment of X follows from (6.7) combined with (6.2) and (6.3) as

$$\mu_n' = \int_0^1 \left(\sum_{k=0}^{\infty} e_k\, u^k\right)^n du = \sum_{k=0}^{\infty} \frac{f_{n,k}}{(k+1)},$$

where (for $n \geq 0$) $f_{n,0} = e_0^n$ and, for $k \geq 1$, $f_{n,k} = (k\, e_0)^{-1} \sum_{r=1}^{k} [r\,(n+1) - k]\, e_r\, f_{n,k-r}$.

7. Moments

Let $Y_{k+1} (k \geq 0)$ be a random variable having the pdf $h_{k+1}(x)$. A first formula for the nth moment of X is obtained from (5.2) as

$$E(X^n) = \sum_{k=0}^{\infty} b_{k+1}\, E(Y_{k+1}^n). \tag{7.1}$$

Moments of some exp-G distributions are given by Nadarajah and Kotz (2006), which can be used to obtain $E(X^n)$.

A second formula for $E(X^n)$ can be expressed from (7.1) as

$$E(X^n) = \sum_{k=0}^{\infty} (k+1)\, b_{k+1}\, \tau(n,k), \tag{7.2}$$

where $\tau(n,k) = \int_0^1 Q_G(u)^n\, u^k du$.

The nth incomplete moment of X is determined from (5.2) as

$$m_n(y) = \int_0^y x^n\, f(x) dx = \sum_{k=0}^{\infty} (k+1)\, b_{k+1} \int_0^{G(y)} Q_G(u)^n\, u^k du.$$

Using (6.2), we obtain

$$m_n(y) = \sum_{i,k=0}^{\infty} \frac{(k+1)\, b_{k+1}\, c_{n,i}}{(k+i+1)}\, G(y)^{k+i+1} \tag{7.3}$$

Equations (7.1)-(7.3) are the main results of this section.

8. Generating function

Here, we provide two formulae for the mgf $M(t) = E(\mathrm{e}^{\mathrm{t\,X}})$ of X. Clearly, the first one simply comes from (5.2) as

$$M(t) = \sum_{k=0}^{\infty} b_{k+1}\, M_{k+1}(t), \tag{8.1}$$

where $M_{k+1}(t)$ is the mgf of Y_{k+1}. Hence, $M(t)$ can be determined from the exp-G generating function. A second formula for $M(t)$ can be derived from (5.2) as

$$M(t) = \sum_{i=0}^{\infty} (k+1)\, b_{k+1}\, \rho(t,k), \tag{8.2}$$

where $\rho(t,k) = \int_0^1 \exp[t\, Q_G(u)]\, u^k du$ can be computed numerically for most G distributions.

So, we can obtain the mgfs of several generated distributions from (3.2) directly from equations (8.1) and (8.2).

9. Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Suppose that $X_1, \ldots, X_n$ is a random sample from X and let $X_{i:n}$ denote the ith order statistic. From equations (5.1) and (5.2), the pdf of $X_{i:n}$ becomes

$$f_{i:n}(x) = C \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \left(\sum_{r=0}^{\infty} (r+1)\, b_{r+1}\ G(x)^r\, g(x) \right) \left(\sum_{k=0}^{\infty} b_k\, G(x)^k \right)^{j+i-1},$$

where $C = n!/[(i-1)!\,(n-i)!]$. Using (6.2) and (6.3), we can write

$$\left(\sum_{k=0}^{\infty} b_k\, G(x)^k \right)^{j+i-1} = \sum_{k=0}^{\infty} e_{j+i-1,k}\ G(x)^k,$$

where $e_{j+i-1,0} = b_0^{j+i-1}$ and

$$e_{j+i-1,k} = (k\, b_0)^{-1} \sum_{m=1}^{k} [m(j+i) - k]\, b_m\, e_{j+i-1,k-m}.$$

Hence,

$$f_{i:n}(x) = \sum_{k=0}^{\infty} d_k\, h_{k+1}(x), \tag{9.1}$$

where $d_k = C \sum_{j=0}^{n-i} \sum_{m=0}^{k} b_{m+1}\, e_{j+i-1,k-m}$.

Equation (9.1) gives the pdf of the KwOLL-G order statistics as a linear combination of exp-G densities.

10. A bivariate extension

Here, we construct a bivariate version of the proposed model. The joint cdf of (X_1, X_2) is given by

$$\begin{aligned} F_{X_1,X_2}(x_1,x_2;a,b,\alpha,\boldsymbol{\xi}) &= \int_0^{\frac{G(x_1,x_2;\boldsymbol{\xi})^\alpha}{G(x_1,x_2;\boldsymbol{\xi})^\alpha+[1-G(x_1,x_2;\boldsymbol{\xi})]^\alpha}} a\,b\,t^{a-1}(1-t^a)^{b-1}dt \\ &= 1-\left\{1-\left[\frac{G(x_1,x_2,\boldsymbol{\xi})^\alpha}{G(x_1,x_2,\boldsymbol{\xi})^\alpha+[1-G(x_1,x_2,\boldsymbol{\xi})]^\alpha}\right]^a\right\}^b, \end{aligned}$$

where $G(x_1, x_2; \boldsymbol{\xi})$ is a bivariate continuous distribution with marginal cdfs $G_1(x_1; \boldsymbol{\xi})$ and $G_2(x_2; \boldsymbol{\xi})$. This distribution is called the *bivariate Kumaraswamy odd log-logistic* (BKwOLL) family of distributions. The marginal cdfs are given by

$$F_{X_i}(x_i;a,b,\alpha,\boldsymbol{\xi}) = 1-\left\{1-\left[\frac{G_i(x_i,\boldsymbol{\xi})^\alpha}{G_i(x_i,\boldsymbol{\xi})^\alpha+\bar{G}_i(x_i,\boldsymbol{\xi})^\alpha}\right]^a\right\}^b, \quad i=1,2.$$

The joint pdf of (X_1, X_2) can be expressed as $f_{X_1,X_2}(x_1,x_2) = \dfrac{\partial^2 F_{X_1,X_2}(x_1,x_2)}{\partial x_1 \partial x_2}$ and then

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2;a,b,\alpha,\boldsymbol{\xi}) &= \frac{ab\alpha A(x_1,x_2;a,b,\alpha,\boldsymbol{\xi})G(x_1,x_2,\boldsymbol{\xi})^{\alpha a-1}\left[1-G(x_1,x_2,\boldsymbol{\xi})\right]^{\alpha-1}}{\left\{G(x_1,x_2,\boldsymbol{\xi})^\alpha+[1-G(x_1,x_2,\boldsymbol{\xi})]^\alpha\right\}^{a+1}} \\ &\times \left\{1-\left[\frac{G(x_1,x_2,\boldsymbol{\xi})^\alpha}{G(x,y,\boldsymbol{\xi})^\alpha+[1-G(x_1,x_2,\boldsymbol{\xi})]^\alpha}\right]^a\right\}^{b-1}, \end{aligned}$$

where

$$\begin{aligned} & A(x_1,x_2;a,b,\alpha,\boldsymbol{\xi}) = g(x_1,x_2;\boldsymbol{\xi}) \\ & +\frac{\partial G(x_1,x_2,\boldsymbol{\xi})}{\partial x_1}\frac{\partial G(x_1,x_2,\boldsymbol{\xi})}{\partial x_2}\left[\frac{a\alpha-1}{G(x_1,x_2,\boldsymbol{\xi})}+\frac{1-\alpha}{1-G(x_1,x_2,\boldsymbol{\xi})}\right] \\ - & \;(a+1)\alpha\frac{\partial G(x_1,x_2,\boldsymbol{\xi})}{\partial x_1}\frac{\partial G(x_1,x_2,\boldsymbol{\xi})}{\partial x_2}\frac{G(x_1,x_2,\boldsymbol{\xi})^{\alpha-1}-[1-G(x_1,x_2,\boldsymbol{\xi})]^{\alpha-1}}{G(x_1,x_2,\boldsymbol{\xi})^\alpha+[1-G(x_1,x_2,\boldsymbol{\xi})]^\alpha} \\ + & \;\frac{a\alpha(1-b)G(x_1,x_2,\boldsymbol{\xi})^{a\alpha-1}\left[1-G(x_1,x_2,\boldsymbol{\xi})\right]^{\alpha-1}}{\left\{G(x_1,x_2,\boldsymbol{\xi})^\alpha+[1-G(x_1,x_2,\boldsymbol{\xi})]^\alpha\right\}} \\ & \times\frac{\frac{\partial G(x_1,x_2,\boldsymbol{\xi})}{\partial x_1}\frac{\partial G(x_1,x_2,\boldsymbol{\xi})}{\partial x_2}}{\left\{\left\{G(x_1,x_2,\boldsymbol{\xi})^\alpha+[1-G(x_1,x_2,\boldsymbol{\xi})]^\alpha\right\}^a-G(x_1,x_2,\boldsymbol{\xi})^{a\alpha}\right\}}. \end{aligned}$$

The marginal pdfs are given by

$$f_{X_i}(x_i) = \frac{ab\alpha g_i(x_i,\boldsymbol{\xi})G_i(x_i,\boldsymbol{\xi})^{\alpha a-1}\bar{G}_i(x_i,\boldsymbol{\xi})^{\alpha-1}}{\left[G_i(x_i,\boldsymbol{\xi})^\alpha+\bar{G}_i(x_i,\boldsymbol{\xi})^\alpha\right]^{a+1}}\left\{1-\left[\frac{G_i(x_i,\boldsymbol{\xi})^\alpha}{G_i(x_i,\boldsymbol{\xi})^\alpha+\bar{G}_i(x_i,\boldsymbol{\xi})^\alpha}\right]^a\right\}^{b-1}, \quad i=1,2.$$

The conditional cdfs are given by

$$F_{X_i|X_j}(x_i|x_j) = \frac{1-\left\{1-\left[\frac{G(x_1,x_2,\boldsymbol{\xi})^\alpha}{G(x_1,x_2,\boldsymbol{\xi})^\alpha+[1-G(x_1,x_2,\boldsymbol{\xi})]^\alpha}\right]^a\right\}^b}{1-\left\{1-\left[\frac{G_j(x_j,\boldsymbol{\xi})^\alpha}{G_j(x_j,\boldsymbol{\xi})^\alpha+\bar{G}_j(x_j,\boldsymbol{\xi})^\alpha}\right]^a\right\}^b} \text{ for } i,j=1,2 \text{ and } i\neq j.$$

The conditional pdfs are given by

$$f_{X_i|X_j}(x_i|x_j) = \frac{A(x_1,x_2;a,b,\alpha,\boldsymbol{\xi})G(x_1,x_2,\boldsymbol{\xi})^{\alpha a-1}\left[1-G(x_1,x_2,\boldsymbol{\xi})\right]^{\alpha-1}}{\left\{G(x_1,x_2,\boldsymbol{\xi})^{\alpha}+\left[1-G(x_1,x_2,\boldsymbol{\xi})\right]^{\alpha}\right\}^{a+1}}$$

$$\times \quad \left\{1-\left[\frac{G(x_1,x_2,\boldsymbol{\xi})^{\alpha}}{G(x,y,\boldsymbol{\xi})^{\alpha}+\left[1-G(x_1,x_2,\boldsymbol{\xi})\right]^{\alpha}}\right]^{a}\right\}^{b-1}$$

$$\times\left\{\frac{g_j(x_j,\boldsymbol{\xi})G_j(x_j,\boldsymbol{\xi})^{\alpha a-1}\bar{G}_j(x_j,\boldsymbol{\xi})^{\alpha-1}}{\left[G_j(x_j,\boldsymbol{\xi})^{\alpha}+\bar{G}_j(x_j,\boldsymbol{\xi})^{\alpha}\right]^{a+1}}\left\{1-\left[\frac{G_j(x_j,\boldsymbol{\xi})^{\alpha}}{G_j(x_i,\boldsymbol{\xi})^{\alpha}+\bar{G}_j(x_j,\boldsymbol{\xi})^{\alpha}}\right]^{a}\right\}^{b-1}\right\}^{-1}$$

for $i,j=1,2$ and $i\neq j$.

11. Estimation

We determine the maximum likelihood estimates (MLEs) of the parameters of the new family from complete samples only. Let $x_1,\ldots,x_n$ be the observed values from the KwOLL-G distribution with parameters a,b, α and $\boldsymbol{\xi}$. Let $\boldsymbol{\theta}=(a,b,\alpha,\boldsymbol{\xi})^{\top}$ be the $r\times 1$ parameter vector. Then, the total log-likelihood function for $\boldsymbol{\theta}$ is given by

$$\begin{aligned}\ell_n(\boldsymbol{\theta}) \quad = \quad & n\,\log[ab\alpha]+\sum_{i=1}^{n}\log\left[g(x_i;\boldsymbol{\xi})\right]+(a\alpha-1)\sum_{i=1}^{n}\log\left[G(x_i;\boldsymbol{\xi})\right]\\ & +(\alpha-1)\sum_{i=1}^{n}\log\left[\bar{G}(x_i;\boldsymbol{\xi})\right]-(a+1)\sum_{i=1}^{n}\log\left\{G(x_i;\boldsymbol{\xi})^{\alpha}+\bar{G}(x_i;\boldsymbol{\xi})^{\alpha}\right\}\\ & +(b-1)\sum_{i=1}^{n}\log\left\{1-\left[\frac{G(x_i,\boldsymbol{\xi})^{\alpha}}{G(x_i,\boldsymbol{\xi})^{\alpha}+\bar{G}(x_i,\boldsymbol{\xi})^{\alpha}}\right]^{a}\right\}.\end{aligned} \tag{11.1}$$

The components of the score function are given by

$$\begin{aligned}U_a(\boldsymbol{\theta}) \quad = \quad & \frac{n}{a}+\sum_{i=1}^{n}\log\left[\frac{G(x_i,\boldsymbol{\xi})^{\alpha}}{G(x_i,\boldsymbol{\xi})^{\alpha}+\bar{G}(x_i,\boldsymbol{\xi})^{\alpha}}\right]\\ + \quad & (1-b)\sum_{i=1}^{n}\frac{\left[\frac{G(x_i,\boldsymbol{\xi})^{\alpha}}{G(x_i,\boldsymbol{\xi})^{\alpha}+\bar{G}(x_i,\boldsymbol{\xi})^{\alpha}}\right]^{a}\log\left[\frac{G(x_i,\boldsymbol{\xi})^{\alpha}}{G(x_i,\boldsymbol{\xi})^{\alpha}+\bar{G}(x_i,\boldsymbol{\xi})^{\alpha}}\right]}{1-\left[\frac{G(x_i,\boldsymbol{\xi})^{\alpha}}{G(x_i,\boldsymbol{\xi})^{\alpha}+\bar{G}(x_i,\boldsymbol{\xi})^{\alpha}}\right]^{a}},\end{aligned}$$

$$U_b(\boldsymbol{\theta}) \quad = \quad \frac{n}{b}+\sum_{i=1}^{n}\log\left\{1-\left[\frac{G(x_i,\boldsymbol{\xi})^{\alpha}}{G(x_i,\boldsymbol{\xi})^{\alpha}+\bar{G}(x_i,\boldsymbol{\xi})^{\alpha}}\right]^{a}\right\},$$

$$\begin{aligned}U_\alpha(\boldsymbol{\theta}) \quad = \quad & \frac{n}{\alpha}+a\sum_{i=1}^{n}\log\left[G(x_i;\boldsymbol{\xi})\right]+\sum_{i=1}^{n}\log\left[\bar{G}(x_i;\boldsymbol{\xi})\right]\\ & -(a+1)\sum_{i=1}^{n}\frac{G(x_i,\boldsymbol{\xi})^{\alpha}\log\left[G(x_i,\boldsymbol{\xi})\right]+\bar{G}(x_i,\boldsymbol{\xi})^{\alpha}\log\left[\bar{G}(x_i,\boldsymbol{\xi})\right]}{G(x_i,\boldsymbol{\xi})^{\alpha}+\bar{G}(x_i,\boldsymbol{\xi})^{\alpha}}\\ & -a(b-1)\sum_{i=1}^{n}\frac{G(x_i,\boldsymbol{\xi})^{a\alpha}\bar{G}(x_i,\boldsymbol{\xi})^{\alpha}\log\left[\frac{G(x_i,\boldsymbol{\xi})}{\bar{G}(x_i,\boldsymbol{\xi})}\right]}{\left\{\left[G(x_i,\boldsymbol{\xi})^{\alpha}+\bar{G}(x_i,\boldsymbol{\xi})^{\alpha}\right]^{a}-G(x_i,\boldsymbol{\xi})^{a\alpha}\right\}\left[G(x_i,\boldsymbol{\xi})^{\alpha}+\bar{G}(x_i,\boldsymbol{\xi})^{\alpha}\right]}\end{aligned}$$

and

$$
\begin{aligned}
U_{\boldsymbol{\xi}}(\boldsymbol{\theta}) \quad = \quad & \sum_{i=1}^{n} \frac{g^{(\xi)}(x_i, \boldsymbol{\xi})}{g(x_i, \boldsymbol{\xi})} + (a\alpha - 1) \sum_{i=1}^{n} \frac{G^{(\xi)}(x_i, \boldsymbol{\xi})}{G(x_i, \boldsymbol{\xi})} + (1 - \alpha) \sum_{i=1}^{n} \frac{G^{(\xi)}(x_i, \boldsymbol{\xi})}{\bar{G}(x_i, \boldsymbol{\xi})} \\
& -\alpha(a+1) \sum_{i=1}^{n} G^{(\xi)}(x_i, \boldsymbol{\xi}) \frac{G(x_i, \boldsymbol{\xi})^{\alpha-1} - \bar{G}(x_i, \boldsymbol{\xi})^{\alpha-1}}{G(x_i, \boldsymbol{\xi})^{\alpha} + \bar{G}(x_i, \boldsymbol{\xi})^{\alpha}} \\
& -a\alpha(b-1) \sum_{i=1}^{n} \frac{G^{(\xi)}(x_i, \boldsymbol{\xi}) G(x_i, \boldsymbol{\xi})^{a\,\alpha-1} \bar{G}(x_i, \boldsymbol{\xi})^{\alpha-1}}{\left\{ \left[G(x_i, \boldsymbol{\xi})^{\alpha} + \bar{G}(x_i, \boldsymbol{\xi})^{\alpha} \right]^{a} - G(x_i, \boldsymbol{\xi})^{a\alpha} \right\} \left[G(x_i, \boldsymbol{\xi})^{\alpha} + \bar{G}(x_i, \boldsymbol{\xi})^{\alpha} \right]}.
\end{aligned}
$$

Numerical maximization of (11.1) is performed by using the RS method (Rigby and Stasinopoulos, 2005) which is available in the gamlss package (R Development Core Team, 2013), SAS (Proc NLMixed) or the Ox program (sub-routine MaxBFGS) (see, Doornik, 2007) or by solving the nonlinear likelihood equations obtained by differentiating (11.1). Setting these equations to zero, $U_a(\boldsymbol{\theta}) = U_b(\boldsymbol{\theta}) = U_\alpha(\boldsymbol{\theta}) = U_{\boldsymbol{\xi}}(\boldsymbol{\theta}) = \mathbf{0}$, and solving them simultaneously yields the MLE $\widehat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$.

For interval estimation and hypothesis tests on the parameters in $\boldsymbol{\theta}$, we require the $(p+3) \times (p+3)$ total observed information matrix $\mathbf{J}(\boldsymbol{\theta}) = -\{U_{rs}\}$, where the elements U_{rs} for $r, s = a, b, \alpha, \boldsymbol{\xi}$ are calculated numerically. The estimated multivariate normal $\mathrm{N}_{p+3}(\boldsymbol{\theta}, \mathbf{J}(\widehat{\boldsymbol{\theta}})^{-1})$ distribution can be used to construct approximate confidence regions for the parameters in $\widehat{\boldsymbol{\theta}}$. An asymptotic confidence interval (ACI) with significance level γ for each parameter θ_r is given by

$$
\mathrm{ACI}(\theta_r, 100(1-\gamma)\%) = (\hat{\theta}_r - z_{\gamma/2}\sqrt{\hat{\kappa}^{\theta_r,\theta_r}}, \hat{\theta}_r + z_{\gamma/2}\sqrt{\hat{\kappa}^{\theta_r,\theta_r}}),
$$

where $\hat{\kappa}^{\theta_r,\theta_r}$ is the rth diagonal element of $\mathbf{J}(\boldsymbol{\theta})^{-1}$ estimated at $\widehat{\boldsymbol{\theta}}$ and $z_{\gamma/2}$ is the quantile $1-\gamma/2$ of the standard normal distribution.

We can compute the maximum values of the unrestricted and restricted log-likelihoods to construct likelihood ratio (LR) statistics for testing some sub-models of the KwOLL-G distribution. For example, we may use LR statistics to check if the fit using the KwOLLW distribution is statistically "superior" to the fits using the KwW, EW, EE and Weibull distributions for a given data set. In any case, considering the partition $\boldsymbol{\theta} = (\boldsymbol{\theta}_1^T, \boldsymbol{\theta}_2^T)^T$, tests of hypotheses of the type $H_0 : \boldsymbol{\theta}_1 = \boldsymbol{\theta}_1^{(0)}$ versus $H_A : \boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_1^{(0)}$ can be performed using the LR statistic $w = 2\{\ell(\widehat{\boldsymbol{\theta}}) - \ell(\widetilde{\boldsymbol{\theta}})\}$, where $\widehat{\boldsymbol{\theta}}$ and $\widetilde{\boldsymbol{\theta}}$ are the estimates of $\boldsymbol{\theta}$ under H_A and H_0, respectively. Under the null hypothesis H_0, $w \xrightarrow{d} \chi^2_q$, where q is the dimension of the vector $\boldsymbol{\theta}_1$ of interest. The LR test rejects H_0 if $w > \xi_\gamma$, where ξ_γ denotes the upper $100\gamma\%$ point of the χ^2_q distribution.

12. Applications

We illustrate the importance of the proposed family in two applications to real data. In the last few years, several extensions of the normal and Weibull distributions have been introduced in the literature. For example, Silva *et al.* (2010) studied the beta modified Weibull (BMW) distribution, Cordeiro *et al.* (2012b) proposed the McDonald normal (McN) distribution, Cordeiro *et al.* (2014b) defined the Libby-Novick beta Weibull (LNBW) distribution, Cordeiro *et al.* (2014c) studied the McDonald Weibull (McW) distribution and Cordeiro *et al.* (2014d) introduced the Kummaraswamy modified Weibull (KwMW) distribution.

We compare the fits of the KwOLLN and KwOLLW distributions with those of other known models, namely the McN, beta normal (BN), Kumaraswamy normal (KwN), McW, BMW, KwMW, LNBW, beta Weibull (BW), Kumaraswamy Weibull (KwW) and their baseline distributions themselves, see Alexander *et al.* (2012) and Cordeiro *et al.* (2010) for more details.

12.1. Aarset data. We consider the lifetimes of 50 industrial devices put on life test at time zero presented by Aarset (1987). These data also reported in Mudholkar and Srivastava (1993), Mudholkar *et al.* (1996) and Silva *et al.* (2010) exhibit a bathtub-shaped failure rate property. These authors consider that the data are generated by a Weibull distribution. So, we adopt this distribution as the baseline model for our family.

Table 2 lists the MLEs and their standard errors (in parentheses) of the parameters from the fitted KwOLLW, McW, KwMW, BMW, LNBW, BW, KwW, EW and Weibull models and the values of the statistics: Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC) and Bayesian Information Criterion (BIC). The computations are performed using the statistical software R. The results indicate that the KwOLLW model has the smallest values of these statistics among the fitted models, and therefore it could be chosen as the best model.

A comparison of the KwOLLW distribution with some of its sub-models using LR statistics is given in Table 3. Clearly, we reject the null hypotheses for the three LR tests in favor of the KwOLLW distribution. In order to assess if the new model is appropriate, Figures 5a and 5b display the histogram of the data and the fitted KwOLLW density function and the densities of some of its sub-models and non-nested models, respectively. Further, Figures 5c and 5d display plots of the empirical and estimated survival functions of the KwOLLW distribution and of some sub-models and non-nested models, respectively. We can conclude that the KwOLLW distribution is a very suitable model to fit to the current data.

We shall apply formal goodness-of-fit tests in order to verify which distribution fits the data better. We consider the Cramér-Von Mises (W^*) and Anderson-Darling (A^*) statistics defined by Chen and Balakrishnan (1995).

The values of these statistics for the fitted models are listed in Table 4. Overall, by comparing the measures of these formal goodness-of-fit tests in Table 4, we conclude that the KwOLLW distribution yields a better fit than the Weibull, EW, KwW, BW and McW distributions and therefore it can be an interesting alternative to these distributions for modeling lifetime data. These results illustrate the importance of the additional shape parameters of the new distribution to analyze real data.

12.2. Respiratory data. Now, we use a real data set to compare the fits of the KwOLLN distribution with those of the McN, BN, KwN and normal distributions. The McN pdf (Cordeiro *et al.*, 2012b) is given by

$$f(x;\mu,\sigma,a,b,c)=\frac{c}{B(a,b)\sigma}\,\phi\left(\sigma^{-1}(x-\mu)\right)\left\{\Phi\left(\sigma^{-1}(x-\mu)\right)\right\}^{ac-1}\left\{1-\Phi\left(\sigma^{-1}(x-\mu)\right)^{c}\right\}^{b-1},$$

Table 2. MLEs and information criteria.

Aarset	λ	β	a	b	α	AIC	CAIC	BIC
KwOLLW	5.4771	0.0203	3.0532	4.9020	0.0514	441.0	442.3	450.5
	(0.0100)	(0.0010)	(1.2819)	(4.5062)	(0.0188)			
KwW	5.5025	0.0165	0.0602	0.2510	1	449.5	450.4	457.2
	(0.0043)	(0.0013)	(0.0205)	(0.0796)	(-)			
EW	4.6978	0.0108	0.1381	1	1	464.3	464.8	470.0
	(0.00002)	(0.0008)	(0.0206)	(-)	(-)			
Weibull	0.9488	0.0222	1	1	1	486.0	486.2	489.8
	(0.1195)	(0.0034)	(-)	(-)	(-)			
	λ	β	a	b	c			
McW	5.4712	0.0202	0.0880	0.0876	0.8457	447.5	448.8	457.0
	(0.0086)	(0.0028)	(0.0195)	(0.0640)	(0.6682)			
BW	5.3386	0.0212	0.0864	0.0731	1	445.7	446.5	453.3
	(0.0146)	(0.0019)	(0.0181)	(0.0306)	(-)			
	λ	β	a_1	b_1	c_1			
LNBW	5.4514	0.0217	0.0838	0.0620	2.1512	447.3	448.7	456.9
	(0.0109)	(0.0041)	(0.0187)	(0.0618)	(14.3571)			
	α_2	λ_2	γ_2	a_2	b_2			
BMW	0.0028	0.0403	1.1337	0.2455	0.1400	453.9	455.2	463.4
	(0.0009)	(0.0125)	(0.2873)	(0.0623)	(0.0671)			
	α_3	λ_3	γ_3	a_3	b_3			
KwMW	0.0038	0.03724	0.9403	0.2654	0.3195	457.7	459.0	467.2
	(0.0020)	(0.0106)	(0.2650)	(0.1058)	(0.1649)			

Table 3. LR tests.

Aarset	Hypotheses	Statistic w	p-value
KwOLLW vs KwW	$H_0 : \alpha = 1$ vs $H_1 : H_0$ is false	10.55	0.0011
KwOLLW vs EW	$H_0 : b = \alpha = 1$ vs $H_1 : H_0$ is false	27.33	<0.0001
KwOLLW vs Weibull	$H_0 : a = b = \alpha = 1$ vs $H_1 : H_0$ is false	51.00	<0.0001

where $x \in \mathbb{R}$, $\mu \in \mathbb{R}$ is the location parameter, $\sigma > 0$ is the scale parameter and a, b and c are positive shape parameters.

We consider 630 observations on respiratory rate (Alexander *et al.*, 2012) and a parent normal distribution. These data were taken from a study by the University of São Paulo, ESALQ (Laboratory of Physiology and Post-Harvest Biochemistry), which evaluate the effects of mechanical damage on banana fruits (genus Musa spp.); see Saavedra del Aguila *et al.* (2010) for more details. The major problem affecting bananas during and after harvest is the susceptibility of the mature fruit to physical damage caused during transport and marketing. The extent of the damage is measured by the respiratory rate.

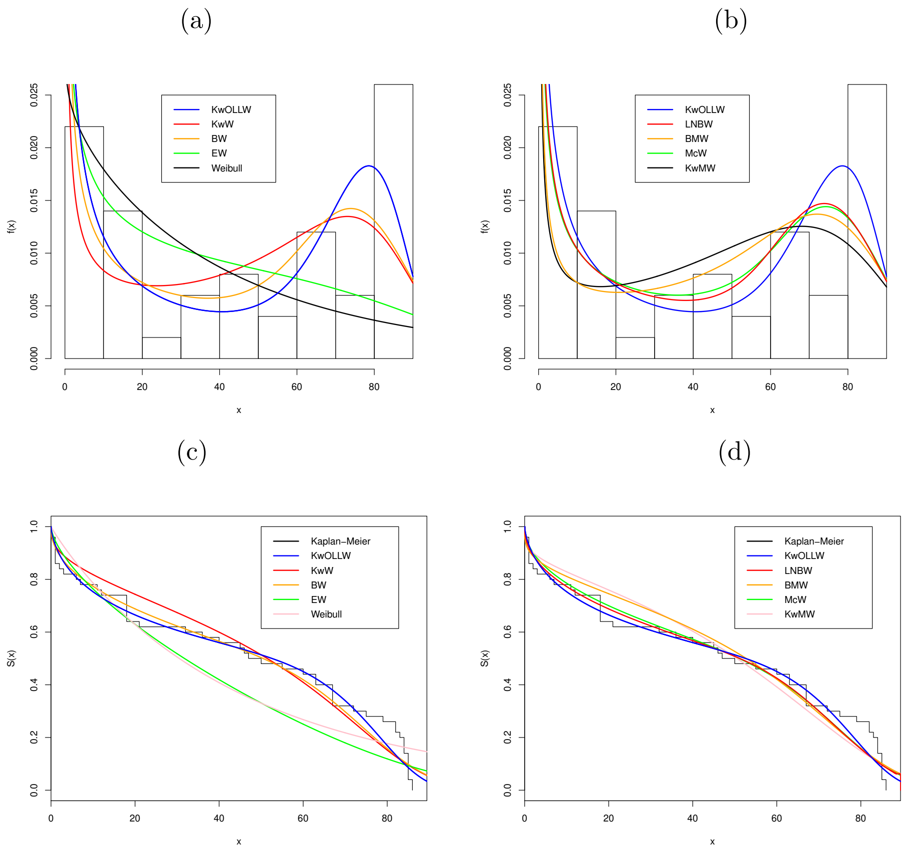

Figure 5. (a) Estimated densities of the KwOLLW, BW, KwW, EW and Weibull models. (b) Estimated densities of the KwOLLW, McW, KwMW, BMW and LNBW models. (c) Empirical and estimated survival functions of the KwOLLWBW, KwW, EW and Weibull. (d) Empirical and estimated survival functions of the KwOLLW, McW, KwMW, BMW and LNBW models.

Initial values for a, b, μ and σ are taken from the fitted KwN model with $\alpha = 1$; see, for example, Cordeiro *et al.* (2012b). The computations are performed using the subroutine NLMIXED in SAS. Table 5 lists the MLEs and their standard errors (in parentheses) of the parameters of the fitted models and the AIC, CAIC and BIC values. The computations are performed using the subroutine NLMixed in SAS. These results indicate that the KwOLLN model has the lowest AIC, CAIC and BIC values among those values of the fitted models, and therefore it could be chosen as the best model.

More information is provided by a visual comparison of the histogram of the data with the fitted densities. In Figure 6, we plot the histogram of the respiratory data and the fitted KwOLLN, McN, BN, KwN and normal densities. The KwOLLN and McN distributions provide reasonable fits, but it is clear that the

Table 4. Formal goodness-of-fit tests for Aarset data.

Model	Statistic W^*	A^*
KwOLLW	0.0833	0.7477
KwW	0.1454	1.1324
EW	0.2740	1.8111
Weibull	0.4963	3.0079
BW	0.1041	0.9043
McW	0.1047	0.9056
LNBW	0.1028	0.8972
BMW	0.1677	1.2697
KwMW	0.1912	1.3995

Table 5. MLEs and information criteria.

Respiratory	μ	σ	a	b	α	AIC	CAIC	BIC
KwOLLN	6.5396	113.18	2.2642	0.2778	11.2953	5547.0	5547.1	5569.3
	(2.7772)	(14.6040)	(0.3356)	(0.0147)	(2.3128)			
KwN	-32.7704	29.4031	13.4721	0.4520	1	5775.1	5775.2	5792.9
	(2.5507)	(0.8140)	(1.4283)	(0.0329)	(-)			
Normal	34.3166	27.7500	1	1	1	5979.3	5979.4	5988.2
	(1.1056)	(0.7818)	(-)	(-)	(-)			
	a	b	c	μ	σ	AIC	CAIC	BIC
McN	10021.0	0.4681	4.6369	-186.04	47.9945	5638.3	5638.4	5660.5
	(8.8561)	(0.0305)	(0.6311)	(7.9203)	(1.7718)			
BN	50.9335	0.4135	1	-56.1790	32.2426	5709.9	5710.0	5727.7
	(2.5794)	(0.0296)	-	(2.1684)	(0.9699)			

KwOLLN model provides a more adequate fit to the histogram and better captures its extreme bathtub shape.

13. Conclusions

We introduce and study a new class of distributions called the Kumaraswamy odd log-logistic-G (KwOLL-G) family, which includes as special cases some classical generators of distributions such as the Kumaraswamy-generalized, exp odd-log logistic, odd-Burr and exponentiated families. For each baseline G distribution, we define the corresponding KwOLL-G distribution with three additional shape parameters using simple formulas to extend widely-known models such as the normal, Weibull and Gumbel distributions in order to provide more flexibility. Some characteristics of the new family, such as the ordinary moments, generating function and mean deviations, have tractable mathematical properties. The role of the generator parameters is related to the skewness and kurtosis of the new family. We estimate the parameters using maximum likelihood and determine the observed information matrix. Inference on the model parameters is conducted based on likelihood ratio statistics for testing nested models and other formal statistics for

(a) (b)

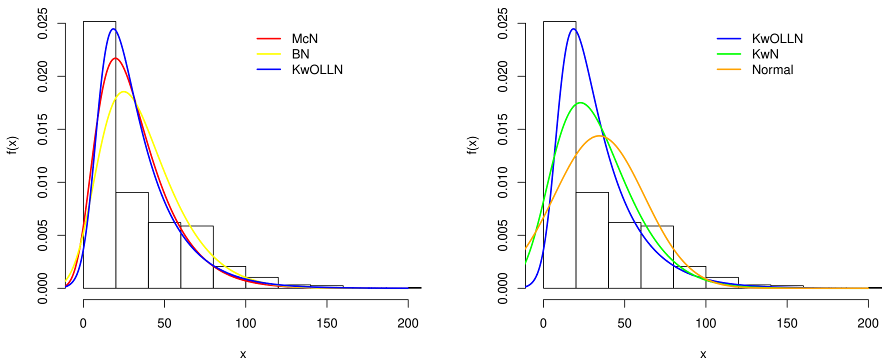

Figure 6. Fitted densities of the KwOLLN, McN and BW models for the respiratory data. (b) Fitted densities of the KwOLLN, KwN and normal models for the respiratory data.

non-nested models. Two applications to real data demonstrate the importance of the new family.

Acknowledgements

We thank two anonymous referees and the editor for their valuable suggestions, which certainly contributed to the improvement of the paper.

Appendix A

We present four power series expansions required for the proof of the general result in Section 4. First, for $b > 0$ real non-integer and $0 < u < 1$, we have the binomial expansion

$$(13.1) \quad (1-u)^a = \sum_{j=0}^{\infty} (-1)^j \binom{a}{j} u^j,$$

where the binomial coefficient is defined for any real.

Second, the following expansion holds for any $\alpha > 0$ real non-integer

$$(13.2) \quad G(x)^\alpha = \sum_{r=0}^{\infty} s_r(\alpha)\, G(x)^r,$$

where $s_r(\alpha) = \sum_{j=r}^{\infty} (-1)^{r+j} \binom{\alpha}{j} \binom{j}{r}$.

Third, by expanding z^λ in Taylor series, we obtain

$$(13.3) \quad z^\lambda = \sum_{k=0}^{\infty} (\lambda)_k \, (z-1)^k / k! = \sum_{i=0}^{\infty} f_i \, z^i,$$

where

$$f_i = f_i(\lambda) = \sum_{k=i}^{\infty} \frac{(-1)^{k-i}\, (\lambda)_k}{k!} \binom{k}{i}$$

and $(\lambda)_k = \lambda(\lambda-1)\dots(\lambda-k+1)$ is the descending factorial.

Fourth, we consider equations (6.2) and (6.3) to obtain an expansion for $[G(x)^a + \bar{G}(x)^a]^c$. We can write from equations (13.1) and (13.2)

$$[G(x)^a + \bar{G}(x)^a] = \sum_{j=0}^{\infty} t_j\, G(x)^j,$$

where $t_j = t_j(a) = s_j(a) + (-1)^j \binom{a}{j}$. Then, using (13.3), we can write

$$[G(x)^a + \bar{G}(x)^a]^c = \sum_{i=0}^{\infty} f_i \left(\sum_{j=0}^{\infty} t_j\, G(x)^j\right)^i,$$

where $f_i = f_i(c)$. Finally, based on equations (6.2) and (6.3), we have

$$[G(x)^a + \bar{G}(x)^a]^c = \sum_{j=0}^{\infty} h_j\, G(x)^j, \qquad (13.4)$$

where $h_j = h_j(a,c) = \sum_{i=0}^{\infty} f_i\, m_{i,j}$ and $m_{i,j} = (j\, t_0)^{-1} \sum_{m=1}^{j} [m(j+1)-j]\, t_m\, m_{i,j-m}$ (for $j \geq 1$) and $m_{i,0} = t_0^i$.

References

[1] Aarset, M.V. (1987). How to identify bathtub hazard rate. *IEEE Transactions Reliability*, **36**, 106–108.

[2] Alexander, C., Cordeiro, G.M., Ortega, E.M.M. and Sarabia, J.M. (2012). Generalized beta-generated distributions. *Computational Statistics and Data Analysis*, **56**, 1880-1897.

[3] Alzaatreh A, Lee C, Famoye, F. (2013). A new method for generating families of distributions.*Metron*, **71**, 63-79

[4] Alizadeh, M., Cordeiro, G.M., Nascimento, A.D.C. and Lima, M.D. (2014). The odd Burr family of distributions: properties and applications. Submitted.

[5] Bourguignon, M., Silva, R.B., Cordeiro, G.M. (2014). The WeibullG family of probability distributions. *Journal of Data Science*, **12**, 53-68.

[6] Chen, G., Balakrishnan, N. (1995). A general purpose approximate goodness-of-fit test. *Journal of Quality Technology*, **27**, 154-161.

[7] Choudhury, A. (2005). A simple derivation of moments of the exponentiated Weibull distribution. *Metrika*, **62**, 17-22.

[8] Cordeiro, G.M., Ortega, E.M.M. and Nadarajah, S. (2010). The Kumaraswamy Weibull distribution with application to failure data. *Journal of the Franklin Institute*, **349**, 11174-1197.

[9] Corderio, G.M. and de Castro, M. (2011). A new family of generalized distributions. *Journal of Statistical Computation and Simulation*, **81**, 883-893.

[10] Cordeiro, G.M., Nadarajah, S. and Ortega, E.M.M. (2012a). The Kumaraswamy Gumbel distribution. *Statistical Methods and Applications*, **21**, 139-168.

[11] Cordeiro, G.M., Cintra, R.J., Rego, L.C. and Ortega, E.M.M. (2012b). The McDonald normal distribution. *Pakistan Journal of Statistics and Operations Research*, **8**, 301-329.

[12] Cordeiro, G.M., Cristino, C.T., Hashimoto, E.M. and Ortega, E.M.M. (2013). The beta generalized Rayleigh distribution with applications to lifetime data. *Statistical Papers*, **54**, 133-161.

[13] Cordeiro, G.M., Alizadeh and Pinho, L.G. (2014a) The exponentiated odd log-logistic family of distributions: Properties and applications. Submitted

[14] Cordeiro, G.M., de Santana, L.H., Ortega, E.M.M. and Pescim, R.R. (2014b). A New Family of Distributions: Libby-Novick Beta. *International Journal of Statistics and Probability*, **3**, 63-80.

[15] Cordeiro, G.M., Hashimoto, E.H. and Ortega, E.M.M. (2014c). The McDonald Weibull model. *Statistics*, **48**, 256-278.

[16] Cordeiro, G.M., Ortega, E.M.M. and Silva, G.O. (2014d). The Kumaraswamy modified Weibull distribution: theory and applications. *Journal of Statistical Computation and Simulation*, **84**, 1387-1411.

[17] Cordeiro, G.M., Alizadeh, M. and Ortega, E.M.M. The Exponentiated Half-Logistic Family of Distributions: Properties and Applications. *Journal of Probability and Statistics*, vol. 2014, Article ID 864396, 21 pages, 2014. doi:10.1155/2014/864396

[18] Da Cruz, J. N., Cordeiro, G.M., Ortega, E.M.M. and Alizadeh, M. (2014). A new family of distributions: Theory and applications in survival analysis. Submitted

[19] Doornik, J. (2007). *Ox: An object-oriented matrix programming language*. International Thompson Business Press.

[20] Eugene, N., Lee, C. and Famoye, F. (2002).Beta-normal distribution and its applications. *Communications in Statistics - Theory and Methods*, **31**,175-195.

[21] Gradshteyn I.S. and Ryzhik I.M. (2007). *Table of integrals, series, and products*. Academic Press, San Diego.

[22] Gupta, R.D. and Kundu, D. (2001). Exponentiated Exponential Family: An Alternative to Gamma and Weibull Distributions. *Biometrical Journal*, **43**, 117–130.

[23] Gupta, R.C., Gupta, R.D. (2007). Proportional revered hazard rate model and its applications. *Journal of Statistical Planning and Inference*, **137**, 3525-3536.

[24] Gupta, R.C., Gupta, P.I., Gupta, R.D. (1998). Modeling failure time data by Lehmann alternatives. *Communications in StatisticsTheory and Methods*, **27**, 887-904.

[25] Lee, C., Famoye, F. and Olumolade, O. (2007). Beta-Weibull distribution: Some properties and applications to censored data. *Journal of Modern Applied Statistical Methods*, **6**, 173-186.

[26] Marshall, A.N. and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families. *Biometrika* , **84**, 641-552.

[27] Mudholkar, G.S. and Srivastava, D.K. (1993). Exponentiated Weibull family for analyzing bathtub failure-rate data. *IEEE Transactions on Reliability*, **42**, 299–302.

[28] Mudholkar, G.S, Srivastava, D.K. and Friemer, M. (1995). The exponential Weibull family: A reanalysis of the bus-motor failure data. *Technometrics*, **37**, 436–445.

[29] Mudholkar, G.S., Srivastava, D.K. and Kollia, G. (1996). A generalization of the Weibull distribution with application to the analysis of survival data. *Journal of the American Statistical Association*, **91**, 1575-1583.

[30] Nadarajah, S. and Kotz, S. (2006). The Exponetiated Type Distributions. *Acta Applicandae Mathematicae*, **92**, 97–111.

[31] Nadarajah, S. and Gupta, A.K. (2007). The exponentiated gamma distribution with application to drought data. *Calcutta Statistical Association Bulletin*, **59**, 29–54.

[32] Nadarajah, S. and Kotz, S. (2006). The beta exponential distribution. *Reliability Engineering and System Safety*, **91**, 689-697.

[33] Nadarajah, S., Cordeiro, G.M. and Ortega, E.M.M. (2013). The exponentiated Weibull distribution: a survey. *Statistical Papers*, **54**, 839-877.

[34] Nassar, M.M. and Eissa, F.H. (2003). On the exponentiated Weibull distribution. *Communications in Statistics Theory and Methods*, **32**, 1317-1336.

[35] R Development Core Team (2013). *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria.

[36] Rigby, R.A. and Stasinopoulos, D.M. (2005). Generalized additive models for location, scale and shape (with discussion). *Applied Statistics*, **54**, 507-554.

[37] Saavedra del Aguila, J., Heiffig-del Aguila, L. S., Sasaki, F. F., Tsumanuma, G. M., Ongarelli, M. G., Spoto, M. H. F., Jacomino, A. P., Ortega, E. M. M. and Kluge, R. A. (2010). Postharvest modifications of mechanically injured bananas. *Revista Iberoamericana de Tecnologia Postcosecha*, **10**, 73-85.

[38] Silva, G.O., Ortega, E.M.M. and Cordeiro, G.M. (2010). The Beta Modified Weibull Distribution. *Lifetime Data Analysis*, **16**, 409-430.