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A new family of maximum scattered linear sets in $\mathrm{PG}(1, q^6)$*

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Abstract

We generalize the example of linear set presented by the last two authors in "Vertex properties of maximum scattered linear sets of $\mathrm{PG}(1, q^n)$" (2019) to a more general family, proving that such linear sets are maximum scattered when q is odd and, apart from a special case, they are new. This solves an open problem posed in "Vertex properties of maximum scattered linear sets of $\mathrm{PG}(1, q^n)$" (2019). As a consequence of Sheekey's results in "A new family of linear maximum rank distance codes" (2016), this family yields to new MRD-codes with parameters $(6, 6, q; 5)$.

Keywords: Scattered linear set, MRD-code, linearized polynomial.

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1 Introduction

Let $\Lambda = \mathrm{PG}(V, \mathbb{F}_{q^n}) = \mathrm{PG}(1, q^n)$, where V is a vector space of dimension 2 over $\mathbb{F}_{q^n}$. If U is a k-dimensional $\mathbb{F}_q$-subspace of V, then the $\mathbb{F}_q$-*linear set* L_U is defined as

$$L_U = \{\langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}} : \mathbf{u} \in U \setminus \{\mathbf{0}\}\},$$

and we say that L_U has *rank* k. Two linear sets L_U and L_W of $\mathrm{PG}(1, q^n)$ are said to be $\mathrm{P\Gamma L}$-*equivalent* if there is an element ϕ in $\mathrm{P\Gamma L}(2, q^n)$ such that $L_U^{\phi} = L_W$. It may happen that two $\mathbb{F}_q$-linear sets L_U and L_W of $\mathrm{PG}(1, q^n)$ are $\mathrm{P\Gamma L}$-equivalent even if the $\mathbb{F}_q$-vector subspaces U and W are not in the same orbit of $\Gamma\mathrm{L}(2, q^n)$ (see [5, 12] for further details). In this paper we focus on *maximum scattered* $\mathbb{F}_q$-linear sets of $\mathrm{PG}(1, q^n)$, that is, $\mathbb{F}_q$-linear sets of rank n in $\mathrm{PG}(1, q^n)$ of size $(q^n - 1)/(q - 1)$.

If $\langle (0, 1) \rangle_{\mathbb{F}_{q^n}}$ is not contained in the linear set L_U of rank n of $\mathrm{PG}(1, q^n)$ (which we can always assume after a suitable projectivity), then $U = U_f := \{(x, f(x)) : x \in \mathbb{F}_{q^n}\}$ for some linearized polynomial (or q-*polynomial*) $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i} \in \mathbb{F}_{q^n}[x]$. In this case we will denote the associated linear set by L_f. If L_f is scattered, then $f(x)$ is called a *scattered* q-polynomial; see [24].

The first examples of scattered linear sets were found by Blokhuis and Lavrauw in [3] and by Lunardon and Polverino in [18] (recently generalized by Sheekey in [24]). Apart from these, very few examples are known, see Section 3.

In [24, Section 5], Sheekey established a connection between maximum scattered linear sets of $\mathrm{PG}(1, q^n)$ and MRD-codes, which are interesting because of their applications to random linear network coding and cryptography. We point out his construction in the last section. By the results of [1] and [2], it seems that examples of maximum scattered linear sets are rare.

In this paper we will prove that any

$$f_h(x) = h^{q-1} x^q - h^{q^2-1} x^{q^2} + x^{q^4} + x^{q^5}, \quad h \in \mathbb{F}_{q^6}, \quad h^{q^3+1} = -1, \quad q \text{ odd} \qquad (1.1)$$

is a scattered q-polynomial. This will be done by considering two cases:

Case 1: $h \in \mathbb{F}_q$, that is, $f_h(x) = x^q - x^{q^2} + x^{q^4} + x^{q^5}$; the condition $h^{q^3+1} = -1$ implies $q \equiv 1 \pmod 4$.

Case 2: $h \notin \mathbb{F}_q$. In this case $h \neq \pm\sqrt{-1}$, otherwise $h \in \mathbb{F}_{q^2}$ and then we have $h^{q+1} = 1$, a contradiction to $h^{q^3+1} = -1$.

Note that in Case 1, this example coincides with the one introduced in [27], where it has been proved that f_h is scattered for $q \equiv 1 \pmod 4$ and $q \leq 29$. In Corollary 3.11 we will prove that the linear set $\mathcal{L}_h$ associated with $f_h(x)$ is new, apart from the case of q a power of 5 and $h \in \mathbb{F}_q$. This solves an open problem posed in [27].

Finally, in Section 4 we prove that the $\mathbb{F}_q$-linear MRD-codes with parameters $(6, 6, q; 5)$ arising from linear sets $\mathcal{L}_h$ are not equivalent to any previously known MRD-code, apart from the case $h \in \mathbb{F}_q$ and q a power of 5; see Theorem 4.1.

2 $\mathcal{L}_h$ is scattered

A q-*polynomial* (or *linearized polynomial*) over $\mathbb{F}_{q^n}$ is a polynomial of the form

$$f(x) = \sum_{i=0}^{t} a_i x^{q^i},$$

where $a_i \in \mathbb{F}_{q^n}$ and t is a positive integer. We will work with linearized polynomials of degree less than or equal to q^{n-1}. For such a kind of polynomial, the *Dickson matrix*[1] $M(f)$ is defined as

$$M(f) := \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1}^q & a_0^q & \cdots & a_{n-2}^q \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{q^{n-1}} & a_2^{q^{n-1}} & \cdots & a_0^{q^{n-1}} \end{pmatrix} \in \mathbb{F}_{q^n}^{n\times n},$$

where $a_i = 0$ for $i > t$.

Recently, different results regarding the number of roots of linearized polynomials have been presented, see [4, 9, 22, 23, 26]. In order to prove that a certain polynomial is scattered, we make use of the following result; see [4, Corollary 3.5].

Theorem 2.1. *Consider the q-polynomial* $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$ *over* $\mathbb{F}_{q^n}$ *and, with* m *as a variable, consider the matrix*

$$M(m) := \begin{pmatrix} m & a_1 & \cdots & a_{n-1} \\ a_{n-1}^q & m^q & \cdots & a_{n-2}^q \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{q^{n-1}} & a_2^{q^{n-1}} & \cdots & m^{q^{n-1}} \end{pmatrix}.$$

The determinant of the $(n-i)\times(n-i)$ *matrix obtained by* $M(m)$ *after removing the first* i *columns and the last* i *rows of* $M(m)$ *is a polynomial* $M_{n-i}(m) \in \mathbb{F}_{q^n}[m]$*. Then the polynomial* $f(x)$ *is scattered if and only if* $M_0(m)$ *and* $M_1(m)$ *have no common roots.*

2.1 Case 1

Let

$$f(x) = x^q - x^{q^2} + x^{q^4} + x^{q^5} \in \mathbb{F}_{q^6}[x].$$

By Theorem 2.1, $f(x)$ is scattered if and only if for each $m \in \mathbb{F}_{q^6}$ the determinants of the following two matrices do not vanish at the same time

$$M_5(m) = \begin{pmatrix} 1 & -1 & 0 & 1 & 1 \\ m^q & 1 & -1 & 0 & 1 \\ 1 & m^{q^2} & 1 & -1 & 0 \\ 1 & 1 & m^{q^3} & 1 & -1 \\ 0 & 1 & 1 & m^{q^4} & 1 \end{pmatrix},$$

$$M_6(m) = \begin{pmatrix} m & 1 & -1 & 0 & 1 & 1 \\ 1 & m^q & 1 & -1 & 0 & 1 \\ 1 & 1 & m^{q^2} & 1 & -1 & 0 \\ 0 & 1 & 1 & m^{q^3} & 1 & -1 \\ -1 & 0 & 1 & 1 & m^{q^4} & 1 \\ 1 & -1 & 0 & 1 & 1 & m^{q^5} \end{pmatrix}.$$

[1] This is sometimes called *autocirculant matrix*.

Theorem 2.2. *The polynomial $f(x)$ is scattered if and only if $q \equiv 1 \pmod 4$.*

Proof. If q is even, then for $m = 0$ the matrix $M_6(0)$ has rank two and $f(x)$ is not scattered.

Suppose now $q \equiv 3 \pmod 4$. Then let $\overline{m} \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ such that $\overline{m}^2 = -4$. So $\overline{m} = \overline{m}^{q^2} = \overline{m}^{q^4} = -\overline{m}^q = -\overline{m}^{q^3} = -\overline{m}^{q^5}$ and, by direct checking,

$$\det(M_5(\overline{m})) = (\overline{m}^2 + 4)^2 = 0, \quad \det(M_6(\overline{m})) = -(\overline{m}^2 + 4)^3 = 0$$

and $f(x)$ is not scattered.

Assume $q \equiv 1 \pmod 4$ and suppose that $f(x)$ is not scattered. Then there exists $m_0 \in \mathbb{F}_{q^6}$ such that

$$(\det(M_5(m_0)))^{q^s} = 0, \quad (\det(M_6(m_0)))^{q^t} = 0, \quad s, t = 0, 1, 2, 3, 4, 5. \tag{2.1}$$

Consider

$$P_1 = \det\begin{pmatrix} 1 & -1 & 0 & 1 & 1 \\ Y & 1 & -1 & 0 & 1 \\ 1 & Z & 1 & -1 & 0 \\ 1 & 1 & U & 1 & -1 \\ 0 & 1 & 1 & V & 1 \end{pmatrix}, \quad P_2 = \det\begin{pmatrix} X & 1 & -1 & 0 & 1 & 1 \\ 1 & Y & 1 & -1 & 0 & 1 \\ 1 & 1 & Z & 1 & -1 & 0 \\ 0 & 1 & 1 & U & 1 & -1 \\ -1 & 0 & 1 & 1 & V & 1 \\ 1 & -1 & 0 & 1 & 1 & W \end{pmatrix}. \tag{2.2}$$

Therefore,

$$X = m_0,\ Y = m_0^q,\ \dots,\ W = m_0^{q^5} \tag{2.3}$$

is a root of $P_1 =: P_1^{(0)}$, $P_2 =: P_2^{(0)}$ and of the polynomials inductively defined by

$$P_i^{(j)}(X, Y, Z, U, V, W) = P_i^{(j-1)}(Y, Z, U, V, W, X), \quad j = 1, 2, 3, 4, 5, \quad i = 1, 2,$$

which arise from Equation 2.1. These polynomials satisfy

$$\left(P_i^{(j-1)}(m_0, m_0^q, m_0^{q^2}, m_0^{q^3}, m_0^{q^4}, m^{q^5})\right)^q = P_i^{(j)}(m_0, m_0^q, m_0^{q^2}, m_0^{q^3}, m_0^{q^4}, m^{q^5}).$$

One obtains a set S of twelve equations in X, Y, Z, U, V, W having a nonempty zero set. The following arguments are based on the fact that taking the resultant R of two polynomials in S with respect to any variable, the equations $S \cup \{R\}$ admit the same solutions.

We have

$$P_1 = YZUV - YZU - 2YZ + 2YU + 4Y - ZUV + 2ZV - 2UV + 4V + 16 = 0. \tag{2.4}$$

Consider the following resultants:

$$\begin{aligned} Q_1 := Res_V(P_1^{(3)}, P_1) = 2(&XY^2ZU - XY^2ZW + XY^2UW + 2XY^2W \\ &- 2XYZU + 2XYZW - 2XYUW + 8XYW + 8XY - 8XW + 16X \\ &- Y^2ZUW - 2Y^2ZU + 2YZUW - 8YZU - 8YZ + 8YU - 8YW \\ &+ 8ZU - 16Z + 16U - 16W), \end{aligned}$$

$$\begin{aligned} Q_2 := Res_V(P_1^{(4)}, P_1) = &XYZW - XYZ - XYW + 2XZ \\ &- 2XW - 2YZ + 2YW + 4Z + 4W + 16, \end{aligned}$$

$$\begin{aligned} Q_3 := Res_V(P_1^{(5)}, P_1) = &XYZU - XYZ - 2XY + 2XZ \\ &+ 4X - YZU + 2YU - 2ZU + 4U + 16. \end{aligned}$$

They all must be zero, as well as

$$Res_W(Res_U(Q_1,Q_3),Q_2) = 8(YZ-4)(Y^2+4)(X-Z)(XZ+4)(XY-4). \quad (2.5)$$

We distinguish a number of cases.

1. Suppose that $Y^2=-4$. Since $q \equiv 1 \pmod 4$, $X=Y=Z=U=V=W$. So

$$P_1 = X^4 - 2X^3 + 8X + 16$$

and the resultant between X^2+4 and P_1 with respect to X is $2^{27} \neq 0$ and then (2.3) is not a root of P_1, a contradiction.

2. Condition $YZ=4$ is clearly equivalent to $XY=4$. This means that $Y=U=W=4/X$, $Z=V=X$. Therefore, by (2.4) we get $X^2+4=0$ and we proceed as above.

3. Case $XZ=-4$. In this case $Z=-4/X$, $U=-4/Y$, $V=-4/Z=X$, $W=Y$, $X=Z$ and therefore $X^2=-4$ and we can proceed as above.

4. Condition $X=Z$ implies $X \in \mathbb{F}_{q^2}$ and so $X=Z=V$ and $Y=U=W$. By substituting in P_1 and P_2,

$$X^3Y^3 + 3X^3Y - 6X^2Y^2 - 12X^2 + 3XY^3 + 24XY - 12Y^2 - 64 = 0,$$
$$X^2Y^2 - X^2Y + 2X^2 - XY^2 - 4XY + 4X + 2Y^2 + 4Y + 16 = 0.$$

Eliminating Y from these two equations one gets

$$8(X^2+4)^6 = 0,$$

and so $X^2+4=0$. We proceed as in the previous cases.

This proves that such $m_0 \in \mathbb{F}_{q^6}$ does not exist and the assertion follows. □

2.2 Case 2

We apply the same methods as in Section 2.1. In the following preparatory lemmas (and in the rest of the paper) q is a power of an arbitrary prime p.

Lemma 2.3. *Let $h \in \mathbb{F}_{q^6}$ be such that $h^{q^3+1}=-1$, $h^4 \neq 1$. Then*

1. *$h^q \neq -h$;*
2. *$h^{q^2+1} \neq 1$;*
3. *$h^{q^2+1} \neq \pm h^q$, if q is odd;*
4. *$h^{4q^2+4} + 14h^{2q^2+2q+2} + h^{4q} = 0$ implies $p=2$ and $h^{q^2-q+1}=1$ or $q=3^{2s}$, $s \in \mathbb{N}^*$, $h^{q^2-q+1} = \pm\sqrt{-1}$.*

Proof. The first three are easy computations. Consider now

$$h^{4q^2+4} + 14h^{2q^2+2q+2} + h^{4q} = 0.$$

For $p=2$ the equation above implies $h^{q^2-q+1}=1$.

Assume now $p \neq 2$. Since $h \neq 0$, it is equivalent to

$$(h^{q^2-q+1})^4 + 14(h^{q^2-q+1})^2 + 1 = 0,$$

that is $(h^{q^2-q+1})^2 = -7 \pm 4\sqrt{3} = (\sqrt{-3} \pm 2\sqrt{-1})^2$. Let $z = -7 \pm 4\sqrt{3}$. Note that $h^{q^2-q+1} = \pm\sqrt{z}$ belongs to $\mathbb{F}_{q^2}$. We distinguish two cases.

- $\sqrt{z} \in \mathbb{F}_q$. Then

$$-1 = h^{q^3+1} = (h^{q^2-q+1})^{q+1} = \left(\pm\sqrt{z}\right)^{q+1} = z = -7 \pm 4\sqrt{3},$$

  a contradiction if $p \neq 3$. Also, $z = -1$, q is an even power of 3, and $h^{q^2-q+1} = \pm\sqrt{-1}$.

- $\sqrt{z} \notin \mathbb{F}_q$. Then

$$-1 = h^{q^3+1} = (h^{q^2-q+1})^{q+1} = \left(\pm\sqrt{z}\right)^{q+1} = -z = 7 \mp 4\sqrt{3},$$

  a contradiction if $p \neq 2$. □

Lemma 2.4. *Let $h \in \mathbb{F}_{q^6}$ be such that $h^{q^3+1} = -1$, $h^4 \neq 1$. If a root σ of the polynomial*

$$h^{q+1}T^{q+1} + (h^{q^2+q+2} + h^{2q^2+2})T^q + (h^{2q^2+2} - h^{q^2+1})T + h^{q^2+2q+1} + h^{2q^2+q+1} - h^{2q} - h^{q^2+q} \in \mathbb{F}_{q^6}[T]$$

belongs to $\mathbb{F}_{q^6}$, then one of the following cases occurs:

- *$p = 2$, $h^{q^2-q+1} = 1$; or*
- *$q = 3^{2s}$, $s > 0$, $h^{q^2-q+1} = \pm\sqrt{-1}$; or*
- *$\sigma = \pm(h^{q^2} + h^q)$; or*
- *$h \in \mathbb{F}_q$.*

Proof. First, note that $\sigma = 0$ would imply $h^q(h^q+h)^q(h^{q^2+1}-1) = 0$ which is impossible by Lemma 2.3. Therefore $\sigma \neq 0$ and $\sigma^{q^i} = \frac{\ell_i(X)}{m_i(X)}$, where

$$\begin{aligned}
\ell_1(X) &= -(h^{q^2+1} - 1)(h^{q^2+1}X + h^{2q} + h^{q^2+q})\\
m_1(X) &= h(h^qX + h^{q^2+q+1} + h^{2q^2+1})\\
\ell_2(X) &= -(h^q + h)(2h^{q^2+q+1}X + h^{2q^2+q+2} + h^{3q^2+2} + h^{3q} + h^{q^2+2q})\\
m_2(X) &= h^{q+1}(h^{2q^2+2}X + h^{2q}X + 2h^{q^2+2q+1} + 2h^{2q^2+q+1})\\
\ell_3(X) &= (h^q + h)^q(3h^{2q^2+q+2}X + h^{3q}X + h^{3q^2+q+3} + h^{4q^2+3} + 3h^{q^2+3q+1}\\
&\quad + 3h^{2q^2+2q+1})\\
m_3(X) &= h^{q^2+q}(h^{3q^2+3}X + 3h^{q^2+2q+1}X + 3h^{2q^2+2q+2} + 3h^{3q^2+q+2} + h^{4q} + h^{q^2+3q})
\end{aligned}$$

$$\begin{aligned}
\ell_4(X) &= (h^{q^2+1}-1)(h^{4q^2+4}X + 6h^{2q^2+2q+2}X + h^{4q}X + 4h^{3q^2+2q+3} + 4h^{4q^2+q+3} \\
&\quad + 4h^{q^2+4q+1} + 4h^{2q^2+3q+1}) \\
m_4(X) &= h^{q^2}(4h^{3q^2+q+3}X + 4h^{q^2+3q+1}X + h^{4q^2+q+4} + h^{5q^2+4} + 6h^{2q^2+3q+2} \\
&\quad + 6h^{3q^2+2q+2} + h^{5q} + h^{q^2+4q}) \\
\ell_5(X) &= -(h^q+h)(h^{5q^2+5}X + 10h^{3q^2+2q+3}X + 5h^{q^2+4q+1}X + 5h^{4q^2+2q+4} \\
&\quad + 5h^{5q^2+q+4} + 10h^{2q^2+4q+2} + 10h^{3q^2+3q+2} + h^{6q} + h^{q^2+5q}) \\
m_5(X) &= 5h^{4q^2+q+4}X + 10h^{2q^2+3q+2}X + h^{5q}X + h^{5q^2+q+5} + h^{6q^2+5} \\
&\quad + 10h^{3q^2+3q+3} + 10h^{4q^2+2q+3} + 5h^{q^2+5q+1} + 5h^{2q^2+4q+1} \\
\ell_6(X) &= (h^q+h)^q(6h^{5q^2+q+5}X + 20h^{q^3+3q+3}X + 6Xh^{q^2+5q+1} + h^{6q^2+q+6} \\
&\quad + h^{7q^2+6} + 15h^{4q^2+3q+4} + 15h^{5q^2+2q+4} + 15h^{2q^2+5q+2} \\
&\quad + 15h^{3q^2+4q+2} + h^{7q} + h^{q^2+6q}) \\
m_6(X) &= h^{6q^2+6}X + 15h^{4q^2+2q+4}X + 15h^{2q^2+4q+2}X + h^{q^6}X + 6h^{5q^2+2q+5} \\
&\quad + 6h^{6q^2+q+5} + 20h^{3q^2+4q+3} + 20h^{4q^2+3q+3} + 6h^{q^2+6q+1} + 6h^{2q^2+5q+1}.
\end{aligned}$$

Since $\sigma^{q^6} = \sigma$, in particular

$$(h^{2q^2+2}+h^{2q})(h^{4q^2+4}+14h^{2q^2+2q+2}+h^{4q})(h^{q^2}-h^q)(\sigma+h^q+h^{q^2})(\sigma-h^q-h^{q^2}) = 0.$$

The claim follows from Lemma 2.3. □

Lemma 2.5. *Let $h \in \mathbb{F}_{q^6}$ be such that $h^{q^3+1} = -1$, $h^4 = 1$. If a root σ of the polynomial*

$$h^{q+1}T^{q^2+1} + (h^q+h)^{q+1} \in \mathbb{F}_{q^6}[T]$$

belongs to $\mathbb{F}_{q^6}$, then

$$\sigma = \pm(h^{q^2}+h^q).$$

Proof. If $\sigma = 0$, then $h^q+h = 0$, a contradiction to Lemma 2.3. So we can suppose $\sigma \neq 0$. Then

$$\begin{aligned}
\sigma^{q^2} &= -\frac{(h^{q-1}+1)^{q+1}}{\sigma} \\
\sigma^{q^4} &= (h^{q-1}+1)^{q^3+q^2-q-1}\sigma \\
\sigma^{q^6} &= -\frac{(h^{q-1}+1)^{q^5+q^4-q^3-q^2+q+1}}{\sigma} = \frac{(h^q+h)^{2q}}{\sigma}.
\end{aligned}$$

So, $\sigma = \pm(h^{q^2}+h^q)$. □

Let $h \in \mathbb{F}_{q^6}$ be such that $h^{q^3+1} = -1$, $h^4 \neq 1$. By Theorem 2.1 the polynomial

$$f_h(x) = h^{q-1}x^q - (h^{q^2-1})x^{q^2} + x^{q^4} + x^{q^5}$$

is scattered if and only if for each $m \in \mathbb{F}_{q^6}$ the determinant of the following two matrices do not vanish at the same time

$$M_6(m) = \begin{pmatrix} m & h^{q-1} & -h^{q^2-1} & 0 & 1 & 1 \\ 1 & m^q & h^{q^2-q} & h^{-q-1} & 0 & 1 \\ 1 & 1 & m^{q^2} & -h^{-q^2-1} & h^{-q^2-q} & 0 \\ 0 & 1 & 1 & m^{q^3} & h^{1-q} & -h^{1-q^2} \\ h^{q+1} & 0 & 1 & 1 & m^{q^4} & h^{q-q^2} \\ -h^{q^2+1} & h^{q^2+q} & 0 & 1 & 1 & m^{q^5} \end{pmatrix}, \tag{2.6}$$

$$M_5(m) = \begin{pmatrix} h^{q-1} & -h^{q^2-1} & 0 & 1 & 1 \\ m^q & h^{q^2-q} & h^{-q-1} & 0 & 1 \\ 1 & m^{q^2} & -h^{-q^2-1} & h^{-q^2-q} & 0 \\ 1 & 1 & m^{q^3} & h^{1-q} & -h^{1-q^2} \\ 0 & 1 & 1 & m^{q^4} & h^{q-q^2} \end{pmatrix}. \tag{2.7}$$

Theorem 2.6. *Let $h \in \mathbb{F}_{q^6}$, $q = 2^s$, be such that $h^{q^3+1} = 1$. Then the polynomial $f_h(x) = h^{q-1}x^q - (h^{q^2-1})x^{q^2} + x^{q^4} + x^{q^5}$ is not scattered.*

Proof. Consider $\overline{m} = h^{q^2} + h^q$. So,

$$\overline{m}^q = \frac{1}{h} + h^{q^2}, \qquad \overline{m}^{q^2} = \frac{1}{h^q} + \frac{1}{h}, \qquad \overline{m}^{q^3} = \frac{1}{h^{q^2}} + \frac{1}{h^q},$$
$$\overline{m}^{q^4} = h + \frac{1}{h^{q^2}}, \qquad \overline{m}^{q^5} = h^q + h.$$

By direct checking, in this case, both $\det(M_6(\overline{m})) = \det(M_5(\overline{m})) = 0$ and therefore $f_h(x)$ is not scattered. □

Theorem 2.7. *Let $h \in \mathbb{F}_{q^6}$, $q = p^s$, $p > 2$, be such that $h^{q^3+1} = -1$ and $h \notin \mathbb{F}_q$. Then the polynomial $f_h(x) = h^{q-1}x^q - (h^{q^2-1})x^{q^2} + x^{q^4} + x^{q^5}$ is scattered.*

Proof. First we note that $h^4 \neq 1$ since q is odd, $h \notin \mathbb{F}_q$, and $h^{q^3+1} = -1$. Suppose that $f(x)$ is not scattered. Then $\det(M_6(m_0)) = \det(M_5(m_0)) = 0$ for some $m_0 \in \mathbb{F}_{q^6}$. Consider

$$X = m_0, \quad Y = m_0^q, \quad Z = m_0^{q^2}, \quad U = m_0^{q^3}, \quad V = m_0^{q^4}, \quad W = m_0^{q^5}.$$

With a procedure similar to the one in the proof of Theorem 2.2, we will compute resultants starting from the polynomials associated with $\det(M_6(m_0))$, $\det(M_5(m_0))^{q^3}$, and $\det(M_5(m_0))^{q^5}$.

Eliminating W using $\det(M_5(m_0))^{q^3} = 0$ and U using $\det(M_5(m_0))^{q^5} = 0$, one gets from $\det(M_6(m_0)) = 0$

$$h^{q^2+2q+1}\varphi_1(X,Y)\varphi_2(X,Y,Z,V)\varphi_3(X,Y,Z,V) = 0,$$

where

$$\begin{aligned}
\varphi_1(X,Y) &= h^{q+1}XY + h^{2q^2+2}X - h^{q^2+1}X + h^{q^2+q+2}Y + h^{2q^2+2}Y \\
&\quad + h^{q^2+2q+1} + h^{2q^2+q+1} - h^{2q} - h^{q^2+q};\\
\varphi_2(X,Y,Z,V) &= h^{q^2+q+2}XYZV - h^{q^2+q+2}XYZ - h^2XY - h^{q+1}XY \\
&\quad - h^{2q^2+q+1}XZV - h^{2q^2+2}XV - h^{2q^2+q+1}XV - h^{q^2+2q+3}YZ \\
&\quad - h^{2q^2+q+3}YZ - h^{q^2+q+2}Y - h^{2q^2+2}Y - h^{q^2+2q+1}Y \\
&\quad - h^{2q^2+q+1}Y - h^{q^2+2q+1}ZV - h^{2q^2+q+1}ZV - h^{2q^2+q+1}V \\
&\quad - h^{3q^2+1}V - h^{2q^2+2q}V - h^{3q^2+q}V + h^{2q^2+q+3} + h^{3q^2+3} \\
&\quad + h^{2q^2+2q+2} + h^{3q^2+q+2} - 2h^{q^2+q+2} - 2h^{2q^2+2} - 2h^{q^2+2q+1} \\
&\quad - 2h^{2q^2+q+1} + h^{q+1} + h^{q^2+1} + h^{2q} + h^{q^2+q};\\
\varphi_3(X,Y,Z,V) &= h^{q^2+q+2}XYZV + h^{q^2+q+2}XYZ - h^2XY - h^{q+1}XY \\
&\quad + h^{2q^2+q+1}XZV - h^{2q^2+2}XV - h^{2q^2+q+1}XV - h^{q^2+2q+3}YZ \\
&\quad - h^{2q^2+q+3}YZ + h^{q^2+q+2}Y + h^{2q^2+2}Y + h^{q^2+2q+1}Y \\
&\quad + h^{2q^2+q+1}Y - h^{q^2+2q+1}ZV - h^{2q^2+q+1}ZV + h^{2q^2+q+1}V \\
&\quad + h^{3q^2+1}V + h^{2q^2+2q}V + h^{3q^2+q}V + h^{2q^2+q+3} + h^{3q^2+3} \\
&\quad + h^{2q^2+2q+2} + h^{3q^2+q+2} - 2h^{q^2+q+2} - 2h^{2q^2+2} - 2h^{q^2+2q+1} \\
&\quad - 2h^{2q^2+q+1} + h^{q+1} + h^{q^2+1} + h^{2q} + h^{q^2+q}.
\end{aligned}$$

- If $\varphi_1(X,Y) = 0$, then by Lemma 2.4 either $q = 3^{2s}$ and $h^{q^2-q+1} = \pm\sqrt{-1}$, or $X = \pm(h^{q^2} + h^q)$.

  In this last case,

  $$\begin{aligned}
  &Y = \pm(-h^{-1} + h^{q^2}), \quad Z = \pm(-h^{-q} - h^{-1}), \quad U = \pm\left(-h^{-q^2} - h^{-q}\right)\\
  &V = \pm(h - h^{-q^2}), \quad W = \pm(h^q + h).
  \end{aligned} \tag{2.8}$$

  By substituting in $\det(M_5(m_0))$ one obtains

  $$4(h + h^q)^{q+1}(h^{q^2+1} - 1)(h^{q^2+1} - h^q) = 0$$

  and

  $$4(h + h^q)^{q+1}(h^{q^2+1} - 1)(h^{q^2+1} + h^q) = 0,$$

  respectively. Both are not possible due to Lemma 2.3.

  Consider now the case $q = 3^{2s}$, $h^{q^2-q+1} = \pm\sqrt{-1}$ and $X \neq \pm(h^{q^2} + h^q)$. So, using $\varphi_1(X,Y) = 0$ and $h^{q^2-q+1} = \pm\sqrt{-1}$,

  $$\begin{aligned}
  &\det(M_5(m_0)) = 0 \Longrightarrow\\
  &\quad h^{q^2+2q+1}(h^{q^2} + h^q)(h^q + h)(h^{q^2+1} - 1)(h^{q^2+q} + h^q)^3(h^{q^2+q} - h^q)^3 \cdot\\
  &\qquad \cdot (h^{2q^2+2} - h^{q^2+1} + h^{2q})(X + h^q + h^{q^2})^2(X - h^q - h^{q^2})^2 = 0.
  \end{aligned}$$

  By Lemma 2.3 we get

  $$h^{2q^2+2} - h^{q^2+1} + h^{2q} = 0,$$

  which yields to a contradiction.

- If $\varphi_2(X,Y,Z,V)=0$ and $\varphi_1(X,Y)\neq 0$, eliminating V in $\det(M_5(m_0))=0$ one gets

$$\begin{aligned}&2h^{3q^2+2q+1}(h^{q+2}YZ-h^{q^2+2}-h^{q^2+q+1}+h^q+h)\cdot\\&\cdot(hXY+h^{q^2+q+1}+h^{2q^2+1}-h^{q^2}-h^q)\cdot\\&\cdot(h^{q+1}XZ+h^{q+1}+h^{q^2+1}+h^{2q}+h^{q^2+q})\cdot\\&\cdot(h^{q+2}YZ+hY+h^qY-h^{q^2+q+1}Z+h^qZ-h^{q^2+2}-h^{q^2+q+1}+h^q+h)=0.\end{aligned}$$

  - If $h^{q+2}YZ-h^{q^2+2}-h^{q^2+q+1}+h^q+h=0$ then, from

$$Z=\frac{h^{q^2+2}+h^{q^2+q+1}-h^q-h}{h^{q+2}Y},$$

    $\det(M_5)=0$ gives

$$(h^q+h)^{q+1}(hY-h^{q^2+1}+1)(hY+h^{q^2+1}-1)=0.$$

    So, (2.8) holds and as in the case $\varphi_1(X,Y)=0$ a contradiction arises.
  - If $hXY+h^{q^2+q+1}+h^{2q^2+1}-h^{q^2}-h^q=0$ then, from

$$Y=\frac{-h^{q^2+q+1}-h^{2q^2+1}+h^{q^2}+h^q}{hX},$$

    the equation $\det(M_5(m_0))=0$ yields

$$(h^q+h)(h^{q^2+1}-1)(X-h^{q^2}-h^q)(X+h^{q^2}+h^q)=0.$$

    So, (2.8) holds and as in the case $\varphi_1(X,Y)=0$, a contradiction.
  - If $h^{q+1}XZ+h^{q+1}+h^{q^2+1}+h^{2q}+h^{q^2+q}=0$ then by Lemma 2.5

$$(X-h^{q^2}-h^q)(X+h^{q^2}+h^q)=0,$$

    again a contradiction as before.
  - If $h^{q+2}YZ+hY+h^qY-h^{q^2+q+1}Z+h^qZ-h^{q^2+2}-h^{q^2+q+1}+h^q+h=0$ then

$$Z=-\frac{(h^q+h)Y-h^{q^2+2}-h^{q^2+q+1}+h^q+h}{h^{q+2}Y-h^{q^2+q+1}+h^q}.$$

    So, substituting $U=Z^q, V=Z^{q^2}, W=Z^{q^3}, X=Z^{q^4}$ in $\det(M_5(m_0))=0$ we get

$$\begin{aligned}&(h-1)^{q+1}(h+1)^{q+1}(h^q+h)^{q+1}(h^{q^2+1}-1)\cdot\\&\quad\cdot(hY-h^{q^2+1}+1)^2(hY+h^{q^2+1}-1)^2=0.\end{aligned}$$

    By Lemma 2.3, $(hY-h^{q^2+1}+1)(hY+h^{q^2+1}-1)=0$. Since $Y=\pm(h^{q^2}-1/h)$ then (2.8) holds and a contradiction arises as in the case $\varphi_1(X,Y)=0$.

- If $\varphi_3(X,Y,Z,V)=0$ and $\varphi_1(X,Y)\neq 0$, eliminating U from $\det(M_5(m_0))=0=\det(M_5(m_0))^{q^5}$ and then eliminating V using $\varphi_3(X,Y,Z,V)=0$ one gets

$$\begin{aligned}2h^{3q^2+q+1}(h^q+h)^q(h^{q+2}YZ-h^{q^2+2}-h^{q^2+q+1}+h^q+h)^2\cdot\\ \cdot(hXY+h^{q^2+q+1}+h^{2q^2+1}-h^{q^2}-h^q)\cdot\\ \cdot(h^{q+1}XZ+h^{q+1}+h^{q^2+1}+h^{2q}+h^{q^2+q})=0.\end{aligned}$$

A contradiction follows as in the case $\varphi_2(X,Y,Z,V)=0$ and $\varphi_1(X,Y)\neq 0$. □

3 The equivalence issue

We will deal with the linear sets $\mathcal{L}_h = L_{f_h}$ associated with the polynomials defined in (1.1). Note that when $h\in\mathbb{F}_q$, such a linear set coincide with the one introduced in [27, Section 5].

3.1 Preliminary results

We start by listing the non-equivalent (under the action of $\Gamma\mathrm{L}(2,q^6)$) maximum scattered subspaces of $\mathbb{F}_{q^6}^2$, i.e. subspaces defining maximum scattered linear sets.

Example 3.1.

1. $U^1 := \{(x,x^q) : x\in\mathbb{F}_{q^6}\}$, defining the linear set of pseudoregulus type, see [3, 11];
2. $U^2_\delta := \{(x,\delta x^q + x^{q^5}) : x\in\mathbb{F}_{q^6}\}$, $\mathrm{N}_{q^6/q}(\delta)\notin\{0,1\}$, defining the linear set of LP-type, see [16, 18, 20, 24];
3. $U^3_\delta := \{(x,x^q+\delta x^{q^4}) : x\in\mathbb{F}_{q^6}\}$, $\mathrm{N}_{q^6/q^3}(\delta)\notin\{0,1\}$, satisfying further conditions on δ and q, see [6, Theorems 7.1 and 7.2] and [23] [2];
4. $U^4_\delta := \{(x,x^q+x^{q^3}+\delta x^{q^5}) : x\in\mathbb{F}_{q^6}\}$, q odd and $\delta^2+\delta=1$, see [10, 21].

In order to simplify the notation, we will denote by L^1 and L^i_δ the $\mathbb{F}_q$-linear set defined by U^1 and U^i_δ, respectively. We will also use the following notation:

$$\mathcal{U}_h := U_{h^{q-1}x^q-h^{q^2-1}x^{q^2}+x^{q^4}+x^{q^5}}.$$

Remark 3.2. Consider the non-degenerate symmetric bilinear form of $\mathbb{F}_{q^6}$ over $\mathbb{F}_q$ defined by

$$\langle x,y\rangle = \mathrm{Tr}_{q^6/q}(xy),$$

for each $x,y\in\mathbb{F}_{q^6}$. Then the *adjoint* $\hat{f}$ of the linearized polynomial $f(x)=\sum_{i=0}^5 a_i x^{q^i}\in\tilde{\mathcal{L}}_{6,q}$ with respect to the bilinear form $\langle\,,\rangle$ is

$$\hat{f}(x)=\sum_{i=0}^{5} a_i^{q^{6-i}} x^{q^{6-i}},$$

i.e.

$$\mathrm{Tr}_{q^6/q}(xf(y)) = \mathrm{Tr}_{q^6/q}(y\hat{f}(x)),$$

for any $x,y\in\mathbb{F}_{q^6}$.

[2] Here $q>2$, otherwise it is not scattered.

In [10, Propositions 3.1, 4.1 and 5.5] the following result has been proved.

Lemma 3.3. *Let L_f be one of the maximum scattered of $\mathrm{PG}(1,q^6)$ listed before. Then a linear set L_U of $\mathrm{PG}(1,q^6)$ is $\mathrm{P\Gamma L}$-equivalent to L_f if and only if U is $\Gamma\mathrm{L}$-equivalent either to U_f or to $U_{\hat{f}}$ Furthermore, L_U is $\mathrm{P\Gamma L}$-equivalent to L^3_δ if and only if U is $\Gamma\mathrm{L}$-equivalent to U^3_δ.*

We will work in the following framework. Let $x_0,\dots,x_5$ be the homogeneous coordinates of $\mathrm{PG}(5,q^6)$ and let

$$\Sigma = \{\langle(x,x^q,\dots,x^{q^5})\rangle_{\mathbb{F}_{q^6}} : x \in \mathbb{F}_{q^6}\}$$

be a fixed canonical subgeometry of $\mathrm{PG}(5,q^6)$. The collineation $\hat{\sigma}$ of $\mathrm{PG}(5,q^6)$ defined by $\langle(x_0,\dots,x_5)\rangle^{\hat{\sigma}}_{\mathbb{F}_{q^6}} = \langle(x_5^q,x_0^q,\dots,x_4^q)\rangle_{\mathbb{F}_{q^6}}$ fixes precisely the points of Σ. Note that if σ is a collineation of $\mathrm{PG}(5,q^6)$ such that $\mathrm{Fix}(\sigma)=\Sigma$, then $\sigma=\hat{\sigma}^s$, with $s\in\{1,5\}$.

Let Γ be a subspace of $\mathrm{PG}(5,q^6)$ of dimension $k\geq 0$ such that $\Gamma\cap\Sigma=\emptyset$, and $\dim(\Gamma\cap\Gamma^\sigma)\geq k-2$. Let r be the least positive integer satisfying the condition

$$\dim(\Gamma\cap\Gamma^\sigma\cap\Gamma^{\sigma^2}\cap\dots\cap\Gamma^{\sigma^r}) > k-2r. \tag{3.1}$$

Then we will call the integer r the *intersection number of* Γ w.r.t. σ and we will denote it by $\mathrm{intn}_\sigma(\Gamma)$; see [27].

Note that if $\hat{\sigma}$ is as above, then $\mathrm{intn}_{\hat{\sigma}}(\Gamma)=\mathrm{intn}_{\hat{\sigma}^5}(\Gamma)$ for any Γ.

As a consequence of the results of [11, 27] we have the following result.

Result 3.4. *Let L be a scattered linear set of $\Lambda=\mathrm{PG}(1,q^6)$ which can be realized in $\mathrm{PG}(5,q^6)$ as the projection of $\Sigma=\mathrm{Fix}(\sigma)$ from $\Gamma\simeq\mathrm{PG}(3,q^6)$ over Λ. If $\mathrm{intn}_\sigma(\Gamma)\neq 1,2$, then L is not equivalent to any linear set neither of pseudoregulus type nor of LP-type.*

3.2 $\mathcal{L}_h$ is new in most of the cases

The linear set $\mathcal{L}_h$ can be obtained by projecting the canonical subgeometry

$$\Sigma = \{\langle(x,x^q,x^{q^2},x^{q^3},x^{q^4},x^{q^5})\rangle_{\mathbb{F}_{q^6}} : x\in\mathbb{F}^*_{q^6}\}$$

from

$$\Gamma\colon \begin{cases} x_0 = 0 \\ h^{q-1}x_1 - h^{q^2-1}x_2 + x_4 + x_5 = 0 \end{cases}$$

to

$$\Lambda\colon \begin{cases} x_1=0\\ x_2=0\\ x_3=0\\ x_4=0. \end{cases}$$

Then

$$\Gamma^{\hat{\sigma}}\colon \begin{cases} x_1 = 0 \\ h^{q^2-q}x_2 + h^{-q-1}x_3 + x_5 + x_0 = 0 \end{cases}$$

and

$$\Gamma^{\hat{\sigma}^2}: \begin{cases} x_2 = 0 \\ -h^{-1-q^2}x_3 + h^{-q^2-q}x_4 + x_0 + x_1 = 0. \end{cases}$$

Therefore,

$$\Gamma \cap \Gamma^{\hat{\sigma}}: \begin{cases} x_0 = 0 \\ x_1 = 0 \\ -h^{q^2-1}x_2 + x_4 + x_5 = 0 \\ h^{q^2-q}x_2 + h^{-q-1}x_3 + x_5 = 0 \end{cases}$$

and

$$\Gamma \cap \Gamma^{\hat{\sigma}} \cap \Gamma^{\hat{\sigma}^2}: \begin{cases} x_0 = 0 \\ x_1 = 0 \\ x_2 = 0 \\ x_4 + x_5 = 0 \\ h^{-q-1}x_3 + x_5 = 0 \\ -h^{-q^2-1}x_3 + h^{-q^2-q}x_4 = 0. \end{cases}$$

Hence, $\dim_{\mathbb{F}_{q^6}}(\Gamma \cap \Gamma^{\hat{\sigma}}) = 1$ and $\dim_{\mathbb{F}_{q^6}}(\Gamma \cap \Gamma^{\hat{\sigma}} \cap \Gamma^{\hat{\sigma}^2}) = -1$, since q is odd and $h^{q^3+1} \neq 1$. So, $\mathrm{intn}_\sigma(\Gamma) = 3$ and hence, by Result 3.4 it follows that $\mathcal{L}_h$ is not equivalent neither to L^1 nor to L^2_δ.

Generalizing [27, Propositions 5.4 and 5.5] we have the following two propositions.

Proposition 3.5. *The linear set* $\mathcal{L}_h$ *is not* $\mathrm{P\Gamma L}$*-equivalent to* L^3_δ.

Proof. By Lemma 3.3, we have to check whether $\mathcal{U}_h$ and U^3_δ are $\Gamma\mathrm{L}$-equivalent, with $\mathrm{N}_{q^6/q^3}(\delta) \notin \{0,1\}$. Suppose that there exist $\rho \in \mathrm{Aut}(\mathbb{F}_{q^6})$ and an invertible matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that for each $x \in \mathbb{F}_{q^6}$ there exists $z \in \mathbb{F}_{q^6}$ satisfying

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x^\rho \\ h^{\rho(q-1)}x^{\rho q} - h^{\rho(q^2-1)}x^{\rho q^2} + x^{\rho q^4} + x^{\rho q^5} \end{pmatrix} = \begin{pmatrix} z \\ z^q + \delta z^{q^4} \end{pmatrix}.$$

Equivalently, for each $x \in \mathbb{F}_{q^6}$ we have[3]

$$\begin{aligned} cx^\rho + d(h^{q-1}x^{\rho q} - h^{q^2-1}x^{\rho q^2} + x^{\rho q^4} + x^{\rho q^5}) = \\ a^q x^{\rho q} + b^q(h^{q^2-q}x^{\rho q^2} + h^{-q-1}x^{\rho q^3} + x^{\rho q^5} + x^\rho) \\ + \delta[a^{q^4}x^{\rho q^4} + b^{q^4}(h^{-q^2+q}x^{\rho q^5} - h^{q+1}x^\rho + x^{\rho q^2} + x^{\rho q^3})]. \end{aligned}$$

This is a polynomial identity in x^ρ and hence we have the following relations:

$$\begin{cases} c = b^q + \delta h^{q+1}b^{q^4} \\ dh^{q-1} = a^q \\ -dh^{q^2-1} = h^{q^2-q}b^q + \delta b^{q^4} \\ 0 = h^{-1-q}b^q + \delta b^{q^4} \\ d = \delta a^{q^4} \\ d = b^q + \delta h^{q-q^2}b^{q^4}. \end{cases} \tag{3.2}$$

[3] We may replace h^ρ by h, since $h^{q^3+1} = -1$ if and only if $(h^\rho)^{q^3+1} = -1$.

From the second and the fifth equations, if $a \neq 0$ then $\delta h^{q-1} = a^{q-q^4}$ and $\mathrm{N}_{q^6/q^3}(\delta) = 1$, which is not possible and so $a = d = 0$ and $b, c \neq 0$. By the last equation, we would get $\mathrm{N}_{q^6/q^3}(\delta) = 1$, a contradiction. □

Proposition 3.6. *The linear set $\mathcal{L}_h$ is $\mathrm{P\Gamma L}$-equivalent to L^4_δ (with $\delta^2 + \delta = 1$) if and only if there exist $a, b, c, d \in \mathbb{F}_{q^6}$ and $\rho \in \mathrm{Aut}(\mathbb{F}_{q^6})$ such that $ad - bc \neq 0$ and either*

$$\begin{cases} c = b^q - \delta k^{q^2+1} b^{q^5} \\ a = -k^{q+1} b^{q^4} - \delta^q b^{q^2} \\ d = k^{-q+1} b^{q^3} + \delta b^{q^5} \\ b^{q^3} + (k^{q-1} + \delta k^{q+q^2}) b^{q^5} = 0 \\ k^{q^2-q} b^q + (1 + k^{q^2-q}) b^{q^3} + \delta k^{q^2-1} b^{q^5} = 0 \\ -\delta b^q + (k^{-q+1} + \delta^2 k^{1-q^2}) b^{q^3} + \delta b^{q^5} = 0 \end{cases} \tag{3.3}$$

or

$$\begin{cases} c = \delta b^q - k^{q^2+1} b^{q^5} \\ a = -\delta^q k^{q+1} b^{q^4} - b^{q^2} \\ d = k^{-q+1} b^{q^3} + b^{q^5} \\ \delta b^{q^3} + (k^{q-1} - \delta k^{q^2+q}) b^{q^5} = 0 \\ \delta k^{q^2-q} b^q + (k^{q^2-q} + 1) b^{q^3} + k^{q^2-1} b^{q^5} = 0 \\ \delta^2 b^q + (k^{-q+1} + \delta^2 k^{-q^2+1}) b^{q^3} + b^{q^5} = 0, \end{cases} \tag{3.4}$$

where $k = h^\rho$.

Proof. By Lemma 3.3 we have to check whether $\mathcal{U}_h$ is equivalent either to U^4_δ or to $(U^4_\delta)^\perp$. Suppose that there exist $\rho \in \mathrm{Aut}(\mathbb{F}_{q^6})$ and an invertible matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that for each $x \in \mathbb{F}_{q^6}$ there exists $z \in \mathbb{F}_{q^6}$ satisfying

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x^\rho \\ h^{\rho(q-1)} x^{\rho q} - h^{\rho(q^2-1)} x^{\rho q^2} + x^{\rho q^4} + x^{\rho q^5} \end{pmatrix} = \begin{pmatrix} z \\ z^q + z^{q^3} + \delta z^{q^5} \end{pmatrix}.$$

Equivalently, for each $x \in \mathbb{F}_{q^6}$ we have

$$\begin{aligned} cx^\rho + d(k^{q-1} x^{\rho q} - k^{q^2-1} x^{\rho q^2} + x^{\rho q^4} + x^{\rho q^5}) = \\ a^q x^{\rho q} + b^q (k^{q^2-q} x^{\rho q^2} + k^{-1-q} x^{\rho q^3} + x^{\rho q^5} + x^\rho) \\ + a^{q^3} x^{\rho q^3} + b^{q^3} (k^{-q+1} x^{\rho q^4} - k^{-q^2+1} x^{\rho q^5} + x^{\rho q} + x^{\rho q^2}) \\ + \delta [a^{q^5} x^{\rho q^5} + b^{q^5} (-k^{1+q^2} x^\rho + k^{q^2+q} x^{\rho q} + x^{\rho q^3} + x^{\rho q^4})]. \end{aligned}$$

This is a polynomial identity in x^ρ which yields to the following equations

$$\begin{cases} c = b^q - \delta k^{q^2+1} b^{q^5} \\ dk^{q-1} = a^q + b^{q^3} + \delta k^{q+q^2} b^{q^5} \\ -dk^{q^2-1} = k^{q^2-q} b^q + b^{q^3} \\ 0 = k^{-q-1} b^q + a^{q^3} + \delta b^{q^5} \\ d = k^{-q+1} b^{q^3} + \delta b^{q^5} \\ d = b^q - k^{-q^2+1} b^{q^3} + \delta a^{q^5} \end{cases}$$

which can be written as (3.3).

Now, suppose that there exist $\rho \in \mathrm{Aut}(\mathbb{F}_{q^6})$ and an invertible matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that for each $x \in \mathbb{F}_{q^6}$ there exists $z \in \mathbb{F}_{q^6}$ satisfying

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x^\rho \\ h^{\rho(q-1)}x^{\rho q} - h^{\rho(q^2-1)}x^{\rho q^2} + x^{\rho q^4} + x^{\rho q^5} \end{pmatrix} = \begin{pmatrix} z \\ \delta z^q + z^{q^3} + z^{q^5} \end{pmatrix}.$$

Equivalently, for each $x \in \mathbb{F}_{q^6}$ we have

$$\begin{aligned} cx^\rho + d(k^{q-1}x^{\rho q} - k^{q^2-1}x^{\rho q^2} + x^{\rho q^4} + x^{\rho q^5}) = \\ \delta[a^q x^{\rho q} + b^q(k^{q^2-q}x^{\rho q^2} + k^{-1-q}x^{\rho q^3} + x^{\rho q^5} + x^\rho)] \\ + a^{q^3}x^{\rho q^3} + b^{q^3}(k^{-q+1}x^{\rho q^4} - k^{-q^2+1}x^{\rho q^5} + x^{\rho q} + x^{\rho q^2}) \\ + a^{q^5}x^{\rho q^5} + b^{q^5}(-k^{1+q^2}x^\rho + k^{q^2+q}x^{\rho q} + x^{\rho q^3} + x^{\rho q^4}). \end{aligned}$$

This is a polynomial identity in x^ρ which yields to the following equations

$$\begin{cases} c = \delta b^q - k^{q^2+1}b^{q^5} \\ dk^{q-1} = \delta a^q + b^{q^3} + k^{q+q^2}b^{q^5} \\ -dk^{q^2-1} = \delta k^{q^2-q}b^q + b^{q^3} \\ 0 = \delta k^{-q-1}b^q + a^{q^3} + b^{q^5} \\ d = k^{-q+1}b^{q^3} + b^{q^5} \\ d = \delta b^q - k^{-q^2+1}b^{q^3} + a^{q^5} \end{cases}$$

which can be written as (3.4). □

We are now ready to prove that when $h \notin \mathbb{F}_{q^2}$, $\mathcal{L}_h$ is new.

Proposition 3.7. *If $h \notin \mathbb{F}_{q^2}$, then $\mathcal{L}_h$ is not* PΓL*-equivalent to L^4_δ (with $\delta^2+\delta=1$).*

Proof. By Proposition 3.6 we have to show that there are no a, b, c and d in $\mathbb{F}_{q^6}$ such that $ad - bc \neq 0$ and (3.3) or (3.4) are satisfied. Note that $b = 0$ in (3.3) and (3.4) yields $a = c = d = 0$, a contradiction. So, suppose $b \neq 0$. Since $h \notin \mathbb{F}_{q^2}$ then $k \notin \mathbb{F}_{q^2}$. We start by proving that the last three equations of (3.3), i.e.

$$\begin{cases} \mathrm{Eq}_1: b^{q^3} + (k^{q-1} + \delta k^{q+q^2})b^{q^5} = 0 \\ \mathrm{Eq}_2: k^{q^2-q}b^q + (1 + k^{q^2-q})b^{q^3} + \delta k^{q^2-1}b^{q^5} = 0 \\ \mathrm{Eq}_3: -\delta b^q + (k^{-q+1} + \delta^2 k^{1-q^2})b^{q^3} + \delta b^{q^5} = 0, \end{cases}$$

yield a contradiction. As in the above section, we will consider the q-th powers of Eq_1, Eq_2 and Eq_3 replacing b^{q^i}, k^{q^j}, and δ^{q^ℓ} (respectively) by X_i, Y_j, and Z_ℓ with $i, j \in \{0,1,2,3,4,5\}$ and $\ell \in \{0,1\}$. Consider the set S of polynomials in the variables X_i, Y_j, and Z_ℓ

$$S := \{\mathrm{Eq}_1^{q^\alpha}, \mathrm{Eq}_2^{q^\beta}, \mathrm{Eq}_3^{q^\gamma} : \alpha, \beta, \gamma \in \{0,1,2,3,4,5\}\}.$$

By eliminating from S the variables X_5, X_4, X_3, and X_2 using Eq_1, Eq_1^q, $\mathrm{Eq}_1^{q^4}$, and $\mathrm{Eq}_1^{q^3}$ respectively we obtain

$$X_0Y_1(Z_1Y_0^2Y_2 - Z_1Y_0Y_2^2 - Z_1Y_0 + Z_1Y_2 - Z_0^2Z_2 - Z_2) = 0.$$

By the conditions on b and k, $X_0Y_1 \neq 0$ and therefore

$$P := Z_1Y_0^2Y_2 - Z_1Y_0Y_2^2 - Z_1Y_0 + Z_1Y_2 - Z_0^2Z_2 - Z_2 = 0.$$

We eliminate Z_1 in S using P, obtaining, w.r.t. b, k, and δ,

$$bk^{q^2+1}(k-k^q)(k+k^q)(k^{q^2+1}-1)(k^{q^2+1}+1) = 0,$$

a contradiction to $k \notin \mathbb{F}_{q^2}$.

Consider now the last three equations of (3.4), i.e.

$$\begin{cases} \mathrm{Eq}_1\colon \delta b^{q^3} + (k^{q-1} - \delta k^{q^2+q})b^{q^5} = 0 \\ \mathrm{Eq}_2\colon \delta k^{q^2-q}b^q + (k^{q^2-q}+1)b^{q^3} + k^{q^2-1}b^{q^5} = 0 \\ \mathrm{Eq}_3\colon \delta^2 b^q + (k^{-q+1} + \delta^2 k^{-q^2+1})b^{q^3} + b^{q^5} = 0. \end{cases}$$

As before, we will consider the q-th powers of Eq_1, Eq_2, and Eq_3 replacing b^{q^i}, k^{q^j}, and δ^{q^ℓ} (respectively) by X_i, Y_j, and Z_ℓ with $i,j \in \{0,1,2,3,4,5\}$ and $\ell \in \{0,1\}$. Consider the set S of polynomials in the variables X_i, Y_j and Z_ℓ

$$S := \{\mathrm{Eq}_1^{q^\alpha}, \mathrm{Eq}_2^{q^\beta}, \mathrm{Eq}_3^{q^\gamma} : \alpha,\beta,\gamma \in \{0,1,2,3,4,5\}\}.$$

We eliminate in S the variables X_5, X_4, X_3, and X_2 using Eq_1, Eq_1^q, $\mathrm{Eq}_1^{q^4}$, and $\mathrm{Eq}_1^{q^3}$ respectively, and we get

$$Y_0X_0(Z_1Y_0^2Y_2^2 + 2Z_1Y_0Y_1^2Y_2 + 2Z_1Y_0Y_2 + Z_1Y_1^2 - Y_0^2Y_2^2 - Y_0Y_1^2Y_2 - Y_0Y_2 - Y_1^2) = 0.$$

Since $b \neq 0$ and $k \notin \mathbb{F}_{q^2}$, $X_0Y_0 \neq 0$ and therefore

$$P := Z_1Y_0^2Y_2^2 + 2Z_1Y_0Y_1^2Y_2 + 2Z_1Y_0Y_2 + Z_1Y_1^2 - Y_0^2Y_2^2 - Y_0Y_1^2Y_2 - Y_0Y_2 - Y_1^2 = 0.$$

Once again we consider the resultants of the polynomials in S and P w.r.t. Z_1 and we obtain

$$bk^{q^2+2q}(k-k^q)(k+k^q)(k^{q^2+1}-1)(k^{q^2+1}+1) = 0,$$

a contradiction to $k \notin \mathbb{F}_{q^2}$. □

As a consequence of the above considerations and Propositions 3.5 and 3.7, we have the following.

Corollary 3.8. *If $h \notin \mathbb{F}_{q^2}$, then $\mathcal{L}_h$ is not $\mathrm{P\Gamma L}$-equivalent to any known scattered linear set in $\mathrm{PG}(1,q^6)$.*

3.3 $\mathcal{L}_h$ may be defined by a trinomial

Suppose that $h \in \mathbb{F}_{q^2}$, then the condition on h becomes $h^{q+1} = -1$. For such h we can prove that the linear set $\mathcal{L}_h$ can be defined by the q-polynomial $(h^{-1}-1)x^q + x^{q^3} + (h-1)x^{q^5}$.

Proposition 3.9. *If $h \in \mathbb{F}_{q^2}$, then the linear set $\mathcal{L}_h$ is $\mathrm{P\Gamma L}$-equivalent to*

$$L_{\mathrm{tri}} := \{\langle (x, (h^{-1}-1)x^q + x^{q^3} + (h-1)x^{q^5})\rangle_{\mathbb{F}_{q^6}} : x \in \mathbb{F}_{q^6}^*\}.$$

Proof. Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2,q^6)$ with $a = -h + h^{-1}, b = 1, c = h^{-1} - 1 - h^3 + h^2$ and $d = h - h^2 - 1$. Straightforward computations show that the subspaces $\mathcal{U}_h$ and $U_{(h^{-1}-1)x^q+x^{q^3}+(h-1)x^{q^5}}$ are $\Gamma\mathrm{L}(2,q^6)$-equivalent under the action of the matrix A. Hence, the linear sets $\mathcal{L}_h$ and L_{tri} are $\mathrm{P\Gamma L}$-equivalent. □

The fact that $\mathcal{L}_h$ can also be defined by a trinomial will help us to completely close the equivalence issue for $\mathcal{L}_h$ when $h \in \mathbb{F}_{q^2}$. Indeed, we can prove the following:

Proposition 3.10. *If* $h \in \mathbb{F}_{q^2}$*, then the linear set* $\mathcal{L}_h$ *is* $\mathrm{P\Gamma L}$*-equivalent to some* L^4_δ *(*$\delta^2 + \delta = 1$*) if and only if* $h \in \mathbb{F}_q$ *and* q *is a power of* 5.

Proof. Recall that by [27, Proposition 5.5] if $h \in \mathbb{F}_q$ and q is a power of 5, then $\mathcal{L}_h$ is $\mathrm{P\Gamma L}$-equivalent to some L^4_δ. As in the proof of Proposition 3.6, by Lemma 3.3 we have to check whether $U_{(h^{-1}-1)x^q+x^{q^3}+(h-1)x^{q^5}}$ is $\Gamma\mathrm{L}$-equivalent either to U^4_δ or to $(U^4_\delta)^\perp$. Suppose that there exist $\rho \in \mathrm{Aut}(\mathbb{F}_{q^6})$ and an invertible matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that for each $x \in \mathbb{F}_{q^6}$ there exists $z \in \mathbb{F}_{q^6}$ satisfying

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x^\rho \\ (h^{-\rho}-1)x^{\rho q} + x^{\rho q^3} + (h^\rho - 1)x^{\rho q^5} \end{pmatrix} = \begin{pmatrix} z \\ z^q + z^{q^3} + \delta z^{q^5} \end{pmatrix}.$$

Let $k = h^\rho$, for which $k^{q+1} = -1$. As in Proposition 3.5, we obtain a polynomial identity, whence

$$\begin{cases} c = b^q(k^q - 1) + b^{q^3} + \delta b^{q^5}(k^{-q} - 1) \\ d(k^{-1} - 1) = a^q \\ 0 = b^q(k^{-q} - 1) + b^{q^3}(k^q - 1) + b^{q^5}\delta \\ d = a^{q^3} \\ 0 = b^q + b^{q^3}(k^{-q} - 1) + b^{q^5}(k^q - 1)\delta \\ d(k - 1) = \delta a^{q^5}. \end{cases} \tag{3.5}$$

By subtracting the fifth equation from the third equation raised to q^2, we get

$$b^q = b^{q^5}(k^q - 1),$$

i.e. either $b = 0$ or $k^q - 1 = (b^q)^{q^4-1}$, whence we get either $b = 0$ or $\mathrm{N}_{q^6/q^2}(k^q - 1) = 1$.

If $b \neq 0$, since $k - 1 \in \mathbb{F}_{q^2}$ and $\mathrm{N}_{q^6/q^2}(k-1) = (k-1)^3 = 1$, then

$$k^3 - 3k^2 + 3k - 2 = 0$$

and, since $\mathrm{N}_{q^6/q^2}(k^q - 1) = 1$ and $k^q = -1/k$,

$$2k^3 + 3k^2 + 3k + 1 = 0,$$

from which we get

$$9k^2 - 3k + 5 = 0. \tag{3.6}$$

- If $k \notin \mathbb{F}_q$ then k and k^q are the solutions of (3.6) and

$$-1 = k^{q+1} = \frac{5}{9},$$

  which holds if and only if q is a power of 7. By (3.6) it follows that $k \in \mathbb{F}_q$, a contradiction.

- If $k \in \mathbb{F}_q$, then $k^2 = -1$ and by (3.6) we have $k = -4/3$, which is possible if and only if q is a power of 5.

Hence, if either $k \notin \mathbb{F}_q$ or $k \in \mathbb{F}_q$ with q not a power of 5, we have that $b = 0$ and hence $c = 0$, $a \neq 0$ and $d \neq 0$.

By combining the second and the fourth equation of (3.5), we get $\mathrm{N}_{q^6/q^2}(k^{-1} - 1) = 1$ and, since $k^q = -1/k$, $\mathrm{N}_{q^6/q^2}(k^q + 1) = -1$. Arguing as above, we get a contradiction whenever $k \notin \mathbb{F}_q$ or $k \in \mathbb{F}_q$ with q not a power of 5.

Now, suppose that there exist $\rho \in \mathrm{Aut}(\mathbb{F}_{q^6})$ and an invertible matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that for each $x \in \mathbb{F}_{q^6}$ there exists $z \in \mathbb{F}_{q^6}$ satisfying

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x^\rho \\ (h^{-\rho} - 1)x^{\rho q} + x^{\rho q^3} + (h^\rho - 1)x^{\rho q^5} \end{pmatrix} = \begin{pmatrix} z \\ \delta z^q + z^{q^3} + z^{q^5} \end{pmatrix}.$$

Let $k = h^\rho$. As before, we get the following equations

$$\begin{cases} c = \delta b^q(k^q - 1) + b^{q^3} + b^{q^5}(k^{-q} - 1) \\ d(k^{-1} - 1) = \delta a^q \\ 0 = \delta b^q(k^{-q} - 1) + b^{q^3}(k^q - 1) + b^{q^5} \\ d = a^{q^3} \\ 0 = \delta b^q + b^{q^3}(k^{-q} - 1) + b^{q^5}(k^q - 1) \\ d(k - 1) = a^{q^5}. \end{cases} \tag{3.7}$$

By subtracting the fifth equation from the third raised to q^2 of the above system we get

$$b^q = b^{q^3}(k^{-q} - 1).$$

If $b \neq 0$, then $\mathrm{N}_{q^6/q^2}(k^{-q} - 1) = 1$. Hence, arguing as above, we get that $b = 0$ and hence $c = 0$, $a, d \neq 0$. By combining the fourth equation with the second and the fifth equation of (3.7) we get $\mathrm{N}_{q^6/q^2}(k - 1) = 1$, which yields again to a contradiction when $k \notin \mathbb{F}_q$ or $k \in \mathbb{F}_q$ with q not a power of 5. □

So, as a consequence of Corollary 3.8 and of the above proposition, we have the following result.

Corollary 3.11. *Apart from the case $h \in \mathbb{F}_q$ and q a power of 5, the linear set $\mathcal{L}_h$ is not* $\mathrm{P\Gamma L}$*-equivalent to any known scattered linear set in* $\mathrm{PG}(1, q^6)$.

By Proposition 3.9, when $h \in \mathbb{F}_{q^2}$, $\mathcal{L}_h$ is a linear set of the family presented in [23, Section 7]. Also, we get an extension of [21, Table 1], where it is shown examples of scattered linear sets which could generalize the family presented in [10]. We do not know whether the linear set $\mathcal{L}_h$, for each $h \in \mathbb{F}_{q^6} \setminus \mathbb{F}_{q^2}$ with $h^{q^3+1} = -1$, may be defined by a trinomial or not.

4 New MRD-codes

Delsarte in [13] (see also [14]) introduced in 1978 rank metric codes as follows. A *rank metric code* (or *RM*-code for short) $\mathcal{C}$ is a subset of the set of $m \times n$ matrices $\mathbb{F}_q^{m \times n}$ over $\mathbb{F}_q$ equipped with the distance function

$$d(A, B) = \mathrm{rk}\,(A - B)$$

for $A,B \in \mathbb{F}_q^{m\times n}$. The *minimum distance* of $\mathcal{C}$ is

$$d = \min\{d(A,B) : A,B \in \mathcal{C},\ A \neq B\}.$$

We will say that a rank metric code of $\mathbb{F}_q^{m\times n}$ with minimum distance d has parameters $(m,n,q;d)$. When $\mathcal{C}$ is an $\mathbb{F}_q$-subspace of $\mathbb{F}_q^{m\times n}$, we say that $\mathcal{C}$ is $\mathbb{F}_q$-linear. In the same paper, Delsarte also showed that the parameters of these codes fulfill a Singleton-like bound, i.e.

$$|\mathcal{C}| \leq q^{\max\{m,n\}(\min\{m,n\}-d+1)}.$$

When the equality holds, we call $\mathcal{C}$ a *maximum rank distance* (*MRD* for short) code. We will consider only the case $m=n$ and we will use the following equivalence definition for codes of $\mathbb{F}_q^{m\times m}$. Two $\mathbb{F}_q$-linear RM-codes $\mathcal{C}$ and $\mathcal{C}'$ are *equivalent* if and only if there exist two invertible matrices $A,B \in \mathbb{F}_q^{m\times m}$ and a field automorphism σ such that $\{AC^\sigma B : C \in \mathcal{C}\} = \mathcal{C}'$, or $\{AC^{T\sigma}B : C \in \mathcal{C}\} = \mathcal{C}'$, where T denotes transposition. Also, the *left* and *right idealisers* of $\mathcal{C}$ are $L(\mathcal{C}) = \{A \in \mathrm{GL}(m,q) : A\mathcal{C} \subseteq \mathcal{C}\}$ and $R(\mathcal{C}) = \{B \in \mathrm{GL}(m,q) : \mathcal{C}B \subseteq \mathcal{C}\}$ [17, 19]. They are important invariants for linear rank metric codes, see also [15] for further invariants.

In [24, Section 5] Sheekey showed that scattered $\mathbb{F}_q$-linear sets of $\mathrm{PG}(1,q^n)$ of rank n yield $\mathbb{F}_q$-linear MRD-codes with parameters $(n,n,q;n-1)$ with left idealiser isomorphic to $\mathbb{F}_{q^n}$; see [7, 8, 25] for further details on such kind of connections. We briefly recall here the construction from [24]. Let $U_f = \{(x,f(x)) : x \in \mathbb{F}_{q^n}\}$ for some scattered q-polynomial $f(x)$. After fixing an $\mathbb{F}_q$-basis for $\mathbb{F}_{q^n}$ we can define an isomorphism between the rings $\mathrm{End}(\mathbb{F}_{q^n},\mathbb{F}_q)$ and $\mathbb{F}_q^{n\times n}$. In this way the set

$$\mathcal{C}_f := \{x \mapsto af(x) + bx : a,b \in \mathbb{F}_{q^n}\}$$

corresponds to a set of $n\times n$ matrices over $\mathbb{F}_q$ forming an $\mathbb{F}_q$-linear MRD-code with parameters $(n,n,q;n-1)$. Also, since $\mathcal{C}_f$ is an $\mathbb{F}_{q^n}$-subspace of $\mathrm{End}(\mathbb{F}_{q^n},\mathbb{F}_q)$ its left idealiser $L(\mathcal{C}_f)$ is isomorphic to $\mathbb{F}_{q^n}$. For further details see [6, Section 6].

Let $\mathcal{C}_f$ and $\mathcal{C}_h$ be two MRD-codes arising from maximum scattered subspaces U_f and U_h of $\mathbb{F}_{q^n} \times \mathbb{F}_{q^n}$. In [24, Theorem 8] the author showed that there exist invertible matrices A, B and $\sigma \in \mathrm{Aut}(\mathbb{F}_q)$ such that $A\mathcal{C}_f^\sigma B = \mathcal{C}_h$ if and only if U_f and U_h are $\Gamma\mathrm{L}(2,q^n)$-equivalent

Therefore, we have the following.

Theorem 4.1. *The $\mathbb{F}_q$-linear MRD-code $\mathcal{C}_{f_h}$ arising from the $\mathbb{F}_q$-subspace $\mathcal{U}_h$ has parameters $(6,6,q;5)$ and left idealiser isomorphic to $\mathbb{F}_{q^6}$, and is not equivalent to any previously known MRD-code, apart from the case $h \in \mathbb{F}_q$ and q a power of 5.*

Proof. From [6, Section 6], the previously known $\mathbb{F}_q$-linear MRD-codes with parameters $(6,6,q;5)$ and with left idealiser isomorphic to $\mathbb{F}_{q^6}$ arise, up to equivalence, from one of the maximum scattered subspaces of $\mathbb{F}_{q^6} \times \mathbb{F}_{q^6}$ described in Section 3. From Corollaries 3.8 and 3.11 the result then follows. □

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