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A new fourth power mean of two-term exponential sums

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Abstract: The main purpose of this paper is to use analytic methods and properties of quartic Gauss sums to study a special fourth power mean of a two-term exponential sums mod$p$, with p an odd prime, and prove interesting new identities. As an application of our results, we also obtain a sharp asymptotic formula for the fourth power mean.

Keywords: Two-term exponential sums; fourth power mean; elementary method; identity; asymptotic formula

MSC: 11L05, 11L07

Dedicated to our supervisor Professor Zhang Wenpeng for his 60th birthday.

1 Introduction

Let $q \geq 3$ be an integer. For any integer m and n, the two-term exponential sum $G(k, h, m, n; q)$ is defined as

$$G(k, h, m, n; q) = \sum_{a=0}^{q-1} e\left(\frac{ma^k + na^h}{q}\right),$$

where as usual, $e(y) = e^{2\pi i y}$, k and h are positive integers with $k \neq h$.

Many scholars have studied various elementary properties of $G(k, h, m, n; q)$ and obtained a series of results. For example, from the A. Weil's important work [2], one can get the general upper bound estimate

$$\left|\sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^k + na}{p}\right)\right| \ll \sqrt{p},$$

where p is an odd prime, χ is any Dirichlet character mod p and $(m, n, p) = 1$.

Zhang Han and Zhang Wenpeng [3] proved the identity

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3 + na}{p}\right)\right|^4 = \begin{cases} 2p^3 - p^2 & \text{if } 3 \nmid p-1, \\ 2p^3 - 7p^2 & \text{if } 3 \mid p-1, \end{cases}$$

where p be an odd prime.

Zhang Han and Zhang Wenpeng [4] also obtained

$$\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1} e\left(\frac{ma^5 + na}{p}\right)\right|^4 = \begin{cases} 3p^3 - p^2\left(8 + 2\left(\frac{-1}{p}\right) + 4\left(\frac{-3}{p}\right)\right) - 3p & \text{if } 5 \nmid p-1, \\ 3p^3 + O\left(p^2\right) & \text{if } 5 \mid p-1, \end{cases}$$

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where $\left(\frac{*}{p}\right) = \chi_2$ denotes the Legendre symbol mod p.

Some other related mean value papers can also be found in [5] - [13]. If someone is interested in this field, please refer to these references. However, regarding the fourth power mean

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^4, \tag{1}$$

it seems that it hasn't been studied yet, at least so far we haven't seen any related papers. We think one of the reasons for this may be that the methods used in the past are not suitable for studying this situation, or perhaps 4 is not a prime number, so it is difficult to study (1).

In this paper we will use analytic methods and properties of quartic Gauss sums to study this problem and solve it completely. That is, we will prove the following two results.

Theorem 1. Let $p > 3$ be a prime with $p \equiv 3 \bmod 4$, then we have

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^4 = \begin{cases} 2p^2(p-2) & \text{if } p = 12h+7, \\ 2p^3 & \text{if } p = 12h+11. \end{cases}$$

Theorem 2. Let p be a prime with $p \equiv 1 \bmod 4$, then we have

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^4 = \begin{cases} 2p\left(p^2-10p-2\alpha^2\right) & \text{if } p = 24h+1, \\ 2p\left(p^2-4p-2\alpha^2\right) & \text{if } p = 24h+5, \\ 2p\left(p^2-6p-2\alpha^2\right) & \text{if } p = 24h+13, \\ 2p\left(p^2-8p-2\alpha^2\right) & \text{if } p = 24h+17, \end{cases}$$

where $\alpha = \alpha(p) = \sum_{a=1}^{\frac{p-1}{2}}\left(\frac{a+\overline{a}}{p}\right)$ is an integer satisfying the identity (state displayed identity), where r is any quadratic non-residue mod$p$: see Theorem 4-11 in [16].

$$p = \alpha^2+\beta^2 \equiv \left(\sum_{a=1}^{\frac{p-1}{2}}\left(\frac{a+\overline{a}}{p}\right)\right)^2 + \left(\sum_{a=1}^{\frac{p-1}{2}}\left(\frac{a+r\overline{a}}{p}\right)\right)^2,$$

and r is any quadratic non-residue mod p.

From these two theorems we may immediately deduce the following:

Corollary. For any odd prime p, we have the asymptotic formula

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^4 = 2p^3 + O\left(p^2\right).$$

2 Several Lemmas

To prove our theorems, we first need to give several necessary lemmas. Hereafter, we will use many properties of the classical Gauss sums, the fourth-order character mod p and the quartic Gauss sums. All of these contents can be found in any Elementary Number Theory or Analytic Number Theory book, such as references [1], [14] or [16]. These contents will not be repeated here. First we have the followings:

Lemma 1. If p is a prime with $p \equiv 1 \bmod 4$, and λ is any fourth-order character mod p, then we have

$$\tau^2(\lambda)+\tau^2\left(\overline{\lambda}\right) = \sqrt{p}\cdot\sum_{a=1}^{p-1}\left(\frac{a+\overline{a}}{p}\right) = 2\sqrt{p}\cdot\alpha.$$

Proof. In fact this is Lemma 2 of [15], so its proof is omitted.

Lemma 2. If p is a prime with $p \equiv 1 \bmod 4$, then we have the identity

$$\sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^2\left(\sum_{c=0}^{p-1} e\left(\frac{-mc^4-c}{p}\right)\right) = \begin{cases} (2+\chi_2(7))\,p^2+2p\alpha & \text{if } p \equiv 5 \bmod 8, \\ (2+\chi_2(7))\,p^2-6p\alpha & \text{if } p \equiv 1 \bmod 8, \end{cases}$$

where $\chi_2 = \left(\frac{*}{p}\right)$ denotes the Legendre's symbol mod p.

Proof. First applying trigonometric identity

$$\sum_{m=1}^{q} e\left(\frac{nm}{q}\right) = \begin{cases} q & \text{if } q \mid n, \\ 0 & \text{if } q \nmid n, \end{cases} \tag{2}$$

we have

$$\begin{aligned} &\sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^2\left(\sum_{c=0}^{p-1} e\left(\frac{-mc^4-c}{p}\right)\right) \\ &= \sum_{m=0}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^2\left(\sum_{c=0}^{p-1} e\left(\frac{-mc^4-c}{p}\right)\right) \\ &= \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{m=0}^{p-1} e\left(\frac{m\left(a^4+b^4-c^4\right)+a+b-c}{p}\right) \\ &= p \sum_{\substack{a=0 \\ a^4+b^4\equiv c^4 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} e\left(\frac{a+b-c}{p}\right) \\ &= p \sum_{\substack{a=0 \\ a^4+b^4\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1} e\left(\frac{a+b}{p}\right) + p \sum_{\substack{a=0 \\ a^4+b^4\equiv 1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=1}^{p-1} e\left(\frac{c(a+b-1)}{p}\right) \\ &= p - p \sum_{\substack{a=0 \\ a^4+1\equiv 0 \bmod p}}^{p-1} 1 + p^2 \sum_{\substack{a=0 \\ a^4+b^4\equiv 1 \bmod p \\ a+b\equiv 1 \bmod p}}^{p-1}\sum_{b=0}^{p-1} 1 - p \sum_{\substack{a=0 \\ a^4+b^4\equiv 1 \bmod p}}^{p-1}\sum_{b=0}^{p-1} 1. \end{aligned} \tag{3}$$

Let λ be a fourth-order character mod p, if $p \equiv 5 \bmod 8$, then note that $\lambda(-1) = -1$ we have

$$-p \sum_{\substack{a=0 \\ a^4+1\equiv 0 \bmod p}}^{p-1} 1 = 0. \tag{4}$$

Noting the identity $\lambda\chi_2 = \overline{\lambda}$ and

$$B(m) = \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) = \chi_2(m)\sqrt{p} + \overline{\lambda}(m)\tau(\lambda) + \lambda(m)\tau\left(\overline{\lambda}\right). \tag{5}$$

Applying (5) and Lemma 1 we have

$$\begin{aligned} &p \sum_{\substack{a=0 \\ a^4+b^4\equiv 1 \bmod p}}^{p-1}\sum_{b=0}^{p-1} 1 = \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{m=0}^{p-1} e\left(\frac{m\left(a^4+b^4-1\right)}{p}\right) \\ &= p^2 + \sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right)^2 e\left(\frac{-m}{p}\right) \end{aligned}$$

$$= p^2 + \sum_{m=1}^{p-1}\left(2\chi_2(m)\sqrt{p}\alpha - p + 2\lambda(m)\sqrt{p}\tau(\lambda) + 2\overline{\lambda}(m)\sqrt{p}\tau\left(\overline{\lambda}\right)\right)e\left(\frac{-m}{p}\right)$$
$$= p^2 + 2p\alpha + p - 2\sqrt{p}\tau^2(\lambda) - 2\sqrt{p}\tau^2\left(\overline{\lambda}\right) = p^2 + p - 2p\alpha. \quad (6)$$

It is clear that the congruences $a^4 + b^4 \equiv 1 \bmod p$ and $a + b \equiv 1 \bmod p$ imply that $ab\left(2a^2 + 3ab + 2b^2\right) \equiv 0 \bmod p$ and $a + b \equiv 1 \bmod p$. So we have

$$p^2 \sum_{\substack{a=0 \\ a^4+b^4\equiv 1 \bmod p \\ a+b\equiv 1 \bmod p}}^{p-1}\sum_{b=0}^{p-1} 1 = p^2 \sum_{\substack{a=0 \\ ab(2a^2+3ab+2b^2)\equiv 0 \bmod p \\ a+b\equiv 1 \bmod p}}^{p-1}\sum_{b=0}^{p-1} 1$$
$$= 2p^2 + p^2 \sum_{\substack{a=1 \\ 2a^2+3ab+2b^2\equiv 0 \bmod p \\ a+b\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1} 1 = 2p^2 + p^2 \sum_{\substack{a=1 \\ 2a^2+3a+2\equiv 0 \bmod p \\ b(a+1)\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1} 1$$
$$= 2p^2 + p^2 \sum_{\substack{a=0 \\ (4a+3)^2\equiv -7 \bmod p}}^{p-1} 1 = 2p^2 + p^2 \sum_{\substack{a=0 \\ a^2\equiv -7 \bmod p}}^{p-1} 1 = \left(3 + \left(\frac{7}{p}\right)\right)\cdot p^2. \quad (7)$$

If $p \equiv 1 \bmod 8$, then noting that $\lambda(-1) = 1$ we have

$$-p \sum_{\substack{a=0 \\ a^4+1\equiv 0 \bmod p}}^{p-1} 1 = -4p. \quad (8)$$

Applying (5), Lemma 1 and note that $\tau(\lambda)\tau\left(\overline{\lambda}\right) = p$ we have

$$p \sum_{\substack{a=0 \\ a^4+b^4\equiv 1 \bmod p}}^{p-1}\sum_{b=0}^{p-1} 1 = \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{m=0}^{p-1} e\left(\frac{m\left(a^4+b^4-1\right)}{p}\right)$$
$$= p^2 + \sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right)^2 e\left(\frac{-m}{p}\right)$$
$$= p^2 + \sum_{m=1}^{p-1}\left(3p + 2\chi_2(m)\sqrt{p}\alpha + 2\lambda(m)\sqrt{p}\tau(\lambda) + 2\overline{\lambda}(m)\sqrt{p}\tau\left(\overline{\lambda}\right)\right)e\left(\frac{-m}{p}\right)$$
$$= p^2 + 2p\alpha - 3p + 2\sqrt{p}\tau^2(\lambda) + 2\sqrt{p}\tau^2\left(\overline{\lambda}\right) = p^2 - 3p + 6p\alpha. \quad (9)$$

Combining (3), (4), (6)-(9) we have the identity

$$\sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^2\left(\sum_{c=0}^{p-1} e\left(\frac{-mc^4-c}{p}\right)\right) = \begin{cases}\left(2+\chi_2(7)\right)p^2 + 2p\alpha & \text{if } p \equiv 5 \bmod 8,\\ \left(2+\chi_2(7)\right)p^2 - 6p\alpha & \text{if } p \equiv 1 \bmod 8.\end{cases}$$

This proves Lemma 2.

Lemma 3. If p is a prime with $p \equiv 3 \bmod 4$, then we have the identity

$$\sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^2\left(\sum_{c=0}^{p-1} e\left(\frac{-mc^4-c}{p}\right)\right) = \left(2-\chi_2(7)\right)p^2.$$

Proof. If $p = 4h + 3$, then $\chi_2(-1) = -1$ and $\tau(\chi_2) = i\sqrt{p}$, $i^2 = -1$. For any integer m with $(m, p) = 1$, we have

$$\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) = 1 + \sum_{a=1}^{p-1}\left(1 + \chi_2(a)\right)e\left(\frac{ma^2}{p}\right) = \sum_{a=0}^{p-1} e\left(\frac{ma^2}{p}\right) = \chi_2(m)\tau(\chi_2) \quad (10)$$

and

$$-p \sum_{\substack{a=0 \\ a^4+1\equiv 0 \bmod p}}^{p-1} 1 = 0. \tag{11}$$

From (10), (11) and the method of proving Lemma 2 we have

$$\sum_{m=1}^{p-1} \left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p} \right) \right)^2 \left(\sum_{c=0}^{p-1} e\left(\frac{-mc^4-c}{p} \right) \right) = \left(2-\chi_2(7)\right) p^2.$$

This proves Lemma 3.

Lemma 4. If p is a prime with $p \equiv 1 \bmod 4$, then we have the identity

$$\begin{aligned} &\sum_{m=1}^{p-1} \left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p} \right) \right)^2 \left(\sum_{c=0}^{p-1} e\left(\frac{-mc^4-c}{p} \right) \right) \left(\sum_{d=1}^{p-1} e\left(\frac{-md^4-d}{p} \right) \right) \\ &= \begin{cases} p\left(2p^2 - 22p - \chi_2(7)p - 4\alpha^2 + 6\alpha\right) & \text{if } p = 24h+1, \\ p\left(2p^2 - 10p - \chi_2(7)p - 4\alpha^2 - 2\alpha\right) & \text{if } p = 24h+5, \\ p\left(2p^2 - 14p - \chi_2(7)p - 4\alpha^2 - 2\alpha\right) & \text{if } p = 24h+13, \\ p\left(2p^2 - 18p - \chi_2(7)p - 4\alpha^2 + 6\alpha\right) & \text{if } p = 24h+17. \end{cases} \end{aligned}$$

Proof. From identity (2) we have

$$\begin{aligned} &\sum_{m=1}^{p-1} \left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p} \right) \right)^2 \left(\sum_{c=0}^{p-1} e\left(\frac{-mc^4-c}{p} \right) \right) \left(\sum_{d=1}^{p-1} e\left(\frac{-md^4-d}{p} \right) \right) \\ &= \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{d=1}^{p-1}\sum_{m=0}^{p-1} e\left(\frac{m\left(a^4+b^4-c^4-d^4\right)+a+b-c-d}{p} \right) \\ &= p \sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{d(a+b-c-1)}{p} \right) \\ &= p^2 \sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p \\ a+b\equiv c+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 - p \sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1. \end{aligned} \tag{12}$$

It is clear that the congruences $a^4+b^4 \equiv c^4+1 \bmod p$ and $a+b \equiv c+1 \bmod p$ imply that $(a-1)(b-1)\left(2a^2+3ab+2b^2-a-b+1\right) \equiv 0 \bmod p$. So we have

$$\begin{aligned} \sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p \\ a+b\equiv c+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} 1 &= \sum_{\substack{a=0 \\ a^4+b^4\equiv (a+b-1)^4+1 \bmod p}}^{p-1}\sum_{b=0}^{p-1} 1 \\ &= \sum_{\substack{a=0 \\ (a-1)(b-1)(2a^2+2b^2+3ab-a-b+1)\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1} 1 \\ &= 2p-1 + \sum_{\substack{a=0 \\ 2a^2+2b^2+3ab-a-b+1\equiv 0 \bmod p \\ a\neq 1,\ b\neq 1}}^{p-1}\sum_{b=0}^{p-1} 1 \end{aligned}$$

$$= 2p - 1 + \sum_{\substack{a=0 \\ 2a^2+2b^2+3ab-a-b+1\equiv 0 \bmod p}}^{p-1} \sum_{b=0}^{p-1} 1 - 2 \sum_{\substack{a=0 \\ a^2+a+1\equiv 0 \bmod p}}^{p-1} 1$$

$$= 2p - 1 + \sum_{\substack{a=0 \\ (4a+3b-1)^2\equiv -7b^2+2b-7 \bmod p}}^{p-1} \sum_{b=0}^{p-1} 1 - 2 \sum_{\substack{a=2 \\ a^3\equiv 1 \bmod p}}^{p-1} 1$$

$$= 2p + 1 + \sum_{\substack{a=0 \\ a^2\equiv -7b^2+2b-7 \bmod p}}^{p-1} \sum_{b=0}^{p-1} 1 - 2 \sum_{\substack{a=1 \\ a^3\equiv 1 \bmod p}}^{p-1} 1$$

$$= 2p + 1 + \sum_{b=0}^{p-1} \left(1 + \left(\frac{-7b^2 + 2b - 7}{p}\right)\right) - 2 \sum_{\substack{a=1 \\ a^3\equiv 1 \bmod p}}^{p-1} 1$$

$$= 3p + 1 + \left(\frac{7}{p}\right) \sum_{b=0}^{p-1} \left(\frac{(7b-1)^2 + 48}{p}\right) - 2 \sum_{\substack{a=1 \\ a^3\equiv 1 \bmod p}}^{p-1} 1$$

$$= 3p + 1 + \left(\frac{7}{p}\right) \sum_{b=0}^{p-1} \left(\frac{b^2 + 48}{p}\right) - 2 \sum_{\substack{a=1 \\ a^3\equiv 1 \bmod p}}^{p-1} 1$$

$$= 3p + 1 - \left(\frac{7}{p}\right) - 2 \sum_{\substack{a=1 \\ a^3\equiv 1 \bmod p}}^{p-1} 1, \tag{13}$$

where we have used the identity $\sum_{b=0}^{p-1} \left(\frac{b^2+48}{p}\right) = -1$.

If $p = 24h + 13$ or $p = 24h + 1$, then $a^3 \equiv 1 \bmod p$ has three solutions. So from (13) we have

$$\sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p \\ a+b\equiv c+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 = 3p - 5 - \left(\frac{7}{p}\right). \tag{14}$$

If $p = 24h + 5$ or $p = 24h + 17$, then $a^3 \equiv 1 \bmod p$ has one solution. So from (13) we have

$$\sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p \\ a+b\equiv c+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 = 3p - 1 - \left(\frac{7}{p}\right). \tag{15}$$

If $p = 8h + 5$, then applying (5), Lemma 1 and noting that $\tau(\lambda)\tau\left(\overline{\lambda}\right) = -p$ we have

$$p \sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 = \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} \sum_{m=0}^{p-1} e\left(\frac{m\left(a^4 + b^4 - c^4 - 1\right)}{p}\right)$$

$$= \sum_{m=0}^{p-1} \left(\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right)^2 \left(\sum_{b=0}^{p-1} e\left(\frac{-mb^4}{p}\right)\right) e\left(\frac{-m}{p}\right)$$

$$= p^3 + \sum_{m=1}^{p-1} \left(3p - \chi_2(m)\tau^2(\lambda) - \chi_2(m)\tau^2\left(\overline{\lambda}\right)\right)$$

$$\times \left(\chi_2(m)\sqrt{p} + \overline{\lambda}(m)\tau(\lambda) + \lambda(m)\tau\left(\overline{\lambda}\right)\right) e\left(\frac{-m}{p}\right)$$

$$= p^3 + 9p^2 + 4p\alpha^2 + 2p\alpha. \tag{16}$$

If $p = 8h + 1$, then applying (5), Lemma 1 and noting that $\tau(\lambda)\tau\left(\overline{\lambda}\right) = p$ we have

$$p \sum_{\substack{a=0 \\ a^4+b^4\equiv c^4+1 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \sum_{c=0}^{p-1} 1 = \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1}\sum_{m=0}^{p-1} e\left(\frac{m\left(a^4+b^4-c^4-1\right)}{p}\right)$$
$$= \sum_{m=0}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right)^2 \left(\sum_{b=0}^{p-1} e\left(\frac{-mb^4}{p}\right)\right) e\left(\frac{-m}{p}\right)$$
$$= p^3 + \sum_{m=1}^{p-1}\left(3p + 2\chi_2(m)\sqrt{p}\alpha + 2\lambda(m)\sqrt{p}\tau(\lambda) + 2\overline{\lambda}(m)\sqrt{p}\tau\left(\overline{\lambda}\right)\right)$$
$$\times \left(\chi_2(m)\sqrt{p} + \overline{\lambda}(m)\tau(\lambda) + \lambda(m)\tau\left(\overline{\lambda}\right)\right) e\left(\frac{-m}{p}\right)$$
$$= p^3 + 17p^2 + 4p\alpha^2 - 6p\alpha. \tag{17}$$

Combining (12), (14) - (17) we have the identity

$$\sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^2 \left(\sum_{c=0}^{p-1} e\left(\frac{-mc^4-c}{p}\right)\right)\left(\sum_{d=1}^{p-1} e\left(\frac{-md^4-d}{p}\right)\right)$$
$$= \begin{cases} p\left(2p^2 - 14p - \chi_2(7)p - 4\alpha^2 - 2\alpha\right) & \text{if } p = 24h+13, \\ p\left(2p^2 - 22p - \chi_2(7)p - 4\alpha^2 + 6\alpha\right) & \text{if } p = 24h+1, \\ p\left(2p^2 - 10p - \chi_2(7)p - 4\alpha^2 - 2\alpha\right) & \text{if } p = 24h+5, \\ p\left(2p^2 - 18p - \chi_2(7)p - 4\alpha^2 + 6\alpha\right) & \text{if } p = 24h+17. \end{cases}$$

This proves Lemma 4.

Lemma 5. If p is a prime with $p \equiv 3 \bmod 4$, then we have the identity

$$\sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^2 \left(\sum_{c=0}^{p-1} e\left(\frac{-mc^4-c}{p}\right)\right)\left(\sum_{d=1}^{p-1} e\left(\frac{-md^4-d}{p}\right)\right)$$
$$= \begin{cases} p^2\left(2p - 6 + \chi_2(7)\right) & \text{if } p = 12h+7, \\ p^2\left(2p - 2 + \chi_2(7)\right) & \text{if } p = 12h+11. \end{cases}$$

Proof. Noting (11) and $\chi_2(-1) = -1$, from the methods of proving Lemma 4 we can easily deduce Lemma 5.

3 Proofs of the theorems

Now we prove our main results. First we prove Theorem 2. If $p = 24h + 1$, then from Lemma 2 and Lemma 4 we have

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^4 = \sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^2 \left(\sum_{c=0}^{p-1} e\left(\frac{-ma^4-c}{p}\right)\right)\left(\sum_{d=1}^{p-1} e\left(\frac{-md^4-d}{p}\right)\right)$$
$$+ \sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^2 \left(\sum_{c=0}^{p-1} e\left(\frac{-ma^4-c}{p}\right)\right)$$
$$= p\left(2p^2 - 22p - \chi_2(7)p - 4\alpha^2 + 6\alpha\right) + 2p^2 + \chi_2(7)p^2 - 6p\alpha$$

$$= 2p\left(p^2 - 10p - 2\alpha^2\right). \tag{18}$$

Similarly, if $p = 24h + 5$, then from Lemma 2 and Lemma 4 we have

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right)\right|^4 = 2p\left(p^2 - 4p - 2\alpha^2\right). \tag{19}$$

If $p = 24h + 13$, then we have

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right)\right|^4 = 2p\left(p^2 - 6p - 2\alpha^2\right). \tag{20}$$

If $p = 24h + 17$, then we have

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right)\right|^4 = 2p\left(p^2 - 8p - 2\alpha^2\right). \tag{21}$$

Now Theorem 2 follows from (18) - (21).

Similarly, from Lemma 3, Lemma 5 and the methods of proving Theorem 2 we can also deduce Theorem 1. This completes the proofs of all of our results.

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