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A New Finance Chaotic Attractor

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Abstract: Based on the mathematical model of a nonlinear finance chaotic system, the complicated dynamical behavior and slow manifold of the model are further investigated. Firstly, the complicated dynamical behaviors of the system are analyzed by studing dynamical behaviors of the subsystem of the system. And then, global dynamical behaviors of the system are discussed, such as symmetry, dissipation and equilibrium points etc. Thirdly, by using different methods, the slow manifold equations of the system are obtained. Finally, the adaptive control of the nonlinear finance chaotic system is presented. We settle the nonlinear finance chaotic system to equilibrium point with only one controller. The results of theoretical analysis and simulation are helpful for better understanding of other similar nonlinear finance chaotic systems.

Key words: nonlinear finance chaotic system; dynamical behavior; slow manifold; bifurcation; chaos

1 Introduction

Nonlinear chaotic dynamical system research is popular problem in the nonlinear science field. Nonlinear chaotic systems have been extensively studied within scientific, engineering and mathematical communities[1-4]. Since the chaotic phenomenon in economics was first found in 1985, great impact has been imposed on the prominent economics at present, because the occurrence of the chaotic phenomenon in the economic system means that the macro economic operation has in itself the inherent indefiniteness. Researches on the complicated economic system by applying nonlinear method have been fruitful [5-7]. In the field of finance, stocks and social economics, due to the interaction between nonlinear factors, with the evolution process from low dimensions to high dimensions, the diversity and complexity have manifested themselves in the internal structure of the system and there exists extremely complicated phenomenon and external characteristics in such a kind of system. So it has become more and more important to make a systematic and deep study in the internal structural characteristics in such a complicated economic system.

In this paper, we study the complicated dynamic behavior and slow manifold of a nonlinear finance chaotic system which was offered by reference [5]. Its mathematical model was the differential equation group model. In order to investigate the complicated dynamical behavior of the nonlinear finance chaotic system further, we introduce a new practical method to distinguish the chaotic, periodic and quasi-periodic orbits [8-9]. In this method, we first select two subsystems in a lower-dimension space, and finally analyze the dynamical behaviors of the whole system based on two subsystems.

In the nonlinear chaotic dynamical systems research, one method is that some chaotic systems can be seen as slow-fast systems to be qualitatively and quantitatively analyzed[9-11]. Firstly, the slow manifold equation of the nonlinear chaotic dynamics system is obtained by considering that the slow manifold is locally

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defined by a plane orthogonal to the tangent system's left fast eigenvector. And secondly, another method consists of defining the slow manifold as the surface generated by the two slow eigenvectors associated with the two eigenvalues $\lambda_2(X)$ and $\lambda_3(X)$ of$J(X)$. Another slow manifold equation of the nonlinear finance chaotic system is obtained.

The controller is to be determined with the purpose of controlling the nonlinear finance chaotic systems when the parameters are unknown[14-18]. We construct the positive Lyapunov function.According to the Lyapunov stable theorem and BarBalat Lemma, we settle the nonlinear finance chaotic systems to the equilibrium point.

This paper is organized as the follows. In Section 2, complicated dynamical behavior of the nonlinear finance chaotic system is further investigated. In Section 3, slow manifold of the nonlinear finance chaotic system is considered. In Section 4, the modified adaptive controlling is applied to control chaos behavior for the finance chaotic system. A brief conclusion is presented in Section 5.

2 Complicated dynamical behavior analysis of the finance chaotic system

The nonlinear finance chaotic system can be described by the following differential equation:

$$\begin{cases} \dot{X} = Z + (Y - a)X \\ \dot{Y} = 1 - bY - X^2 \\ \dot{Z} = -X - cZ \end{cases} \tag{1}$$

where variable X represents the interest rate in the model; variable Y represents the investment demand and variable$Z$ is the price exponent. $\dot{X} = \frac{dX}{dt}, \dot{Y} = \frac{dY}{dt}, \dot{Z} = \frac{dZ}{dt}$. The parameter a is the saving. b is the per-investment cost. c is the elasticity of demands of commercials. And they are positive constants.

When $c - b$-abc≤0, system (1) has only equilibrium point (0,1/b,0). When $c - b - abc$>0, system (1) has another two equilibrium points ($\pm m$, (1+ac)/c, $\mp m/c$), where$m = \sqrt{(c - b - abc)/c}$.

Using the linear transformation $x(t) = X(t)$,$y(t) = Y(t) - 1/b$, $z(t) = Z(t)$, then system (1) becomes system (2) as follows:

$$\begin{cases} \frac{dx}{dt} = (1/b - a)\,x + z + xy \\ \frac{dy}{dt} = -by - x^2 \\ \frac{dz}{dt} = -x - cz \end{cases} \tag{2}$$

System (1) and system (2) are topologically equivalent. We study the dynamical behaviors of system (1) by studying the dynamical behaviors of system (2).

To study the long-term dynamical behavior of the finance chaotic system (2), the system is divided into subsystems.

$$\begin{cases} \frac{dx}{dt} = (1/b - a)x + xy \\ \frac{dy}{dt} = -by - x^2 \end{cases} \tag{3}$$

Let x be a known function of the time t, and the second subsystem is obtained:

$$\begin{cases} \frac{dy}{dt} = -by - x^2 \\ \frac{dz}{dt} = -x - cz \end{cases} \tag{4}$$

When $t = t_0$, $x$is a constant number, then the system (4) is a two dimensional linear system with constant coefficients. Therefore its dynamical behavior is very simple and global.

2.1 Dynamical analysis of the subsystem (3)

Let

$$\begin{cases} (1/b - a)x + xy = 0, \\ -by - x^2 = 0. \end{cases}$$

When ab≥1, subsystem (3) has only one equilibrium point O (0,0). When ab¡1, subsystem (3) has three equilibrium points O (0, 0), S ($\pm x_0$, y_0), where x_0^2=1-ab, y_0= -x_0^2/b.

Proposition 1 *The equilibrium point O (0, 0) of subsystem (3) is a Hopf bifurcation point.*

Proof: The Jacobian matrix of subsystem (3) at$O$ (0, 0) is:

$$J_0 = \begin{bmatrix} 1/b - a & 0 \\ 0 & -b \end{bmatrix}$$

The eigenvalues of J_0 are: $\lambda_1 = -b<0$, λ_2=1/$b - a$. If *ab* >1, then λ_2<0, so the equilibrium point O (0, 0) is a stable point. If *ab*<1, then λ_2>0, so the equilibrium point O (0, 0) is an unstable focus point. If *ab*=1, thenλ_2=0. So the equilibrium point O (0, 0) is a Hopf bifurcation point. □

The dynamical behaviors of system (3) at S $(\pm x_0, y_0)$ are similar to discussion of the equilibrium point O (0, 0). We have

Proposition 2 *The equilibrium points S ($\pm x_0$, y_0) of subsystem (3) are Hopf bifurcation points.*

2.2 Dynamical analysis of the subsystem (4)

Let

$$\begin{cases} -by - x^2 = 0, \\ -x - cz = 0. \end{cases}$$

When x= 0, the subsystem (4) has only one equilibrium point O(0, 0). When $x \neq 0$, subsystem (3) has an equilibrium point S (-x^2/b, -x/c).

Proposition 3 *The equilibrium points $O(0,0)$ andS (-x^2/b, -x/c) of subsystem (4) are both stable points.*

2.3 Complicated dynamical behavior analysis of the finance chaotic system

We analyze the global complicated dynamical behavior of the nonlinear finance chaotic system (2) in this section.

Firstly, we discuss the symmetry of the nonlinear finance chaotic system. For any arbitrary parameters a, b, c, system (2) is invariant under the transformation (x, y, z) $\rightarrow$ (-x, y, -z), which indicates that the nonlinear finance chaotic system is symmetric with respect to the y-axis.

Secondly, we discuss the dissipation of the nonlinear finance chaotic system. Let$f = (f_1, f_2, f_3)^T = (\dot{x}, \dot{y}, \dot{z})^T$, then we have

$$\text{div}\ f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = 1/b - a - b - c$$

If $1/b - a - b - c$< 0, i.e. $a + b + c >$ 1/b, then we have div $f < 0$. This shows that the finance chaotic system is a dissipative system. The volume of any attractor of the system must absolutely contract to zero.

Thirdly, we discuss the equilibrium points of the nonlinear finance chaotic system.

When $c - b$ - *abc* ≤0, system (2) has the only equilibrium point O (0,0,0). When $c - b$- *abc* >0, system (1) has other equilibrium points S ($\pm m$, - m^2/b, $\mp m/c$), where $m = \sqrt{(c - b - abc)/c}$.

The dynamical behaviors of system (2) at S ($\pm m$, - m^2/b, $\mp m/c$) are similar to discussion of the equilibrium point O (0,0,0). They are Hopf bifurcation points.

2.4 A new chaotic attractor of the nonlinear finance chaotic system

When a= 0.00001, b= 0.1, c= 1, we discover a new chaotic attractor of the nonlinear finance chaotic system.

The chaotic attractor is a strange attractor like butterfly wings, and also one similar to Lorenz attractor. The chaotic attractor is shown in Fig.1 when a= 0.00001, b= 0.1, c= 1 . The initial states of system (2) are x(0)=0.1, y(0)=0.23, z(0)=0.31. Using the computer symbolic system *Matlab* Software, the value of Lyapunov exponents of this system is obtained as (0.1735, 0.0015, -0.8030). Because the maximum Lyapunov exponent of this system is positive, it shows that the system is a nonlinear finance chaotic system in theory. On the other hand, because of the parameter a is the saving, when a= 0.00001 become little, the inflation happen, the chaos appear. So this shows that the nonlinear finance system is a chaotic system, no matter in theory, practice or numeral simulation.

We fix parameters a=0.00001, b= 0.1, c= 1, and the time series of $x(t)$, $y(t)$, $z(t)$ are generated by the *Matlab* Software as shown in Fig.2. Also may obviously see from the time series Fig. 2, the nonlinear finance system (2) is a nonlinear finance chaotic system.

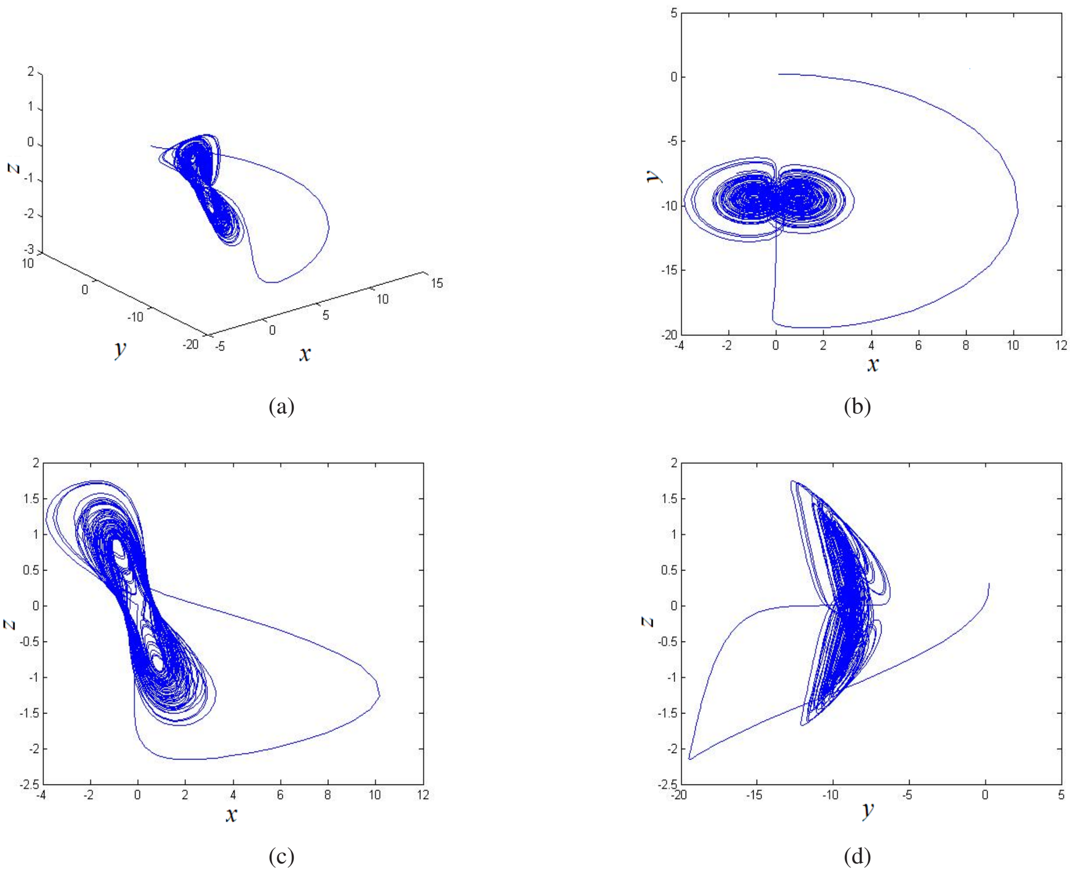


Figure 1: The finance chaotic attractor of system (*a*) *x-y-z*; (*b*) *x-y*; (*c*) *x-z*; (*d*) *y-z*.

3 Slow manifold analysis of the nonlinear finance chaotic system

3.1 The first slow manifold equation of the nonlinear finance chaotic system

Theorem 1 *In the attractive parts of the phase space of the nonlinear finance chaotic system, the first slow manifold equation of the nonlinear finance chaotic system (2) is:*

$$\begin{aligned} &(9.99999x + z + xy)[(\lambda_1(x,y,z) + 1)(\lambda_1(x,y,z) + 0.1)] + \\ &x(\lambda_1(x,y,z) + 0.1)(-0.y - x^2) - (x + z)(\lambda_1(x,y,z) + 0.1) = 0 \end{aligned} \tag{5}$$

where $\lambda_1(x,\, y,\, z)$ *is the left fast eigenvalue of* $J(X)$.

Proof: When parameter a = 0.00001, b= 0.1, c= 1, the nonlinear finance chaotic system can be written as

$$\begin{cases} \dot{x} = 9.99999x + y + xy \\ \dot{y} = -0.1y - x^2 \\ \dot{z} = -x - z \end{cases} \tag{6}$$

For a point $X = (x,\, y,\, z)^T$ in the attractive parts of the phase space, the Jacobian matrix of the nonlinear finance chaotic system at the point $X = (x,\, y,\, z)^T$,

$$J(X) = J(x,y,z) = \begin{bmatrix} 9.99999 + y & x & 1 \\ -2x & -0.1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

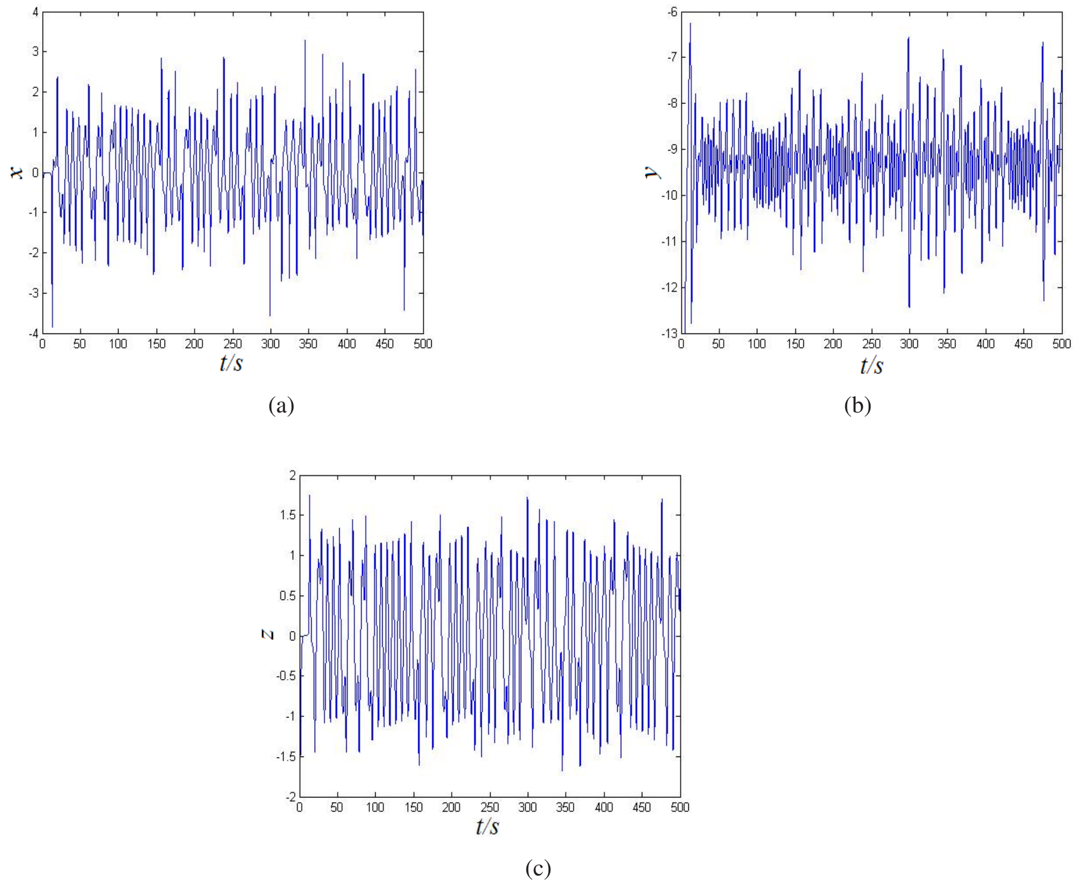


Figure 2: The chaotic time series of trajectory: (*a*) *x-t*; (*b*) *y-t*; (*c*) *z-t*.

Let $\lambda_1(X) = \lambda_1(x,\ y,\ z)$ be the fast eigenvalue and $\lambda_2\ (X)$, $\lambda_3(X)$ the two slow ones. The left fast eigenvector, i.e. the eigenvector of $J^t\ (X)$ (the superscript "t" denote "transpose") is given by

$$z_{\lambda_1}(x,y,z) = \left(\begin{array}{l} (\lambda_1(x,y,z)+1)(\lambda_1(x,y,z)+0.1) \\ x(\lambda_1(x,y,z)+1) \\ \lambda_1(x,y,z)+0.1 \end{array} \right) \tag{7}$$

In the attractive parts of the phase space (i.e. where $J(X)$ have a fast eigenvalue), the equation of the slow manifold is given by

$$z^T_{\lambda_1}(x,y,z) \cdot \left(\begin{array}{c} \dot{x} \\ \dot{y} \\ \dot{z} \end{array} \right) = 0$$

If we replace $z_{\lambda_1}(x,y,z)$ by its expression given by (7) and the velocities by the equation (6), we can obtain the slow manifold equation of the nonlinear finance chaotic system as (5). □

3.2 The second slow manifold equation of the nonlinear finance chaotic system

Theorem 2 *In the attractive parts of the phase space of the nonlinear finance chaotic system, the second slow manifold equation of the nonlinear finance chaotic system (2) is:*

$$F_1(x,y,z)x + F_2(x,y,z)y + F_3(x,y,z)z + 10F_2(x,y,z)x^2 + F_4(x,y,z)xy = 0, \tag{8}$$

where

$F_1(x,y,z) = 9.99999[Re[u_2(x,y,z)](Re[v_3(x,y,z)]-Im[v_2(x,y,z)])-Re[v_2(x,y,z)](Re[u_3(x,y,z)]-Im[u_2(x,y,z)])] + Re[u_2(x,y,z)] - (Re[u_3(x,y,z)] - Im[u_2(x,y,z)])$;

$F_2(x,y,z) = 0.1((Re[v_3(x,y,z)] - Im[v_2(x,y,z)]) - Re[v_2(x,y,z)])$;

$F_3(x,y,z) = [Re[u_2(x,y,z)](Re[v_3(x,y,z)] - Im[v_2(x,y,z)]) - Re[v_2(x,y,z)](Re[u_3(x,y,z)] - Im[u_2(x,y,z)]) + Re[u_2(x,y,z)] - (Re[u_3(x,y,z)] - Im[u_2(x,y,z)])$;

$F_4(x,y,z) = Re[u_2(x,y,z)](Re[v_3(x,y,z)] - Im[v_2(x,y,z)]) - Re[v_2(x,y,z)](Re[u_3(x,y,z)] - Im[u_2(x,y,z)])$;

Proof: For a point $X = (x, y, z)^T$ in the attractive parts of the phase space. Let $\lambda_2\ (X)$, $\lambda_3(X)$ be the two slow eigenvalues of the nonlinear finance chaotic system at the point $X = (x, y, z)^T$. It is easy to show that for $k \in \{2, 3\}$, it is possible to write

$$z_{\lambda_r}(x,y,z) = \begin{bmatrix} 1 \\ u_k(x,y,z) \\ v_k(x,y,z) \end{bmatrix}$$

The equation of the slow manifold can be derived from

$$\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ 1 & u_2(x,y,z) & v_2(x,y,z) \\ 1 & u_3(x,y,z) & v_3(x,y,z) \end{vmatrix} = 0$$

It is known that for some points $X = (x, y, z)^t$, the two eigenvalues $\lambda_2(X)$ and $\lambda_3(X)$ are complex conjugate numbers and so are $z_{\lambda_2}(x,y,z)$ and $z_{\lambda_3}(x,y,z)$ (in fact they are the second and third components). So we have$u_3(x, y, z)$ = ($u_2(x, y, z)$*, $v_3(x, y, z)$ = ($v_2(x, y, z)$*, where "*" denotes complex conjugate operation.

The general case for both real and complex eigenvalues can be combined with a unique equation of the slow manifold as follows

$$\begin{vmatrix} 9.99999x + z + xy & -0.1y - x^2 & -z - x \\ 1 & Re[u_2(x,y,z)] & Re[v_2(x,y,z)] \\ 1 & Re[u_3(x,y,z)] - Im[u_2(x,y,z)] & Re[v_3(x,y,z)] - Im[v_2(x,y,z)] \end{vmatrix} = 0 \quad (9)$$

If we expand equation (9), in terms of these last notations, we would obtain the second slow manifold equation of the nonlinear finance chaotic system as (8). □

It is easy to verify that coordinates of balance points of O (0,0,0) and S ($\pm$ m, - m^2/b, $\mp$ m/c) satisfy slow manifold equation, so they are in slow manifold. The slow manifold equation present a formula among x(rate of interest), y(investment coefficient), z(resilience cofficient), so we can further research the internal and interdependent relation among them with the aid of slow manifold equation.

4 Adaptive control of the nonlinear finance chaotic system

In this section we will control the nonlinear finance chaotic system(2) to equilibrium point (0,0,0). When parameter a= 0.00001, b= 0.1, c= 1, the nonlinear finance chaotic system (2) has a new chaotic attractor that is a strange attractor like butterfly wings, and also one similar to Lorenz attractor.

Lemma 1 ***Barbalat Lemma***. *If* $f(t) \in L_2 \cup \quad L_\infty$ *and* $\dot{f}(t) \in L_\infty$,*then* $\lim_{t\to\infty} f(t) = 0$.

Theorem 3 *Consider the controlled system as follows:*

$$\begin{cases} \dot{x} = (1/b - a)\,x + z + xy - \hat{k}x \\ \dot{y} = -by - x^2 \\ \dot{z} = -x - cz \end{cases} \quad (10)$$

where $\hat{k} = \beta x^2, \beta > 0$*, then system(10) will converge to the equilibrium point (0,0,0).*

Proof: the Jacobi matrix of system (10) is:

$$J = \begin{pmatrix} 1/b - a - \hat{k} & 0 & 1 \\ 0 & -b & 0 \\ -1 & 0 & -c \end{pmatrix}$$

Construct the Lyapunov function :

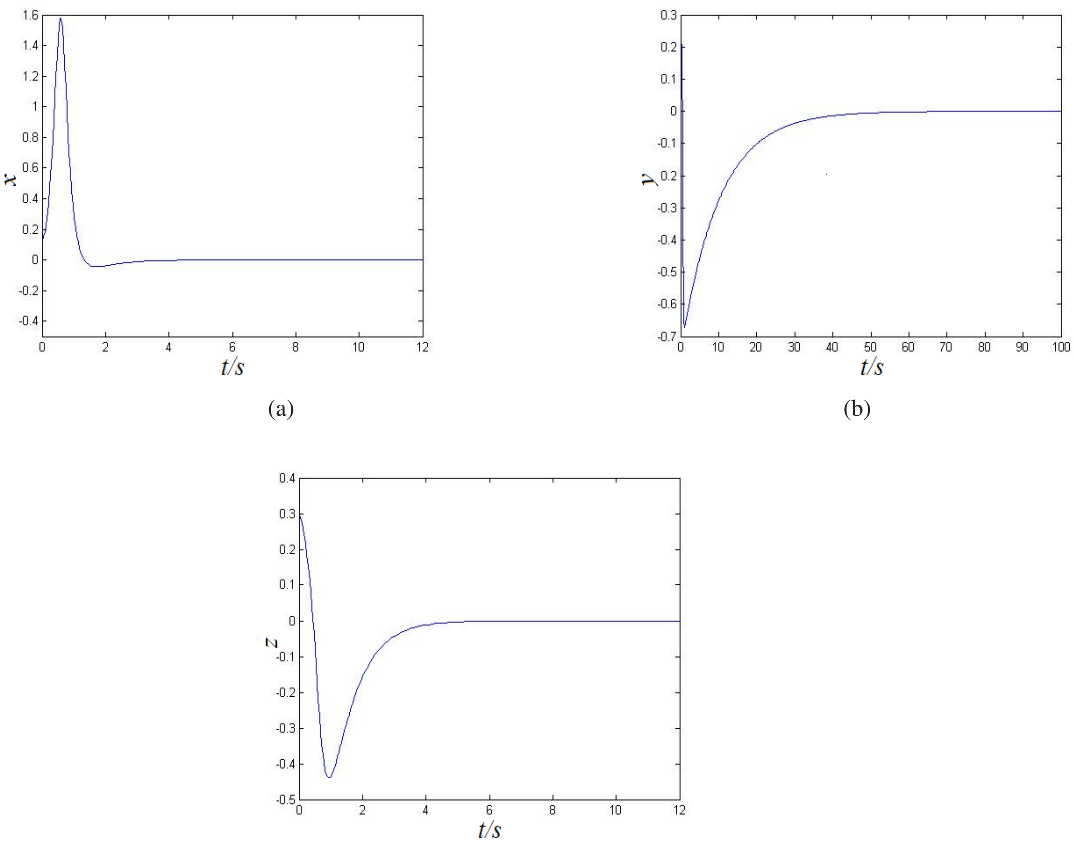
$$V(x, y, z) = \frac{1}{2}(x^2 + y^2 + z^2) + \frac{1}{2\beta}(\hat{k} - k^*)^2$$

The time derivative of V along trajectories (10) is

$$\dot{V} = x\dot{x} + y\dot{y} + z\dot{z} + (\hat{k} - k^*)x^2 = -(k^* - (1/b - a))x^2 - by^2 - cz^2$$

When $k^* > 1/b - a$, $\dot{V} \leq 0$, $x, y, z, \dot{k} - k^* \in L_\infty$,$\int_0^t \dot{V} \leq 0$ and $x, y, z \in L_2$,$\dot{x}, \dot{y}, \dot{z} \in L_\infty$.According to BarBalat Lemma, x,y,$z$will gradually converge to zero which is negative definite. According to the Lyapunov stable theorem, the trajector of the controlled system is asmptotically stabilized to the equilibrium point (0,0,0). The proof is thus completed. □

Numerical experiment is carried out to inegrate the controlled system (10) by using fourth-order Runge-Kutta method with time step 0.001. It has been proved that nonlinear finance chaotic system (2) has chaos behavior when$a$= 0.00001, b= 0.1, c= 1. The initial states of the controlled system (10) are x(0)=0.1, y(0)=0.23, z(0)=0.31. Feedback gain β=10, Fig .3 (*a*)-(*c*) shows that the nonlinear finance chaotic system is controlled to equilibrium point (0,0,0).



(a) (b) (c)

Figure 3: The states of controlled system (10): (*a*) *x-t*; (*b*) *y-t*; (*c*) *z-t*.

Chaotic control idea in the section indicates that it is an effective way to adjust and control *x* (interest rate) when inflation occurs and chaotic phenomenon appears in nonlinear finance chaotic system. It has been proved to be an effective measure of reviving economy and it is frequently used in a lot of countries.

5 Conclusion

This paper investigates the dynamical behavior and slow manifold of the nonlinear finance chaotic system. the complicated dynamical behaviors of the system are analyzed by studied dynamical behaviors of the subsystem of the system, the slow manifold equations of the system are obtained with different methods. We stable the nonlinear finance chaotic system to equilibrium point with the adaptive control method. Theoretical analysis and numerical simulations are given to validate the control approach. The results of and simulation are helpful for better understanding of other similar nonlinear finance chaotic systems.

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