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A New Form of Bäcklund Transformations and Its Relation to the Inverse Scattering Problem

Ryogo HIROTA

RCA Research Laboratories, Inc.
*971-2 Aza 4-go, Zushi-machi, Machida-City, Tokyo 194-02**)

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We present a new form of Bäcklund transformations which gives a unified method of finding Bäcklund transformations for solutions of certain nonlinear evolution equations and constructing the inverse scattering problem for solving the initial-value problem. We consider the sine-Gordon equation, the modified Korteweg-de Vries equation and the Korteweg-de Vries equation as examples.

§ 1. Introduction

Recent advances in the theory of solitons[1] have shown that multi-soliton solutions of certain nonlinear evolution equations can be obtained by three different methods; the inverse scattering method,[2)~4)] the Bäcklund transformation[5),6)] and the method[7),8)] found by the present author. The inverse scattering method is a well-developed mathematical theory which provides a means of solving the initial-value problem for a broad class of nonlinear evolution equations.[9] However, it is very difficult to set up an appropriate inverse scattering problem which depends seemingly on the existence of an infinite number of independent conservation laws for the evolution equation. The Bäcklund transformation, which is used in connection with a transformation problem in the differential geometry, also prevides a means of finding multi-soliton solutions of some nonlinear equations[5] and gives an infinite number of independent conservation laws.[1] The method found by the author is rather heuristic, but it gives multi-soliton solutions for a wide class of nonlinear equations in a unified way. We have used the dependent variable transformations which provide simplifications of many nonlinear equations having multi-soliton solutions. The simplified equations have, in general, the following special form:

$$F\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}, \frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)a(x,t)\,b(x',t')_{\text{at } x=x',t=t'}=0\,. \tag{1·1}$$

We have investigated properties of this type of equation and found Bäcklund transformations for solutions of some equations of this type. The Bäcklung trans-

*) New Address (after April 1, 1974): Department of Mathematics and Physics, Faculty of Science and Engineering, Ritsumeikan University, Kitamachi, Tojiin, Kitaku, Kyoto.

formations obtained are linear partial differential equations, which are equivalent to the inverse scattering operators for the original nonlinear equations. Hence, our method provides, in a unified way, a means of finding inverse scattering problems for a wide class of nonlinear evolution equations. We consider the sine-Gordon equation, the modified Korteweg-de Vries (K-dV) equation, and the Korteweg-de Vries (K-dV) equation, as examples.

§ 2. Properties of new operator

In previous papers,[7),8)] we transformed certain nonlinear evolution equations into the following type of differential equations:

$$\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)\left[\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)+\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^3\right]a(x,t)a(x',t')_{\text{at } x=x',t=t'}=0\,, \tag{2·1}$$

$$\left[\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)+\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^3\right]a(x,t)\,b(x',t')_{\text{at } x=x',t=t'}=0\,, \tag{2·2}$$

$$\left\{\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^2-4\sinh^2\left[\left(\frac{\partial}{\partial n}-\frac{\partial}{\partial n'}\right)\Big/2\right]\right\}a(n,t)\,b(n',t')_{\text{at } n=n',t=t'}=0\,, \tag{2·3}$$

etc.

For convenience, we define new operators D_x, D_t, D_n, etc. by

$$D_x{}^m a(x)\cdot b(x)\equiv\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m a(x)\,b(x')_{\text{at } x=x'}, \quad (m \text{ is an integer}) \tag{2·4}$$

$$D_t{}^m a(t)\cdot b(t)\equiv\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^m a(t)\,b(t')_{\text{at } t=t'} \quad (m \text{ is an integer}) \tag{2·5}$$

and

$$\sinh(D_n/2)\,a(n)\cdot b(n)\equiv\sinh\left[\left(\frac{\partial}{\partial n}-\frac{\partial}{\partial n'}\right)\Big/2\right]a(n)\,b(n')_{\text{at } n=n'}\,. \tag{2·6}$$

With this notation, Eqs. (2·1), (2·2) and (2·3) are expressed as

$$D_x(D_t+D_x{}^3)\,a\cdot a=0\,, \tag{2·7}$$

$$(D_t+D_x{}^3)\,a\cdot b=0 \tag{2·8}$$

and

$$[D_t{}^2-4\sinh^2(D_n/2)\,]\,a\cdot b=0\,. \tag{2·9}$$

We list some properties of this operator which are used in the following sections:

(I) $D_x{}^m a\cdot b=(-1)^m D_x{}^m b\cdot a\,,$

(I-1) $D_x{}^m a\cdot a=0$ for odd $m\,,$

(I-2) $\exp(\varepsilon D_x)\,a(x)\cdot b(x)=a(x+\varepsilon)\,b(x-\varepsilon)$ for a constant $\varepsilon\,.$

Some relations to the usual differential operators are listed below:

(III) $$[\exp(\varepsilon\partial/\partial x)+\exp(-\varepsilon\partial/\partial x)]\log f(x)=\log[\exp(\varepsilon D_x)f(x)\cdot f(x)],$$

(III-1) $$\frac{\partial^2}{\partial x^2}\log f=\frac{1}{2f^2}(D_x^2 f\cdot f),$$

(III-2) $$\frac{\partial^4}{\partial x^4}\log f=\frac{1}{2f^2}(D_x^4 f\cdot f)-6\left[\frac{1}{2f^2}(D_x^2 f\cdot f)\right]^2,$$

(IV) $$\exp(\varepsilon\partial/\partial x)\log(f+ig)+\exp(-\varepsilon\partial/\partial x)\log(f\pm ig)$$
$$=\log[\exp(\varepsilon D_x)(f+ig)\cdot(f\pm ig)].$$

Let $\tan\phi=g/f$, then we obtain

(IV-1) $$\frac{\partial\phi}{\partial x}=\frac{D_x g\cdot f}{f^2+g^2},$$

(IV-2) $$\frac{\partial^2\phi}{\partial x^2}=\frac{(D_x^2 g\cdot f)(f^2-g^2)-gf[D_x^2(f\cdot f-g\cdot g)]}{(f^2+g^2)^2},$$

(IV-3) $$\frac{\partial^3\phi}{\partial x^3}=\frac{D_x^3 g\cdot f}{f^2+g^2}-3\frac{(D_x g\cdot f)[D_x^2(f\cdot f+g\cdot g)]}{(f^2+g^2)^2}-8\left[\frac{D_x g\cdot f}{f^2+g^2}\right]^3,$$

(IV-4) $$\frac{\partial^2}{\partial x^2}\log(f+ig)=\frac{1}{2(f+ig)^2}D_x^2(f+ig)\cdot(f+ig),$$

(IV-5) $$\frac{\partial^2}{\partial x^2}\log(f^2+g^2)=\frac{D_x^2(f\cdot f+g\cdot g)}{f^2+g^2}+4\left(\frac{D_x g\cdot f}{f^2+g^2}\right)^2.$$

(V) $$\exp(\varepsilon D_x)a(x)b(x)\cdot c(x)=[\exp(\varepsilon\partial/\partial x)a(x)][\exp(\varepsilon D_x)b(x)\cdot c(x)],$$

(V-1) $$D_x ab\cdot c=(\partial a/\partial x)bc+a(D_x b\cdot c),$$

(V-2) $$D_x^2 ab\cdot c=(\partial^2 a/\partial x^2)bc+2(\partial a/\partial x)(D_x b\cdot c)+a(D_x^2 b\cdot c)$$

and

(V-3) $$D_x^3 ab\cdot c=(\partial^3 a/\partial x^3)bc+3(\partial^2 a/\partial x^2)(D_x b\cdot c)$$
$$+3(\partial a/\partial x)(D_x^2 b\cdot c)+a(D_x^3 b\cdot c).$$

The following properties of the operators are of special importance in deriving a new form of the Bäcklund transformation:

(VI) $$\exp(\varepsilon D_t)[\exp(\delta D_x)a(x,t)\cdot b(x,t)]\cdot[\exp(\delta D_x)c(x,t)\cdot d(x,t)]$$
$$=[\exp(\delta D_x+\varepsilon D_t)a(x,t)\cdot d(x,t)][\exp(\delta D_x-\varepsilon D_t)c(x,t)\cdot b(x,t)],$$

(VI-1) $$D_t ab\cdot cd=(D_t a\cdot d)cb-ad(D_t c\cdot b),$$

(VI-2) $$(D_t a\cdot b)cd-ab(D_t c\cdot d)=(D_t a\cdot c)bd-ac(D_t b\cdot d),$$

(VI-3) $$D_t[(D_x a\cdot b)\cdot cd+ab\cdot(D_x c\cdot d)]=(D_t D_x a\cdot d)cb-ad(D_t D_x c\cdot b)$$
$$+(D_t a\cdot d)(D_x c\cdot b)-(D_x a\cdot d)(D_t c\cdot b).$$

(VII) $\{\exp[(\varepsilon+\delta)D_x]a\cdot b\}\{\exp[(\varepsilon-\delta)D_x c\cdot d]\}$

$$=\exp(\delta D_x)[\exp(\varepsilon D_x)a\cdot d]\cdot[\exp(\varepsilon D_x)c\cdot b],$$

(VII-1) $(D_x^2 a\cdot b)cd-ab(D_x^2 c\cdot d)=D_x\{(D_x a\cdot d)\cdot cb+ad\cdot(D_x c\cdot b)\},$

(VII-2) $(D_x^4 a\cdot a)cc-aa(D_x^4 c\cdot c)=2D_x(D_x^3 a\cdot c)\cdot ca+6D_x(D_x^2 a\cdot c)\cdot(D_x c\cdot a).$

(VIII) $[\exp(\delta D_x)a\cdot b]cd+ab[\exp(\delta D_x)c\cdot d]$

$$=[\exp(\delta D_x)a\cdot d]cb+ad[\exp(\delta D_x)c\cdot b]$$
$$-\exp(\delta D_x/2)[2\sinh(\delta D_x/2)a\cdot c][2\sinh(\delta D_x/2)b\cdot d],$$

(VIII-1) $(D_x a\cdot b)cd+ab(D_x c\cdot d)=(D_x a\cdot d)cb+ad(D_x c\cdot b),$

(VIII-2) $(D_x^2 a\cdot b)cd+ab(D_x^2 c\cdot d)$

$$=(D_x^2 a\cdot d)cb+ad(D_x^2 c\cdot b)-2(D_x a\cdot c)(D_x b\cdot d),$$

(VIII-3) $(D_x^3 a\cdot b)cd+ab(D_x^3 c\cdot d)$

$$=(D_x^3 a\cdot d)cb+ad(D_x^3 c\cdot b)-3D_x(D_x a\cdot c)\cdot(D_x b\cdot d).$$

All of these properties can be verified very simply, so that we will verify only Eqs. (IV), (VI) and (VIII) which are seemingly complicated.

For the case of Eq. (IV), we have

$$\exp(\varepsilon\partial/\partial x)\log(f(x)+ig(x))+\exp(-\varepsilon\partial/\partial x)\log(f(x)\pm ig(x))$$
$$=\log(f(x+\varepsilon)+ig(x+\varepsilon))+\log(f(x-\varepsilon)\pm ig(x-\varepsilon)). \qquad (2\cdot 10)$$

On the other hand, from Eq. (I-2) we have

$$\log[\exp(\varepsilon D_x)(f(x)+ig(x))\cdot(f(x)\pm ig(x))]$$
$$=\log[(f(x+\varepsilon)+ig(x+\varepsilon))(f(x-\varepsilon)\pm ig(x-\varepsilon))], \qquad (2\cdot 11)$$

which is equal to Eq. (2·10), and hence Eq. (IV) is proved. Equations (IV-1)～(IV-5) are obtained from Eq. (IV) by expanding it in power series of ε.

For the case of Eq. (VI), we have, from Eq. (II-2)

$$\exp(\varepsilon D_t)[\exp(\delta D_x)a(x,t)\cdot b(x,t)]\cdot[\exp(\delta D_x)c(x,t)\cdot d(x,t)]$$
$$=a(x+\delta,t+\varepsilon)b(x-\delta,t+\varepsilon)c(x+\delta,t-\varepsilon)d(x-\delta,t-\varepsilon). \qquad (2\cdot 12)$$

On the other hand, the right-hand side of Eq. (VI) becomes

$$[\exp(\delta D_x+\varepsilon D_t)a(x,t)\cdot d(x,t)][\exp(\delta D_x-\varepsilon D_t)c(x,t)\cdot b(x,t)]$$
$$=a(x+\delta,t+\varepsilon)d(x-\delta,t-\varepsilon)c(x+\delta,t-\varepsilon)b(x-\delta,t+\varepsilon), \qquad (2\cdot 13)$$

which is equal to Eq. (2·12). Equations (VI-1)～(VI-3) are obtained from Eq. (VI).

For the case of Eq. (VIII), we have

$$[\exp(\delta D_x)a(x)\cdot b(x)]c(x)d(x)+a(x)b(x)[\exp(\delta D_x)c(x)\cdot d(x)]$$

$$=a(x+\delta)b(x-\delta)c(x)d(x)+a(x)b(x)c(x+\delta)d(x-\delta)$$
$$=a(x+\delta)d(x-\delta)c(x)b(x)+a(x)d(x)c(x+\delta)b(x-\delta)$$
$$-[a(x+\delta)c(x)-a(x)c(x+\delta)][b(x)d(x-\delta)-b(x-\delta)d(x)]$$
$$=a(x+\delta)d(x-\delta)c(x)b(x)+a(x)d(x)c(x+\delta)b(x-\delta)$$
$$-\exp(\delta D_x/2)[a(x+\delta/2)c(x-\delta/2)-a(x-\delta/2)c(x+\delta/2)]$$
$$\times[b(x+\delta/2)d(x-\delta/2)-b(x-\delta/2)d(x+\delta/2)],$$

which is equal to the right-hand side of Eq. (VIII). Equations (VIII-1) ~ (VIII-3) are obtained from Eq. (VIII).

§ 3. New forms of the Bäcklund transformations for the solution of the sine-Gordon equation, the modified K-dV equation and the K-dV equation

In this section, we transform the sine-Gordon equation, the modified K-dV equation and the K-dV equation into the type of Eq. (1·1), using the new operators defined in the previous section, and find Bäcklund transformations in new forms for solutions of these equations. From them we derive the usual forms of Bäcklund transformations for solutions of the original equations.

(a) *The sine-Gordon equation*

We transform the sine-Gordon equation[10]

$$\left(\frac{\partial^2}{\partial X^2}-\frac{\partial^2}{\partial T^2}\right)\varphi=\sin\varphi\,, \tag{3·1}$$

using the relation

$$\varphi(X,T)=4\tan^{-1}[g(X,T)/f(X,T)]. \tag{3·2}$$

From Eq. (IV-2) we have

$$\left(\frac{\partial^2}{\partial X^2}-\frac{\partial^2}{\partial T^2}\right)\left(\frac{\varphi}{4}\right)=\frac{[(D_X^2-D_T^2)g\cdot f](f^2-g^2)-gf[(D_X^2-D_T^2)(f\cdot f-g\cdot g)]}{(f^2+g^2)^2}. \tag{3·3}$$

On the other hand, from Eq. (3·2) we have

$$\sin\varphi=\frac{4gf(f^2-g^2)}{(f^2+g^2)^2}. \tag{3·4}$$

Hence, $\varphi(x,t)$ defined by Eq. (3·2) is a solution of the sine-Gordon equation provided that g and f satisfy the following equations:

$$(D_X^2-D_T^2)g\cdot f=gf \tag{3·5}$$

and

$$(D_X^2-D_T^2)(f\cdot f-g\cdot g)=0, \tag{3·6}$$

which are the same equations as those obtained in the previous paper.

For convenience, we write the sine-Gordon equation (3·1) as

$$\varphi_{xt}=\sin\varphi\,, \tag{3·7}$$

where subscript indicates the partial differentiation, by the use of the transformation

$$x=\tfrac{1}{2}(X+T),\qquad t=\tfrac{1}{2}(X-T). \tag{3·8}$$

Equations (3·5) and (3·6) are, then, written as

$$D_tD_x g\cdot f=gf\,, \tag{3·9}$$

$$D_tD_x(f\cdot f-g\cdot g)=0\,, \tag{3·10}$$

which are combined as one equation,

$$D_tD_x(f+ig)\cdot(f+ig)=2ifg\,. \tag{3·11}$$

Now we have a Bäcklund transformation for solutions of Eqs. (3·9) and (3·10): Let a pair of g and f be any solution of Eqs. (3·9) and (3·10). A pair of different solution g' and f' is then defined by a Bäcklund transformation,

$$D_x(f'+ig')\cdot(f+ig)=-(a/2)(f'-ig')(f-ig), \tag{3·12}$$

$$D_x(f'-ig')\cdot(f-ig)=-(a/2)(f'+ig')(f+ig), \tag{3·13}$$

$$D_t(f'+ig')\cdot(f-ig)=-(1/2a)(f'-ig')(f+ig) \tag{3·14}$$

and

$$D_t(f'-ig')\cdot(f+ig)=-(1/2a)(f'+ig')(f-ig), \tag{3·15}$$

where a is an arbitrary parameter. We note that if a, f, f', g and g' are real, Eqs. (3·13) and (3·15) are redundant, because they are complex conjugate of Eqs. (3·12) and (3·14), respectively.

We will verify that g' and f' defined by Eqs. (3·12)$\sim$(3·15) satisfy Eq. (3·11).

By the use of Eqs. (VI-3) and (I-1), we have

$$\begin{aligned}&[D_tD_x(f'+ig')\cdot(f'+ig')](f+ig)^2-(f'+ig')^2[D_tD_x(f+ig)\cdot(f+ig)]\\&\quad=D_t\{[D_x(f'+ig')\cdot(f+ig)]\cdot(f+ig)(f'+ig')\\&\qquad+(f'+ig')(f+ig)\cdot[D_x(f+ig)\cdot(f'+ig')]\},\end{aligned}$$

which is converted, by means of Eqs. (3·12), (VI-1) and (I), to

$$\begin{aligned}&-a\{[D_t(f'-ig')\cdot(f+ig)](f'+ig')(f-ig)\\&\qquad-(f'-ig')(f+ig)[D_t(f'+ig')\cdot(f-ig)]\},\end{aligned}$$

and reduces, by the use of Eqs. (3·14) and (3·15), to

$$\begin{aligned}&(1/2)(f'+ig')(f-ig)(f'+ig')(f-ig)-(1/2)(f'-ig')(f+ig)(f'-ig')(f+ig)\\&\quad=2if'g'(f+ig)^2-(f'+ig')^2 2ifg\,,\end{aligned}$$

verifying that g' and f' satisfy Eq. (3·11) provided that g and f satisfy the same eqation.

We show that the present form of Bäcklund transformation equations (3·12) ~(3·15) leads to the well-known form of Bäcklund transformation:[6]

By the use of Eqs. (VI-2), (3·12), and (3·13), we have

$$\begin{aligned}&[D_x(f'+ig')\cdot(f'-ig')](f+ig)(f-ig)\\&\qquad -(f'+ig')(f'-ig')[D_x(f+ig)\cdot(f-ig)]\\&= -(a/2)[(f'-ig')^2(f-ig)^2-(f'+ig')^2(f+ig)^2]\\&=a[((f')^2-(g')^2)(2igf)+2ig'f'(f^2-g^2)],\end{aligned}$$

which yields one of the Bäcklund transformations

$$(\varphi_x'-\varphi_x)/2=a\sin[(\varphi'+\varphi)/2] \tag{3·16}$$

by means of the relations

$$\frac{i}{2}\frac{\partial\varphi'}{\partial x}=\frac{D_x(f'+ig')\cdot(f'-ig')}{(f'+ig')(f'-ig')}, \tag{3·17}$$

$$\frac{i}{2}\frac{\partial\varphi}{\partial x}=\frac{D_x(f+ig)\cdot(f-ig)}{(f+ig)(f-ig)}, \tag{3·18}$$

$$\sin(\varphi'/2)=\frac{2g'f'}{(f'+ig')(f'-ig')}, \tag{3·19}$$

$$\cos(\varphi'/2)=\frac{(f')^2-(g')^2}{(f'+ig')(f'-ig')}, \tag{3·20}$$

$$\sin(\varphi/2)=\frac{2gf}{(f+ig)(f-ig)} \tag{3·21}$$

and

$$\cos(\varphi/2)=\frac{f^2-g^2}{(f+ig)(f-ig)}. \tag{3·22}$$

Similarly we have, by the use of Eqs. (VI-2), (3·14) and (3·15),

$$\begin{aligned}&[D_t(f'+ig')\cdot(f'-ig')](f-ig)(f+ig)-(f'+ig')(f'-ig')[D_t(f-ig)\cdot(f+ig)]\\&= -(1/2a)[(f'-ig')^2(f+ig)^2-(f'+ig')^2(f-ig)^2],\end{aligned}$$

which yields the Bäcklund transformation

$$(\varphi_t'+\varphi_t)/2=a^{-1}\sin[(\varphi'-\varphi)/2]. \tag{3·23}$$

(b) *The modified K-dV equation*

We transform the modified K-dV equation[11)]

$$v_t+24v^2v_x+v_{xxx}=0, \tag{3·24}$$

using the relations

$$v=\phi_x \tag{3·25}$$

and

$$\tan\phi=g/f. \tag{3·26}$$

Integrating Eq. (3·24) with respect to x, we have

$$\phi_t+8\phi_x{}^3+\phi_{xxx}=0, \tag{3·27}$$

where we have chosen an integration constant to be zero.

By the use of Eqs. (IV-1) and (IV-3), Eq. (3·27) is transformed into

$$[(D_t+D_x{}^3)g\cdot f]-3(D_xg\cdot f)[D_x{}^2(f\cdot f+g\cdot g)](f^2+g^2)^{-2}=0,$$

which shows that $\phi(x,t)$ defined by Eq. (3·26) is a solution of Eq. (3·27), hence, $v(x,t)$ defined by Eq. (3·25) is a solution of the modified K-dV equation (3·24) provided that g and f satisfy the following equations:

$$(D_t+D_x{}^3)g\cdot f=0 \tag{3·28}$$

and

$$D_x{}^2(f\cdot f+g\cdot g)=0, \tag{3·29}$$

which are rewritten as

$$(D_t+D_x{}^3)(f+ig)\cdot(f-ig)=0 \tag{3·30}$$

and

$$D_x{}^2(f+ig)\cdot(f-ig)=0. \tag{3·31}$$

Here we have used the same letters g and f as those used for the case of the sine-Gordon equation, in order to emphasize the similarity in the procedures for both cases.

We have a Bäcklund transformation for solutions of Eqs. (3·30) and (3·31). Let a pair, g and f, be any solution of Eqs. (3·30) and (3·31). A different solution pair, g' and f', is then defined by a Bäcklund transformation:

$$(D_t+3\lambda^2D_x+D_x{}^3)(f'+ig')\cdot(f+ig)=0, \tag{3·32}$$

$$(D_t+3\lambda^2D_x+D_x{}^3)(f'-ig')\cdot(f-ig)=0, \tag{3·33}$$

$$D_x(f'+ig')\cdot(f-ig)=-\lambda(f'-ig')(f+ig) \tag{3·34}$$

and

$$D_x(f'-ig')\cdot(f+ig)=-\lambda(f'+ig')(f-ig), \tag{3·35}$$

where λ is an arbitrary parameter. We note that if λ, f, f', g and g' are real, Eqs. (3·33) and (3·35) are redundant. They are complex conjugate of Eqs. (3·32) and (3·34), respectively.

We will verify that g' and f' defined by Eqs. (3·32)～(3·35) satisfy Eqs. (3·30) and (3·31).

By the use of Eqs. (VIII-1) and (VIII-3), we have

$$\begin{aligned}&[(D_t+D_x^3)(f'+ig')\cdot(f'-ig')](f-ig)(f+ig)\\&\quad+(f'+ig')(f'-ig')[(D_t+D_x^3)(f-ig)\cdot(f+ig)]\\&=[(D_t+D_x^3)(f'+ig')\cdot(f+ig)](f-ig)(f'-ig')\\&\quad+(f'+ig')(f+ig)[(D_t+D_x^3)(f-ig)\cdot(f'-ig')]\\&\quad-3D_x[D_x(f'+ig')\cdot(f-ig)]\cdot[D_x(f'-ig')\cdot(f+ig)].\end{aligned}$$

By the use of Eqs. (3·34) and (3·35), the last term is converted to

$$\begin{aligned}&-3\lambda^2D_x(f'-ig')(f+ig)\cdot(f'+ig')(f-ig)\\&=3\lambda^2\{[D_x(f'+ig')\cdot(f+ig)](f'-ig')(f-ig)\\&\quad+(f'+ig')(f+ig)[D_x(f-ig)\cdot(f'-ig')]\},\end{aligned}$$

where we have used Eqs. (I) and (VI-1). Hence we have

$$\begin{aligned}&[(D_t+D_x^3)(f'+ig')\cdot(f'-ig')](f-ig)(f+ig)\\&\quad+(f'+ig')(f'-ig')[(D_t+D_x^3)(f-ig)\cdot(f+ig)]\\&=[(D_t+3\lambda^2D_x+D_x^3)(f'+ig')\cdot(f+ig)](f-ig)(f'-ig')\\&\quad+(f'+ig')(f+ig)[(D_t+3\lambda^2D_x+D_x^3)(f-ig)\cdot(f'-ig')],\end{aligned}$$

which vanishes by virtue of Eqs. (3·32) and (3·33).

Thus, we have verified that g' and f' satisfy Eq. (3·30) provided that g and f satisfy the same equation. We have completed the first half of the proof.

By the use of Eq. (VII-1), we have

$$\begin{aligned}&[D_x^2(f'+ig')\cdot(f'-ig')](f+ig)(f-ig)\\&\quad-(f'+ig')(f'-ig')[D_x^2(f+ig)\cdot(f-ig)]\\&=D_x\{[D_x(f'+ig')\cdot(f-ig)]\cdot(f+ig)(f'-ig')\\&\quad+(f'+ig')(f-ig)\cdot[D_x(f+ig)\cdot(f'-ig')]\},\end{aligned}$$

which vanishes by virtue of Eqs. (3·34), (3·35) and (I-1).

Thus, we have verified that g' and f' satisfy Eq. (3·31) provided that g and f satisfy the same equation.

Here we note that Eqs. (3·34) and (3·35) lead to the following relation:

$$D_x^2(f'+ig')\cdot(f+ig)=\lambda^2(f'+ig')(f+ig). \qquad (3\cdot36)$$

Equation (3·36) is obtained as follows:

By the use of Eq. (VII-1), we have

$$[D_x^2(f'+ig')\cdot(f+ig)](f'-ig')(f-ig)$$
$$-(f'+ig')(f+ig)[D_x^2(f'-ig')\cdot(f-ig)]$$
$$=D_x\{[D_x(f'+ig')\cdot(f-ig)]\cdot(f'-ig')(f+ig)$$
$$+(f'+ig')(f-ig)\cdot[D_x(f'-ig')\cdot(f+ig)]\},$$

which vanishes by virtue of Eqs. (3·34), (3·35) and (I-1).

On the other hand, by the use of Eq. (VIII-2), we have

$$[D_x^2(f'+ig')\cdot(f+ig)](f-ig)(f'-ig')$$
$$+(f'+ig')(f+ig)[D_x^2(f-ig)\cdot(f'-ig')]$$
$$=[D_x^2(f'+ig')\cdot(f'-ig')](f-ig)(f+ig)$$
$$+(f'+ig')(f'-ig')[D_x^2(f-ig)\cdot(f+ig)]$$
$$-2[D_x(f'+ig')\cdot(f-ig)][D_x(f+ig)\cdot(f'-ig')],$$

where the first two terms vanish, and we have

$$[D_x^2(f'+ig')\cdot(f+ig)](f-ig)(f'-ig')$$
$$+(f'+ig')(f+ig)[D_x^2(f-ig)\cdot(f'-ig')]$$
$$=2\lambda^2(f'-ig')(f+ig)(f-ig)(f'+ig')$$

by virtue of Eqs. (3·34) and (3·35). Hence, we have

$$D_x^2(f'+ig')\cdot(f+ig)=\lambda^2(f'+ig')(f+ig).$$

Following the same procedure as before, we show that Eqs. (3·32) ~ (3·35) lead to the usual form of Bäcklund transformation:

By the use of Eqs. (VI-2), (3·34) and (3·35), we have

$$[D_x(f'+ig')\cdot(f'-ig')](f-ig)(f+ig)$$
$$-(f'+ig')(f'-ig')[D_x(f-ig)\cdot(f+ig)]$$
$$=-\lambda[((f')^2-(g')^2)(4igf)-4ig'f'(f^2-g^2)],$$

which yields a Bäcklund transformation

$$\phi_x'+\phi_x=\lambda\sin(2\phi'-2\phi) \tag{3·37}$$

by the use of the relations

$$\phi=\tan^{-1}(g/f)$$

and

$$\phi'=\tan^{-1}(g'/f').$$

Differentiating Eq. (3·37) twice with respect to x and by means of Eq. (3·27), we have

$$\phi_t'+\phi_t=-8\lambda(\phi_x')^2\sin(2\phi'-2\phi)-4\lambda^2(\phi_x'-\phi_x)\cos(4\phi'-4\phi)+4\lambda\phi_{xx}\cos(2\phi'-2\phi). \tag{3·38}$$

Equations (3·37) and (3·38) form the Bäcklund transformation for solutions of the modified K-dV equation.

(c) *The K-dV Equation*

We transform the K-dV equation[12)]

$$u_t+6uu_x+U_{xxx}=0\,, \tag{3·39}$$

using the relation

$$u=2\frac{\partial^2}{\partial x^2}\log f\,. \tag{3·40}$$

Integrating Eq. (3·39) with respect to x, we have

$$2\frac{\partial^2}{\partial t\partial x}\log f+3\left(2\frac{\partial^2}{\partial x^2}\log f\right)^2+2\frac{\partial^4}{\partial x^4}\log f=0\,, \tag{3·41}$$

where we have chosen an integration constant to be zero.

By the use of Eqs. (III-1) and (III-2), Eq. (3·41) is transformed into

$$D_x(D_t+D_x^3)f\cdot f=0\,. \tag{3·42}$$

We have a Bäcklund transformation for solutions of Eq. (3·42). Let f be any solution of Eq. (3·42). A different solution f' is then defined by a Bäcklund transformation:

$$(D_t+3k^2D_x+D_x^3)f'\cdot f=0 \tag{3·43}$$

and

$$D_x^2f'\cdot f=k^2f'f\,, \tag{3·44}$$

where k is an arbitrary parameter. These equations have the same form as Eqs. (3·32) and (3·36), respectively. Following a similar procedure as before, we will prove that f' defined by Eqs. (3·43) and (3·44) satisfies Eqs. (3·42).

By means of Eqs. (VI-3) and (VII-2), we have

$$[D_x(D_t+D_x^3)f\cdot f]f'f'-ff[D_x(D_t+D_x^3)f'\cdot f']$$
$$=2D_x[(D_t+D_x^3)f\cdot f']\cdot f'f+6D_x(D_x^2f\cdot f')\cdot(D_xf'\cdot f)\,.$$

By the use of Eq. (3·44), the last term is converted to

$$6k^2D_x(D_xf\cdot f')\cdot f'f\,.$$

Hence, we have

$$[D_x(D_t+D_x^3)f\cdot f]f'f'-ff[D_x(D_t+D_x^3)f'\cdot f']$$
$$=2D_x[(D_t+3k^2D_x+D_x^3)f\cdot f']\cdot ff',$$

which vanishes by virtue of Eq. (3·43). Thus, we have verified that f' satisfies Eq. (3·42) provided that f satisfies the same equation.

We will show that Eqs. (3·43) and (3·44) lead to the Bäcklund transformation obtained by Wahlquist and Estabrook.[13]

By means of Eq. (VIII-2), we have

$$(D_x^2 f'\cdot f')ff+f'f'(D_x^2 f\cdot f)$$
$$=(D_x^2 f'\cdot f)ff'+f'f(D_x^2 f\cdot f')-2(D_x f'\cdot f)(D_x f'\cdot f).$$

Dividing this equation by a factor $2(f'f)^2$ and by the use of Eqs. (3·44) and (III-1), we have

$$w_x'+w_x=-k^2+(w'-w)^2, \tag{3·45}$$

where w' and w are defined by

$$w'=-\frac{\partial}{\partial x}\log f' \tag{3·46}$$

and

$$w=-\frac{\partial}{\partial x}\log f, \tag{3·47}$$

and satisfy the differential equations

$$w_t'-6(w_x')^2+w'_{xxx}=0 \tag{3·48}$$

and

$$w_t-6(w_x)^2+w_{xxx}=0, \tag{3·49}$$

respectively.

Differentiating Eq. (3·45) twice with respect to x and by the use of Eqs. (3·48) and (3·49), we have

$$w_t'+w_t=4[-k^2w_x'+w_x^2+w_x(w'-w)^2+w_{xx}(w'-w)]. \tag{3·50}$$

Equations (3·45) and (3·50) are the Bäcklund transformations found by Wahlquist and Estabrook.

§ 4. Relation to the inverse methods for solving the sine-Gordon equation, the modified K-dV equation and the K-dV equation

In this section, we show that the new forms of the Bäcklund transformations described in the previous section lead to the known inverse scattering methods of solutions of the initial-value problem for the respective nonlinear evolution equations.

(a) *The sine-Gordon equation*

We rewrite Bäcklund transformation equations (3·12) and (3·14) as

$$\psi_x(f-ig)(f+ig)+\psi[D_x(f-ig)\cdot(f+ig)]=-(a/2)\psi^*(f+ig)(f-ig) \tag{4·1}$$

and

$$\psi_t(f-ig)^2 = -(1/2a)\psi^*(f+ig)^2 \qquad (4\cdot2)$$

by means of Eq. (V-1), where new functions ψ and ψ^* are defined by

$$f'+ig' = \psi(f-ig); \qquad f'-ig' = \psi^*(f+ig). \qquad (4\cdot3)$$

If we recall the relations $\varphi_x/4 = (D_x g\cdot f)/(f^2+g^2)$ and $f+ig = (f^2+g^2)^{1/2} \times \exp(i\varphi/4)$, we find that Eqs. (4·1) and (4·2) give

$$\psi_x - (i/2)\varphi_x\psi + (a/2)\psi^* = 0 \qquad (4\cdot4)$$

and

$$\psi_t + (1/2a)\psi^* \exp(i\varphi) = 0\,, \qquad (4\cdot5)$$

which yield the inverse scattering method for solving the initial value problem for the sine-Gordon equation found by Ablowitz et al.:[14)]

$$\partial v_1/\partial x + (a/2)v_1 = -(\varphi_x/2)v_2\,, \qquad (4\cdot6)$$

$$\partial v_2/\partial x - (a/2)v_2 = (\varphi_x/2)v_1\,, \qquad (4\cdot7)$$

$$\partial v_1/\partial t = -(1/2a)(v_1\cos\varphi + v_2\sin\varphi) \qquad (4\cdot8)$$

and

$$\partial v_2/\partial t = -(1/2a)(v_1\sin\varphi - v_2\cos\varphi), \qquad (4\cdot9)$$

where v_1 and v_2 are defined by

$$\psi = v_1 + iv_2; \qquad \psi^* = v_1 - iv_2\,. \qquad (4\cdot10)$$

(b) *The modified K-dV equation*

We rewrite a Bäcklund transformation equations (3·32) and (3·34) as

$$\psi_t(f+ig)^2 + 3\lambda^2\psi_x(f+ig)^2 + \psi_{xxx}(f+ig)^2 + 3\psi_x D_x^2(f+ig)\cdot(f+ig) = 0\,, \qquad (4\cdot11)$$

$$\psi_x(f+ig)(f-ig) + \psi[D_x(f+ig)\cdot(f-ig)] = -\lambda\psi^*(f-ig)(f+ig), \qquad (4\cdot12)$$

by the use of Eqs. (V-1) ~ (V-3), where new functions ψ and ψ^* are defined by

$$f'+ig' = \psi(f+ig); \qquad f'-ig' = \psi^*(f-ig). \qquad (4\cdot13)$$

By means of Eqs. (IV-1), (IV-4), (IV-5), (3·25) and (3·29), we have

$$D_x^2(f+ig)\cdot(f+ig) = 2(f+ig)^2\{\log[(f^2+g^2)^{1/2}\exp(i\phi)]\}_{xx}$$
$$= (f+ig)^2(4v^2 + 2iv_x).$$

Then, Eqs. (4·11) and (4·12) become

$$\psi_t + 3\lambda^2\psi_x + \psi_{xxx} + 3\psi_x(4v^2 + 2iv_x) = 0 \qquad (4\cdot14)$$

and

$$\psi_x + 2iv\psi + \lambda\psi^* = 0\,, \qquad (4\cdot15)$$

which yield the inverse scattering problem for the modified K-dV equation found by Wadati,[15] and by Tanaka.[16] We write the form found by Ablowitz et al. Substituting Eq. (4·15) and its second derivative into Eq. (4·14), we have

$$\psi_t = 2i(4\lambda^2 v + 8v^3 + v_{xx})\psi + 4\lambda(\lambda^2 + 2v^2 + iv_x)\psi^*,$$

which, together with Eq. (4·15), gives the form found by Ablowitz et al.:[9]

$$\partial v_1/\partial x + \lambda v_1 = (2v)v_2,$$
$$\partial v_2/\partial x - \lambda v_2 = -(2v)v_1,$$
$$\partial v_1/\partial t = Av_1 + Bv_2$$

and

$$\partial v_2/\partial t = Cv_1 - Av_2,$$

where

$$A = 4\lambda^3 + 2\lambda(2v)^2,$$
$$B = 2\lambda(2v_x) - (2v_{xx}) - [4\lambda^2 + 2(2v)^2](2v),$$
$$C = 2\lambda(2v_x) + (2v_{xx}) + [4\lambda^2 + 2(2v)^2](2v),$$

and where v_1 and v_2 are defined by

$$\psi = v_1 + iv_2, \qquad \psi^* = v_1 - iv_2.$$

(c) *The K-dV equation*

Following a similar procedure to the case of the modified K-dV equation, we transform Bäcklund transformation equations (3·43) and (3·44) into

$$\psi_t + 3(u(x) + k^2)\psi_x + \psi_{xxx} = 0 \tag{4·16}$$

and

$$\psi_{xx} + u(x)\psi = k^2\psi, \tag{4·17}$$

where ψ is defined by $f' = \psi f$.

Substituting a derivative of Eq. (4·17) with respect to x into Eq. (4·16), we have

$$\psi_t = -4\psi_{xxx} - 3\left(\frac{\partial}{\partial x}u(x) + u(x)\frac{\partial}{\partial x}\right)\psi, \tag{4·18}$$

which, together with Eq. (4·17), forms the inverse scattering problem for the K-dV equation found by Gardner et al,[2] and formulated by Lax.[3]

We note that Wahlquist and Estabrook[13] obtained the Schrödinger equation (4·17) from their form of the Bäcklund transformation for the K-dV equation (3·45) via a Riccati equation, and pointed out a contact between the Bäcklund transformation for the K-dV equation and the inverse scattering method. However, they did not obtain the time evolution equation (4·16), and did not develop the complete relationship between the Bäcklund transformation and the inverse scattering method as has been carried out in the present paper.*)

We have presented a unified method of finding Bäcklund transformations for solutions of certain nonlinear evolution equations and of constructing the inverse scattering problems for solving the initial-value problems. Applications of the present method to other nonlinear equations will be published elsewhere.

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