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A NEW FAMILY OF PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS WITH SOME LATIN SQUARE DESIGN PROPERTIES[1]

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0. Summary. An extensive family of two-class PBIB designs is defined in Section 1 and given the title of NL designs. Several analogies of NL designs with Latin square type designs are discussed in Sections 1 and 2. Some of these designs have already appeared in scattered contexts. Section 3 gives a construction based on finite geometry for an infinite class of NL designs, and others can be constructed by methods to be taken up elsewhere. An extension to designs with more than two associate classes is indicated in Section 4.

1. NL designs. We adopt the standard ([4], [13], [5]) definitions and terminology for m-class partially balanced incomplete block (PBIB) designs and association schemes. P_i denotes the $m \times m$ matrix whose element in the j, k position is p^i_{jk}. If N is the $v \times b$ incidence matrix of objects and blocks, then NN^T has $m + 1$ distinct characteristic roots $\theta_0 = rk, \theta_1, \cdots, \theta_m$ with respective multiplicities $\alpha_0, \cdots, \alpha_m$. For a two-class design we define

$$(1.1)\qquad \begin{aligned} \gamma &= p^2_{12} - p^1_{12}; \\ \Delta &= \gamma^2 + 2p^1_{12} + 2p^2_{12} + 1; \\ \sigma &= (\Delta^{\frac{1}{2}} - \gamma - 1)/2; \\ \tau &= (\Delta^{\frac{1}{2}} + \gamma - 1)/2. \end{aligned}$$

For later reference we list the known relations [10]

$$(1.2)\qquad \theta_1 = r + \lambda_1\tau + \lambda_2(-\tau - 1), \qquad \theta_2 = r + \lambda_1(-\sigma - 1) + \lambda_2\sigma;$$

$$(1.3)\qquad \alpha_1 = [\sigma n_1 + (\sigma + 1)n_2]/\Delta^{\frac{1}{2}}, \qquad \alpha_2 = [(\tau + 1)n_1 + \tau n_2]/\Delta^{\frac{1}{2}};$$

and [12]

$$(1.4)\qquad vn_1n_2 = \Delta\alpha_1\alpha_2;$$

$$(1.5)\qquad p^1_{12} = \sigma(\tau + 1), \qquad p^2_{12} = \tau(\sigma + 1).$$

Two two-class association schemes are *complements* of each other if and only if each can be obtained from the other by interchanging the designation of first and second associates.

Most known two-class PBIB designs have been classified by Bose and Shimamoto [5] into five types, distinguished primarily by the structure of their associ-

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ation schemes. We recall that known cyclic type schemes have parameters which may be expressed as follows in terms of an integer t:

$$\begin{gathered} v = 4t + 1, \\ n_1 = n_2 = \alpha_1 = \alpha_2 = 2t, \\ P_1 = \begin{bmatrix} t-1 & t \\ t & t \end{bmatrix}, \qquad P_2 = \begin{bmatrix} t & t \\ t & t-1 \end{bmatrix}. \end{gathered} \tag{1.6}$$

Association schemes with parameters (1.6), whatever their combinatorial structure, will be called *pseudo-cyclic.* An association scheme of *Latin square* type with $v = n^2$ objects and g constraints, which we denote as an $L_g\,(n)$ scheme, is defined by an $n \times n$ square array of the objects and a set of $g - 2$ pairwise orthogonal Latin squares of order n. Two objects are first associates if and only if they occur in the same row or column of the array or in positions occupied by the same letter in any of the Latin squares. If to the set of Latin squares we adjoin two more $n \times n$ arrays of n letters, one in which the ith letter occupies all positions in the ith row and another in which the ith letter occupies all positions in the ith column, we have g pairwise orthogonal squares (not all Latin) and may define the first associates somewhat more symmetrically as objects which occur in positions occupied by the same letter in any of the squares. Finite nets ([6], [7]) and orthogonal arrays [8] may be used as the basis for equivalent definitions. $L_g(n)$ parameters are given by

$$\begin{gathered} v = n^2, \qquad n_1 = g(n-1), \qquad n_2 = (n-g+1)(n-1), \\ P_1 = \begin{bmatrix} (g-1)(g-2) + n - 2 & (n-g+1)(g-1) \\ (n-g+1)(g-1) & (n-g+1)(n-g) \end{bmatrix}, \\ P_2 = \begin{bmatrix} g(g-1) & g(n-g) \\ g(n-g) & (n-g)(n-g-1) + n - 2 \end{bmatrix}. \end{gathered} \tag{1.7}$$

These lead to further parameters

$$\begin{gathered} \sigma = g - 1, \qquad \tau = n - g, \qquad \Delta = n^2, \\ \alpha_1 = g(n-1), \qquad \alpha_2 = (n-g+1)(n-1). \end{gathered} \tag{1.8}$$

Association schemes with parameters (1.7), whatever their combinatorial structure, will be called *pseudo-Latin square.*

Since there can be at most $n - 1$ pairwise orthogonal Latin squares of order n, g cannot exceed $n + 1$; moreover, if $g = n + 1$, all pairs of objects are first associates and the design reduces to a BIB design. The result is the same with $g = 0$. A Latin square association scheme with $g = 1$ constraint is a special case of a group divisible scheme, while it is easy to show that its complement has this structure in the case $g = n$. It is therefore convenient to assume

$$2 \leqq g \leqq n - 1. \tag{1.9}$$

We observe that the complement of a Latin square association scheme with g constraints is a pseudo-Latin square scheme with $n + 1 - g$ constraints. As a result, any pseudo-Latin square association scheme may be reduced by choice of notation to a pseudo-Latin square scheme with $2 \leqq g \leqq (n + 1)/2$. These are simply the schemes of this family for which $n_1 \leqq n_2$.

An example will show that not all pseudo-Latin square schemes have Latin square structure. An $L_3(6)$ scheme can be constructed from any 6×6 Latin square. Its complement then has $L_4(6)$ parameters but cannot have Latin square structure since no set of $4 - 2 = 2$ orthogonal 6×6 Latin squares exists. On the other hand, it is known ([16], [7], [11]) for a wide range of values of n and g that an association scheme with parameters (1.7) necessarily corresponds to a set of $g - 2$ pairwise orthogonal Latin squares of order n.

While minor infringements of inequality (1.9) lead only to trivial special cases, we now obtain something interesting by committing a major violation. Negative values of n and g lead in many cases to parameters (1.7) which are non-negative integers, but differ from the parameters of any of the types of association schemes in the Bose-Shimamoto classification. This suggests the existence of a new series of 2-class PBIB designs, based on association schemes with such parameters. We shall designate this as the NL family of designs and association schemes. The simplest case is $n = -4$, $g = -1$, giving the following, which could be termed $L_{-1}(-4)$ parameters.

$$v = 16, \qquad n_1 = 5, \qquad n_2 = 10,$$

$$P_1 = \begin{bmatrix} 0 & 4 \\ 4 & 6 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}.$$

Designs are known with these parameters, showing that the NL family is not vacuous.

Instead of using (1.7) with negative arguments for NL parameters it is convenient to have expressions in terms of positive arguments, which we shall still denote, however, by the same letters n and g. Then using the negative integers $-n$ and $-g$ in (1.7) we arrive at

$$(1.10) \qquad v = n^2, \qquad n_1 = g(n + 1), \qquad n_2 = (n - g - 1)(n + 1),$$

$$P_1 = \begin{bmatrix} (g + 1)(g + 2) - n - 2 & (n - g - 1)(g + 1) \\ (n - g - 1)(g + 1) & (n - g - 1)(n - g) \end{bmatrix},$$

$$P_2 = \begin{bmatrix} g(g + 1) & g(n - g) \\ g(n - g) & (n - g)(n - g + 1) - n - 2 \end{bmatrix}.$$

In terms of the positive integers n and g, we denote these as $\mathrm{NL}_g(n)$ parameters. Using (1.10) in (1.1) and (1.3),

$$(1.11) \qquad \sigma = n - g - 1, \qquad \tau = g, \qquad \Delta = n^2,$$

$$\alpha_1 = (n - g - 1)(n + 1), \qquad \alpha_2 = g(n + 1).$$

Alternatively, values of σ, τ, α_i could be obtained by using the negative integers $-n$ and $-g$ in (1.8). This amounts to using the negative square root of Δ in (1.1) and leads to negative values of σ and τ, finally giving values of θ_i and α_i which differ from those of (1.2) and (1.11) by an interchange of indices 1 and 2. In adopting expressions (1.11) we are following the customary [10] notation for θ_i and α_i.

The abbreviations $L_g(n)$ and $\mathrm{NL}_g(n)$ will sometimes be shortened to L_g and NL_g when it is not necessary to specify the value of n.

Like the pseudo-Latin square family, which is also defined in terms of the form of its parameters, the NL family of association schemes contains the complement of each of its members; specifically, the complement of an $\mathrm{NL}_g(n)$ scheme is an $\mathrm{NL}_{n-g-1}(n)$ scheme. As a result, any NL scheme may be reduced by choice of notation to one for which $g \leqq \frac{1}{2}(n-1)$, or equivalently, $n_1 \leqq n_2$. The well-known relation for sums of parameters n_i and p_{jk}^i and for products $n_i p_{jk}^i$ are satisfied identically in n and g. The requirement that p_{11}^1 is non-negative places a lower bound on g.

If n is odd, we note that $L_{\frac{1}{2}(n+1)}(n)$ parameters are identical with $\mathrm{NL}_{\frac{1}{2}(n-1)}(n)$ parameters and that both agree with pseudo-cyclic parameters (1.6) with argument $t = (n^2-1)/4$. These are the only L_g or NL_g schemes for which $n_1 = n_2$ and the only pseudo-cyclic schemes for which v is a square. No other schemes are common to any two of these three families.

We conclude this section with an outline of the present state of knowledge about $\mathrm{NL}_g(n)$ schemes in the range $n \leqq 10$, subject to $n_1 \leqq n_2$. The schemes $\mathrm{NL}_1(3)$, $\mathrm{NL}_2(5)$, $\mathrm{NL}_3(7)$ and $\mathrm{NL}_4(9)$ are of the pseudo-cyclic and Latin square type just discussed and are known. The schemes $\mathrm{NL}_1(4)$ can be constructed [9] by deducing the table of first associates directly from the definition and parameter values; there are also several geometrical and algebraic methods of constructing this scheme. The scheme $\mathrm{NL}_2(8)$ is known [14]; its construction will be described in Section 4. A new method of construction in Section 4 will provide solutions for $\mathrm{NL}_1(4)$, $\mathrm{NL}_3(8)$ and $\mathrm{NL}_2(9)$. Methods to be presented in later papers give solutions for some of the foregoing as well as for $\mathrm{NL}_3(9)$ and $\mathrm{NL}_2(10)$. The schemes $\mathrm{NL}_2(6)$, $\mathrm{NL}_2(7)$, $\mathrm{NL}_3(10)$, and $\mathrm{NL}_4(10)$ are still unknown. Some designs are known for each of the known schemes.

2. A characterizing property.

THEOREM 2.1. *In order for the parameters α_1, α_2 in a two-class association scheme to be equal in some order to the parameters n_1, n_2, it is necessary and sufficient that the scheme be of pseudo-cyclic, pseudo-Latin square, or* NL *type.*

PROOF. Sufficiency follows trivially from (1.6), (1.7), (1.8), (1.10), and (1.11).

Let the parameters α_1, α_2 of a two-class scheme be equal in some order to n_1, n_2. We first assert that this is the case if and only if $v = \Delta$. This assertion follows from (1.4) and the relation $\alpha_1 + \alpha_2 = n_1 + n_2$.

If the scheme is of pseudo-cyclic type, there is nothing to prove. For schemes of other types, Theorems 5.3 and 5.5 of [10] may be applied to show that $\Delta = n^2$

for some integer n. Therefore $v = n^2$. Then

$$n_2 = n^2 - 1 - n_1 . \tag{2.1}$$

Using (1.1),

$$\sigma + \tau + 1 = \Delta^{\frac{1}{2}} = n, \tag{2.2}$$

partially identifying σ and τ.

CASE I. Suppose $n_1 = \alpha_1$. Then from (1.3) and (2.1),

$$n_1 = [\sigma n_1 + (\sigma + 1)(n^2 - 1 - n_1)]/n,$$

reducing to $n_1 = (\sigma + 1)(n - 1)$. This identifies σ and τ completely. If we set $\sigma + 1 = g$ we have

$$n_1 = g(n - 1), \qquad n_2 = (n - g + 1)(n - 1)$$

and from (1.5)

$$p_{12}^1 = (g - 1)(n - g + 1), \qquad p_{12}^2 = g(n - g).$$

The parameters v, n_i, p_{12}^i are of the form of (1.7), and it follows from standard identities that the same is true of the remaining p_{jk}^i. Therefore the scheme is of pseudo-Latin square type.

CASE II. Suppose $n_1 = \alpha_2$. Then using (1.3) and (2.1) as in Case I we find $n_1 = \tau(n + 1)$ and setting $\tau = g$ we again use (1.5), this time arriving at parameters of the form of (1.10). Therefore the scheme is of NL type and the proof is complete.

It is clear from the proof that the condition on n_i and α_i in Theorem 2.1 could be replaced by the condition $v = \Delta$. The fact that v is a square is a distinctive property of the Latin square and NL schemes but is not peculiar to them, as shown for example by numerous group divisible schemes and by the triangular scheme with $v = 36$. However, inspection of a list of arithmetically possible parameters for two-class association schemes leads to the interesting conjecture that when v is a square, a high proportion of these parameters fall in the group divisible, L_g and NL_g series. As an illustration, in the range $v \leqq 100$, v a square, there are at most 65 sets of two-class parameters, of which 59 are in these three series.

3. Constructions in finite geometry which give NL designs. In this section infinitely many PBIB designs are constructed from finite geometries by a generalization of a method due to Ray-Chaudhuri [14]. Some of the designs are shown to be of NL type.

Let $\Sigma = PG(n, s)$ be a projective space of dimension n and order s, where s is a prime power. Let Γ be a fixed hyperplane, i.e., a subspace of dimension $n - 1$, and let Δ be the complement of Γ in Σ. Thus Δ is an affine space of dimension n, containing s^n points. It will sometimes be convenient to refer to points of Δ and Γ as affine and ideal points respectively. Each line of Δ contains s points and is contained in a unique line l of Σ, along with a unique point of Γ. We say that this is the ideal point of l. For any set $R \subset \Gamma$, define $\mathcal{L}(R)$ to be the incomplete block design whose objects are the s^n points of Δ and whose blocks are those lines of Δ whose ideal points are contained in R. Define two points of Δ to be first associates if the line containing them is a block of $\mathcal{L}(R)$ and second associates otherwise. While $\mathcal{L}(R)$ does not in general satisfy the conditions of partial balance, the following parameters are determined. $|R|$ denotes the cardinality of set R.

$$\begin{aligned} & v = s^n, \qquad n_1 = (s-1)|R|, \\ (3.1) \qquad & r = |R|, \qquad k = s, \qquad b = s^{n-1}|R|, \\ & \lambda_1 = 1, \qquad \lambda_2 = 0. \end{aligned}$$

In order for $\mathcal{L}(R)$ to be a two-class PBIB design, it is now necessary and sufficient [3] that the following condition be satisfied for some integers p_{11}^1 and p_{11}^2.

Given any two objects which are ith associates, $i = 1, 2$, there are exactly p_{11}^i other objects which are first associates of both. We proceed to interpret this as a condition on set R.

$\bar{R}$ will denote the complement of R in Γ. Define $T_\nu(R)$, $\nu = 0, \cdots, s+1$, to be the set of lines of Γ which contain exactly N points of R. For two points A and B of Δ, we denote by D the ideal point of line AB. If A and B are ith associates, $i = 1, 2$, we denote by $p_{11}^i(A, B)$ the number of points C which are common first associates of A and B. The required points C are of two different types which will be enumerated separately. Define C to be a c-point of A and B (collinear common first associate) if C is on line AB. The number of c-points of A and B is clearly $s - 2$ if $D \varepsilon R$ and zero if $D \varepsilon \bar{R}$. Define C to be a d-point of A and B (diagonal common first associate) if C is not on line AB. In this case lines AC and BC of Σ intersect Γ in points $E \varepsilon R$ and $F \varepsilon R$ respectively, where D, E, F are distinct points of the line m of intersection of plane ABC and hyperplane Γ. Suppose that m contains ν points of R. Then the ordered pair of points E, F can be chosen in $(\nu - 1)(\nu - 2)$ ways if $D \varepsilon R$ and in $\nu(\nu - 1)$ ways if $D \varepsilon \bar{R}$, and in each case a like number of d-points of A and B occur in the plane determined by A, B and m. The total number of d-points of A and B can be obtained by summing over all lines m which are in Γ and contain D. Define $f_D(\nu)$, $\nu = 0, 1, \cdots, s+1$, as the number of lines of $T_\nu(R)$ which contain D. The preceding remarks on c-points and d-points imply the following statement.

If A and B are first associates, so that $D \varepsilon R$,

$$p_{11}^1(A, B) = s - 2 + \sum_{\nu=0}^{s+1} (\nu - 1)(\nu - 2) f_D(\nu), \tag{3.2}$$

while if A and B are second associates, so that $D \varepsilon \bar{R}$,

$$p_{11}^2(A, B) = \sum_{\nu=0}^{s+1} \nu(\nu - 1) f_D(\nu). \tag{3.3}$$

This is enough to prove

LEMMA 3.1. *$\mathcal{L}(R)$ is a two-class* PBIB *design if and only if the right hand side of* (3.2) *has the same value for all points $D \varepsilon R$ and the right hand side of* (3.3) *has the same value for all points $D \varepsilon \bar{R}$. In this case $\mathcal{L}(R)$ will have parameters* (3.1), *along with $p_{11}^1 = p_{11}^1(A, B)$, $p_{11}^2 = p_{11}^2(A, B)$.*

The condition of Lemma 3.1 will be recognized as essentially a condition on the variance of the numbers ν. It is implied by the condition the following lemma places on their frequency distribution.

LEMMA 3.2. *$\mathcal{L}(R)$ is a two-class* PBIB *design if for fixed $\nu = 0, 1, \cdots, s + 1$, the frequencies $f_D(\nu)$ are equal for all $D \varepsilon R$ and are equal for all $D \varepsilon \bar{R}$. In this case $\mathcal{L}(R)$ will have parameters as stated in Lemma* 3.1.

In our first application of these lemmas we take $R = Q$, a non-degenerate quadric in $\Gamma = PG(n - 1, s)$, denoting $\bar{Q} = \bar{R}$. The assertions of this and the three succeeding paragraphs about the sets $T_\nu(Q)$ and the integers $|Q|$ and $f_D(\nu)$ are adapted from [15] and from Chapter 2 of [2]. All lines of Γ fall into the following sets $T_\nu(Q)$:

$$T_0(Q): \text{ non-intersectors, containing no points of } Q,$$

$$T_1(Q): \text{ tangents, each containing 1 point of } Q, \tag{3.4}$$

$$T_2(Q): \text{ secants, each containing 2 points of } Q,$$

$$T_{s+1}(Q): \text{ rulings, each containing } s+1 \text{ points of } Q.$$

Thus non-zero frequencies $f_D(\nu)$ can occur only for $\nu = 0, 1, 2, s + 1$, and (3.2) and (3.3) reduce to

$$p_{11}^1(A, B) = s - 2 + s(s - 1)f_D(s + 1), \qquad D \varepsilon Q, \tag{3.5}$$

$$p_{11}^2(A, B) = 2f_D(2), \qquad D \varepsilon \bar{Q}. \tag{3.6}$$

In a particular non-degenerate quadric Q in $PG(n - 1, s)$, the number $f_D(s + 1)$ of rulings on D is the same for all points $D \varepsilon Q$, so that $p_{11}^1(A, B)$ has a uniform value p_{11}^1 for all pairs A, B of first associates in $\mathcal{L}(Q)$.

We must specify the dimension n before proceeding further. If $n = 2t$, so that Γ has odd dimension $2n - 1$, the number $f_D(2)$ of secant lines on D is the same for all points $D \varepsilon \bar{Q}$, so that $p_{11}^2(A, B)$ has a uniform value p_{11}^2 for all pairs A, B of second associates in $\mathcal{L}(Q)$. If n is odd, so that Q is a non-degenerate quadric in a space Γ of even dimension, the points $D \varepsilon \bar{Q}$ are of different types which are contained in different numbers $f_D(2)$ of secant lines. In this case $p_{11}^2(A, B)$ does not have the same value for all pairs A, B of second associates.

We conclude that if Q is a non-degenerate quadric in Γ, the design $\mathcal{L}(Q)$ is a two-class partially balanced design if and only if the dimension n of Σ is even.

Let $n = 2t$. In $\Gamma = PG(2t - 1, s)$ there are two types of non-degenerate quadrics, which we shall call hyperbolic and elliptic, differing in the number of points, ruling lines, and secants. In the following formulas, the upper signs hold for hyperbolic quadrics and the lower signs hold for elliptic quadrics.

$$
\begin{aligned}
|Q| &= (s^{t-1} \pm 1)(s^t \mp 1)/(s-1), && \\
f_D(s+1) &= (s^{t-2} \pm 1)(s^{t-1} \mp 1)/(s-1), && D \varepsilon Q, \\
f_D(2) &= s^{t-1}(s^{t-1} \pm 1)/2, && D \varepsilon \bar{Q}.
\end{aligned} \tag{3.7}
$$

The parameters of $\mathfrak{L}(Q)$ can now be computed in both cases and compared with (1.7) and (1.10) to complete the proof of

THEOREM 3.1. *If $n = 2t$ and Q is a non-degenerate quadric in Γ, the design $\mathfrak{L}(Q)$ is a two-class* PBIB *design with association scheme parameters*

$$
\begin{aligned}
&v = s^{2t}, \qquad n_1 = (s^{t-1} \pm 1)(s^t \mp 1), \\
&p_{11}^1 = s^{t-1}(s^{t-1} \mp 1) \pm s^t - 2, \qquad p_{11}^2 = s^{t-1}(s^{t-1} \pm 1),
\end{aligned} \tag{3.8}
$$

and design parameters

$$
r = (s^{t-1} \pm 1)(s^t \mp 1)/(s-1), \quad k = s, \quad b = s^{2t-1}r, \quad \lambda_1 = 1, \quad \lambda_2 = 0. \tag{3.9}
$$

If Q is hyperbolic the upper signs hold and $\mathfrak{L}(Q)$ is of pseudo-Latin square type $L_g(s^t)$, $g = s^{t-1} + 1$. If Q is elliptic the lower signs hold and $\mathfrak{L}(Q)$ is of type $\mathrm{NL}_g(s^t)$, $g = s^{t-1} - 1$.

Since the required projective spaces and quadrics exist for every s which is a prime or a power of a prime and for every positive integer t, our construction gives a doubly infinite family of designs having NL_g association schemes. The following schemes with $v \leqq 100$ are included.

$$
\begin{aligned}
s &= 2, & t &= 2, & &\mathrm{NL}_1(4), \\
s &= 2, & t &= 3, & &\mathrm{NL}_3(8), \\
s &= 3, & t &= 2, & &\mathrm{NL}_2(9).
\end{aligned}
$$

The spaces Σ, Γ, and Δ and the quadric Q may be used to construct other designs which have the same association scheme as $\mathfrak{L}(Q)$.

We note that each block of $\mathfrak{L}(Q)$ is the intersection $l \cap \Delta$ of Δ with a line l of Σ where l intersects Γ in a point of Q. We define a more general design $\mathfrak{L}_\nu(Q)$, $\nu = 0, 1$, with sets of blocks constructed as follows from the set of all lines l which are in Σ but not in Γ.

Design	Blocks
$\mathfrak{L}_0(Q)$	$\{l \cap \Delta \mid l \cap \Gamma \varepsilon \bar{Q}\}$
$\mathfrak{L}_1(Q)$	$\{l \cap \Delta \mid l \cap \Gamma \varepsilon Q\}$

The subscript ν may be interpreted as the number of points of Q contained in l. The following theorem is now obvious.

THEOREM 3.2. *If $n = 2t$ and Q is a non-degenerate quadric in Γ, then $\mathfrak{L}_\nu(Q)$, $\nu = 0, 1$, is partially balanced with the same association scheme as $\mathfrak{L}(Q)$ and with association scheme parameters* (3.8). *$\mathfrak{L}_1(Q)$ is identical with $\mathfrak{L}(Q)$. $\mathfrak{L}_0(Q)$ has design parameters*

$$
r = |\bar{Q}|, \qquad k = s, \qquad b = s^{2t-1}|\bar{Q}|, \qquad \lambda_1 = 0, \qquad \lambda_2 = 1. \tag{3.10}
$$

Let π be a plane in Σ but not in Γ. π intersects Δ in a set of s^2 points which we shall use as a block of a design, and intersects Γ in a line which falls in one of the classes $T_\nu(Q)$. We define designs $\mathcal{P}_\nu(Q)$ with sets of blocks constructed as follows from the set of all planes π which are in Σ but not in Γ.

Design	Blocks
$\mathcal{P}_\nu(Q)$	$\{\pi \cap \Delta \mid \pi \cap \Gamma \, \varepsilon \, T_\nu(Q)\}, \quad \nu = 0, 1, 2, s+1$

The subscript ν may be interpreted as the number of points of Q contained in π.

If A is a point of Δ, planes containing A are determined by the lines of $T_\nu(Q)$, and these planes lead to the blocks of $\mathcal{P}_\nu(Q)$ which contain A. Therefore A is contained in $|T_\nu(Q)|$ blocks.

If A and B are two points of Δ and D is the intersection of line AB with Γ, planes containing A and B are determined by the lines of $T_\nu(Q)$ which contain D, and these planes lead to the blocks of $\mathcal{P}_\nu(Q)$ which contain both A and B. Therefore A and B occur together in $f_D(\nu)$ blocks. We now use the fact, stated in part in (3.7), that for a non-degenerate Q in Γ of odd dimension, all of the frequencies $f_D(\nu)$ satisfy the uniformity condition of Lemma 3.2. This gives us the following theorem.

THEOREM 3.3. *If $n = 2t$ and Q is non-degenerate, then $\mathcal{P}_\nu(Q)$, $\nu = 0, 1, 2, s + 1$, is partially balanced with the same association scheme as $\mathcal{L}(Q)$. $\mathcal{P}_\nu(Q)$ has association scheme parameters* (3.8) *and design parameters*

$$r = |T_\nu(Q)|, \qquad k = s^2, \qquad b = s^{2t-2}|T_\nu(Q)|, \tag{3.11}$$
$$\lambda_1 = f_D(\nu), \quad D \, \varepsilon \, Q; \qquad \lambda_2 = f_D(\nu), \quad D \, \varepsilon \, \bar{Q}.$$

Explicit formulas for $|T_\nu(Q)|$ and $f_D(\nu)$ are known but will not be reproduced here.

Taking $\Sigma = PG(3, 2^t)$ and R as the set of points of a non-degenerate conic together with the common point of the tangent lines, Ray-Chaudhuri [14] constructs the equivalent of $\mathcal{L}(R)$ and shows that it is a two-class PBIB design. The design for $t = 2$ is of $\mathrm{NL}_2(8)$ type. This illustrates that there are sets R other than quadrics for which $\mathcal{L}(R)$ is partially balanced. As a further illustration, our final construction of this section used an interesting set whose properties have been investigated by Bose [1].

Take $\Sigma = PG(3, q)$, $\Gamma = PG(2, q)$, where $q = s^2$, and represent the points of Γ by homogeneous coordinates (y_1, y_2, y_3), $y_i \, \varepsilon \, GF(q)$. Take $R = W$, where W is the set of points of Γ for which the equation

$$y_1^{s+1} + y_2^{s+1} + y_3^{s+1} = 0 \tag{3.12}$$

is satisfied. Bose shows that

$$\begin{array}{lll} & |W| = s^3 + 1; & \\ (3.13) & f_D(1) = 1 \quad \text{and} \quad f_D(s+1) = s^2, & D \, \varepsilon \, W; \\ & f_D(1) = s + 1 \quad \text{and} \quad f_D(s+1) = s^2 - s, & D \, \varepsilon \, \bar{W}; \\ & f_D(\nu) = 0, & \text{otherwise.} \end{array}$$

We prove the following theorem by applying Lemma 3.2 and comparing parameters with (1.10).

THEOREM 3.4. *$\mathcal{L}(W)$ is a two-class* PBIB *design with parameters*

$$v = s^6, \qquad n_1 = (s^2 - 1)(s^3 + 1),$$

$$p_{11}^1 = s_2(s^2 + 1) - s^3 - 2, \qquad p_{11}^2 = s_2(s^2 - 1), \tag{3.14}$$

$$r = s^3 + 1, \qquad k = s^2, \qquad b = s^4(s^3 + 1),$$

$$\lambda_1 = 1, \qquad \lambda_2 = 0.$$

This design is of type $\mathrm{NL}_g\,(s^3)$, $g = s^2 - 1$.

Three other designs $\mathcal{L}_0(W)$, $\mathcal{P}_1(W)$, $\mathcal{P}_{s+1}(W)$ with the same association scheme can be constructed from W by the methods of Theorems (3.2) and (3.3). These designs have the same association scheme parameters as the NL designs of Theorem 3.1, $t = 3$, but have different design parameters r, k, b, λ_i.

Only a few of the designs presented in this section have parameters in or near the range $r \leqq 10$, $k \leqq 10$. Additional designs exist, of course, for all of the association schemes which have been constructed here.

4. Generalized L_g and NL_g designs with m associate classes. The Latin square family of two-class association schemes and designs can be generalized in a natural way to a larger number of associate classes. The three-class case has been discussed [17] by Singh and Shukla, who were aware of the full generalization. In this section we describe the family of m-class Latin square association schemes, then define an m-class NL scheme.

The definition in Section 2 of the Latin square type association scheme is modified by arranging the set of g orthogonal squares into $m - 1$ disjoint nonempty subsets, where, denoting by g_i the number of squares in the ith set,

$$g_1 + \cdots + g_{m-1} = g. \tag{4.1}$$

We assume $g \leqq n$ and denote $g_m = n + 1 - g$. The objects in two positions of the $n \times n$ array are defined as ith associates if the positions are occupied by the same letter in an orthogonal square of the ith subset, $i = 1, \cdots, m - 1$, and are defined as mth associates otherwise. It can be shown that this association relation is a partially balanced m-class scheme with the following parameters:

$$v = n^2, \qquad n_i = g_i(n - 1),$$

$$p_{ii}^i = (g_i - 1)(g_i - 2) + n - 2, \qquad p_{ij}^i = p_{ji}^i = g_j(g_i - 1), \tag{4.2}$$

$$p_{jj}^i = g_j(g_j - 1), \qquad p_{jk}^i = g_j\, g_k, \qquad i, j, k \text{ distinct}, \ 1 \leqq i, j, k \leqq m.$$

We define an m-class association scheme to be of generalized pseudo-Latin square type if it has parameters (4.2) but does not necessarily have the combinatorial structure just described.

It is now completely straightforward to define a generalized NL family of association schemes by using negative integers $n, g_1, \cdots, g_m$ in expressions (4.1) and (4.2). In terms of positive parameters n^*, g_1^*, we take $n = -n^*$, $g_i = -\,g_i^*$ and substitute in (4.1) and (4.2). Dropping the stars, we have

$$g_1 + \cdots + g_m = n - 1, \tag{4.3}$$

$$v = n^2, \quad n_i = g_i(n+1),$$

$$p^i_{ii} = (g_i + 1)(g_i + 2) - n - 2, \qquad p^i_{ij} = p^i_{ji} = g_j(g_i + 1), \tag{4.4}$$

$$p^i_{jj} = g_j(g_j + 1), \qquad p^i_{jk} = g_j g_k,$$

$$i, j, k \text{ distinct}, 1 \leqq i, j, k \leqq m.$$

These parameters are integers satisfying the standard relations on $\sum n_i$, $\sum p^i_{jk}$, and n_i, p^i_{jk}, and all except possibly p^i_{ii} are non-negative. The requirement $p^i_{ii} \geqq 0$ places a lower bound on g_i for a given n, $i = 1, \cdots, m$, and (4.3) then places an upper bound on the number m of associate classes for a given n.

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