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A NEW FRACTIONAL DERIVATIVE WITHOUT SINGULAR KERNEL
Application to the Modelling of the Steady Heat Flow

by

Xiao-Jun YANG [a]*, Hari M. SRIVASTAVA[b,c], and J. A. Tenreiro MACHADO [d]

[a] Department of Mathematics and Mechanics, China University of Mining and Technology, Xuzhou, China
[b] Department of Mathematics and Statistics, University of Victoria, Victoria, B. C., Canada
[c] China Medical University, Taichung, Taiwan, China
[d] Department of Electrical Engineering, Institute of Engineering, Polytechnic of Porto, Porto, Portugal

Original scientific paper
DOI: 10.2298/TSCI151224222Y

In this article we propose a new fractional derivative without singular kernel. We consider the potential application for modeling the steady heat-conduction problem. The analytical solution of the fractional-order heat flow is also obtained by means of the Laplace transform.

Key words: *heat conduction, steady heat flow, analytical solution, Laplace transform, fractional derivative without singular kernel*

Introduction

Fractional derivatives with singular kernel [1], namely, the Riemann-Liouville [2, 3], Caputo [4, 5], and other derivatives, see [6-8] and the references therein, have nowadays a wide application in the field of heat-transfer engineering.

More recently, the fractional Caputo-Fabrizio derivative operator without singular kernel was given [1, 9-12]:

$$ {}^{CF}\mathrm{D}_x^{(\nu)}T(x) = \frac{(2-\nu)\Im(\nu)}{2(1-\nu)} \int_0^x \exp\left[-\frac{\nu}{1-\nu}(x-\lambda)\right] T^{(1)}(\lambda)\mathrm{d}\lambda \tag{1} $$

where $\Im(\nu)$ is a normalization constant depending on ν $(0<\nu<1)$.

Following eq. (1), Losada and Nieto [10] suggested the new fractional Caputo-Fabrizio derivative operator [11, 12]:

$$ {}^{CF}_{\ *}\mathrm{D}_x^{(\nu)}T(x) = \frac{1}{1-\nu} \int_0^x \exp\left[-\frac{\nu}{1-\nu}(x-\lambda)\right] T^{(1)}(\lambda)\mathrm{d}\lambda \tag{2} $$

where ν $(0<\nu<1)$ is a real number and $\Im(\nu) = 2/(2-\nu)$.

* Corresponding author; e-mail: dyangxiaojun@163.com

Equations (1) and (2) represent an extension of the Caputo fractional derivative with singular kernel. However, an analog of the Riemann-Liouville fractional derivative with singular kernel has not yet been formulated. The main aim of the article is to propose a new fractional derivative without singular kernel, which is an extension of the Riemann-Liouville fractional derivative with singular kernel, and to study its application in the modeling of the fractional-order heat flow.

Mathematical tools

The Riemann-Liouville fractional derivative of fractional order ν of the function $T(x)$ is defined [1]:

$$ {}^{RL}\mathrm{D}_{a^+}^{(\nu)}T(x) = \frac{1}{\Gamma(1-\nu)}\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x \frac{T(\lambda)}{(x-\lambda)^{\nu}}\mathrm{d}\lambda \tag{3}$$

where $a \le x$ and ν $(0<\nu<1)$ is a real number.

Replacing the function $1/(x-\lambda)^{\nu}\Gamma(1+\nu)$ by $\Re(\nu)\exp\{[-\nu/(1-\nu)](x-\lambda)\}/(1-\nu)$, we obtain a new fractional derivative given by:

$$ \mathrm{D}_{a^+}^{(\nu)}T(x) = \frac{\Re(\nu)}{1-\nu}\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x \exp\left[-\frac{\nu}{1-\nu}(x-\lambda)\right]T(\lambda)\mathrm{d}\lambda \tag{4}$$

where $a \le x$, ν $(0<\nu<1)$ is a real number, and $\Re(\nu)$ is a normalization function depending on ν such that $\Re(0)=\Re(1)=1$.

Taking $\psi = 1/\nu - 1$, with $0<\psi<+\infty$, eq. (4) can be re-written:

$$ \mathrm{D}_{a^+}^{\left(\frac{1}{\psi+1}\right)}T(x) = \aleph(\psi)\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x \Pi(\lambda)T(\lambda)\mathrm{d}\lambda \tag{5}$$

where $\aleph(\psi) = (\psi+1)\Re[1/(\psi+1)]$, and $\Pi(\lambda) = \exp[-(x-\lambda)/\psi]/\psi$.

With the help of the following approximation to the identity [9, 13]:

$$ \lim_{\psi\to 0}\Pi(\lambda) = \delta(x-\lambda) \tag{6}$$

where $\nu \to 1$ (or $\psi \to 0$), eq. (4) becomes:

$$ \lim_{\nu\to 1}\mathrm{D}_{a^+}^{(\nu)}T(x) = \lim_{\psi\to 0}\aleph(\psi)\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x \Pi(\lambda)T(\lambda)\mathrm{d}\lambda = T^{(1)}(x) \tag{7}$$

When $\nu \to 0$ (or $\psi \to +\infty$), eq. (4) can be written:

$$ \lim_{\nu\to 0}\mathrm{D}_{a^+}^{(\nu)}T(x) = \lim_{\nu\to 0}\frac{\Re(\nu)}{1-\nu}\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x \exp\left[-\frac{\nu}{1-\nu}(x-\lambda)\right]T(\lambda)\mathrm{d}\lambda = T(x) \tag{8}$$

Taking the Laplace transform of the new fractional derivative without singular kernel for the parameter $a = 0$, we have:

$$ L[\mathrm{D}_0^{(\nu)}T(x)] = \frac{\Re(\nu)s}{\nu(1-s)+s}T(s) \tag{9}$$

where $L[\xi(x)] := \int_0^x \exp(-sx)\xi(x)\mathrm{d}x = \xi(s)$ represents the Laplace transform of the function $\xi(x)$, [14].

We now consider:

$$T(s)=\left[\frac{\nu}{\Re(\nu)s}+\frac{1-\nu}{\Re(\nu)}\right]\Xi(s) \tag{10}$$

where $\mathrm{D}_0^{(\nu)}T(x)=\Xi(x)$ and $L[\Xi(x)]=\Xi(x)$.

Taking the inverse Laplace transform of eq. (10) we obtain:

$$T(x)=\frac{1-\nu}{\Re(\nu)}\Xi(x)+\frac{\nu}{\Re(\nu)}\int_0^x \Xi(x)\mathrm{d}x, \qquad x>0, \qquad 0<\nu<1 \tag{11}$$

If $0<\nu<1$ and $\Re(\nu)=1$, then eqs. (4) and (11) can be written:

$$ {}_*\mathrm{D}_{a^+}^{(\nu)}T(x)=\frac{1}{1-\nu}\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x \exp\left[-\frac{\nu}{1-\nu}(x-\lambda)\right]T(\lambda)\mathrm{d}\lambda \tag{12}$$

and

$$T(x)=(1-\nu)\Xi(x)+\nu\int_0^x \Xi(x)\mathrm{d}x, \qquad x>0, \qquad 0<\nu<1 \tag{13}$$

respectively.

Modelling the fractional-order steady heat flow

The fractional-order Fourier law in 1-D case is suggested:

$$K\mathrm{D}_0^{(\nu)}T(x)=-H(x) \tag{14}$$

where K is the thermal conductivity of the material and $H(x)$ – the heat flux density.

The heat flow of the fractional-order heat conduction is presented:

$$H(x)=g \tag{15}$$

where g is the heat flow (a constant) of the material.

By submitting eq. (13) into eq. (14), and taking the Laplace transform, it results:

$$\frac{\Re(\nu)s}{\nu+(1-\nu)s}T(s)=-\frac{g}{K} \tag{16}$$

which leads to:

$$T(s)=\frac{-g[\nu+(1-\nu)s]}{K\Re(\nu)s} \tag{17}$$

Taking the inverse Laplace transform of eq. (17), we obtain:

$$T(x)=-C\left[\frac{g\nu x}{K\Re(\nu)}+\frac{g(1-\nu)}{K\Re(\nu)}\right] \tag{18}$$

where C is a constant depending on the initial value $T(x)$.

The corresponding graphs with different orders $\nu=\{0.3,\ 0.6,\ 1\}$ are shown in fig. 1.

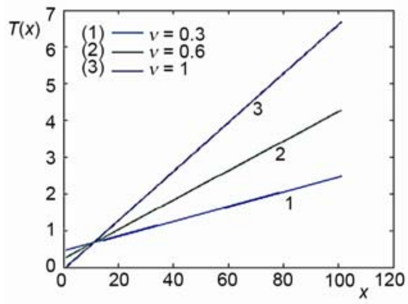

Figure 1. The plots of *T*(*x*) with the parameters ν = {0.3, 0.6, 1}, *C* = –1, *g* = 2, *K* = 3, and ℜ(ν) = 1

Conclusion

In this work a new fractional-order operator without singular kernel, which is an analog of the Riemann-Liouville fractional derivative with singular kernel, was proposed for the first time. An illustrative example for modelling the fractional-order steady heat flow was given and the analytical solution for the governing equation involving the fractional derivative without singular kernel was discussed.

Nomenclature

$D_0^{(\nu)}$ – fractional derivative without singular kernel, [–]
$H(x)$ – heat flux density, [Wm^{-2}]
K – thermal conductivity, [Wm^{-1}K^{-1}]
$L(\bullet)$ – Laplace transform, [–]
$T(x)$ – temperature distribution, [K]
x – space co-ordinate, [m]

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Paper submitted: December 24, 2015
Paper revised: December 25, 2015
Paper accepted: December 25, 2015