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A NEW GENERALIZATION OF GAMMA, BETA, HYPERGEOMETRIC AND CONFLUENT HYPERGEOMETRIC FUNCTIONS

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The main object of this paper is to present new generalizations of gamma, beta, hypergeometric and confluent hypergeometric functions. Some integral representations, Mellin transform, operation formulas, differentiation formulas,summation formula,beta distribution and transformation formulas are obtained for these new generalizations.

1. Introduction

In recent years, several extensions of the well known special functions have been considered by several authors [1–7]. In 1994, Chaudhry and Zubair [1] have introduced the following extension of gamma function.

$$\Gamma_p(x) = \int_0^\infty t^{x-1} \exp(-t - \frac{p}{t})dt \qquad (\Re(p) > 0, \Re(x) > 0) \tag{1}$$

In 1997, Chaudhry et al. [2] presented the following extension of Euler's beta function

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$$B_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\exp\left[-\frac{p}{t(1-t)}\right]dt$$
$$(\Re(p) > 0, \Re(x) > 0, \Re(y) > 0) \quad (2)$$

Afterwards, Chaudhry et al. [8] used $B_p(x,y)$ to extend the hypergeometric and confluent hypergeometric functions as follows:

$$F_p(a,b;c;z) = \sum_{n=0}^{\infty} \frac{B_p(b+n,c-b)}{B(b,c-b)}(a)_n \frac{z^n}{n!}$$
$$(p \geq 0; |\, z \,| < 1; \Re(c) > \Re(b) > 0) \quad (3)$$

$$\Phi_p(b;c;z) = \sum_{n=0}^{\infty} \frac{B_p(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!}$$
$$(p \geq 0; \Re(c) > \Re(b) > 0) \quad (4)$$

and gave the Euler type integral representation

$$F_p(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\exp\left[-\frac{p}{t(1-t)}\right]dt$$
$$(p > 0; p = 0 \text{ and } |\, arg(1-z) \,| < \pi; \Re(c) > \Re(b) > 0) \quad (5)$$

$$\Phi_p(b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\exp\left[zt - \frac{p}{t(1-t)}\right]dt$$
$$(p > 0; p = 0 \text{ and } \Re(b) > \Re(c) > 0) \quad (6)$$

They have obtained the integral representation, differentiation properties, Mellin transforms, transformation formulas, recurrence relations, summation and asymptotic formulas for these function.

Recently Lee et al. [11] generalized the beta, hypergeometric and confluent hypergeometric function as

$$B_p^m(x,y) = B_p(x,y;m) = \int_0^1 t^{x-1}(1-t)^{y-1}\exp\left[-\frac{p}{t^m(1-t)^m}\right]dt$$
$$(\Re(p) > 0, \Re(x) > 0, \Re(y) > 0), \Re(m) > 0) \quad (7)$$

$$F_p^m(a,b;c;z) = F_p(a,b;c;z;m) = \sum_{n=0}^{\infty} \frac{B_p^m(b+n,c-b)}{B(b,c-b)}(a)_n \frac{z^n}{n!}$$
$$(p \geq 0; |z| < 1; \Re(c) > \Re(b) > 0, \Re(m) > 0) \quad (8)$$

$$\Phi_p^m(b;c;z) = \Phi_p(b;c;z;m) = \sum_{n=0}^{\infty} \frac{B_p^m(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!}$$
$$(p \geq 0; \Re(c) > \Re(b) > 0, \Re(m) > 0) \quad (9)$$

and gave the Euler type integral representation

$$F_p^m(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \exp\left[-\frac{p}{t^m(1-t)^m}\right] dt$$
$$(p > 0;\ p = 0 \text{ and } |arg(1-z)| < \pi;\ \Re(c) > \Re(b) > 0,\ \Re(m) > (0) \quad (10)$$

$$\Phi_p^m(b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \exp\left[zt - \frac{p}{t^m(1-t)^m}\right] dt$$
$$(p > 0; p = 0; \Re(c) > \Re(b) > 0, \Re(m) > 0) \quad (11)$$

and Ozergin et al. [10] generalized the gamma, beta, hypergeometric and confluent hypergeometric function as

$$\Gamma_p^{(\alpha,\beta)}(x) = \int_0^{\infty} t^{x-1} {}_1F_1\left(\alpha;\beta;-t-\frac{p}{t}\right) dt$$
$$(\Re(\alpha) > 0, \Re(\beta) > 0, \Re(p) > 0, \Re(x) > 0) \quad (12)$$

$$B_p^{(\alpha,\beta)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} {}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right) dt$$
$$(\Re(p) > 0, \Re(x) > 0, \Re(y) > 0, \Re(\alpha) > 0, \Re(\beta) > 0) \quad (13)$$

$$F_p^{(\alpha,\beta)}(a,b;c;z) = \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!}$$
$$(p \geq 0; |z| < 1; \Re(c) > \Re(b) > 0, \Re(\alpha) > 0, \Re(\beta) > 0) \quad (14)$$

$$\Phi_p^{(\alpha,\beta)}(b;c;z) = \sum_{n=0}^{\infty} \frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!}$$
$$(p \geq 0; \Re(c) > \Re(b) > 0, \Re(\alpha) > 0, \Re(\beta) > 0) \quad (15)$$

and gave the Euler type integral representation

$$F_p^{(\alpha,\beta)}(a,b;c;z)$$
$$= \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right) dt$$
$$(p > 0;\ p = 0 \text{ and } |arg(1-z)| < \pi; \Re(c) > \Re(b) > 0, \Re(\alpha) > 0, \Re(\beta) > 0) \quad (16)$$

$$\Phi_p^{(\alpha,\beta)}(b;c;z)$$
$$= \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} e^{zt} {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right) dt$$
$$(p > 0;\ p = 0; \Re(c) > \Re(b) > 0, \Re(\alpha) > 0, \Re(\beta) > 0) \quad (17)$$

respectively.

In this paper, the new generalizations of gamma, Euler's beta functions, hypergeometric function and confluent hypergeometric function is considered. The plan of this paper is as follows:

The present paper is divided into three sections. In Section 2 , a new generalization of the gamma and beta functions is introduced. Some properties, Mellin transform representation, various integral representations, functional relation, summation relations and beta distribution are obtained for these newly generalized gamma and beta functions. In Section 3, a new generalization of the hypergeometric function and confluent hypergeometric function is defined and some integral representations are obtained. Furthermore differentiation properties, Mellin transforms, transformation formulas, recurrence relations, summation formulas for these newly generalized hypergeometric and confluent hypergeometric functions are obtained.

2. Generalized Gamma and Beta Function

In this section the new generalized gamma and beta functions are defined as:

$$\Gamma_p^{(\alpha,\beta;m)}(x) = \int_0^\infty t^{x-1}{}_1F_1(\alpha;\beta;-t-\frac{p}{t^m})dt$$
$$(\Re(\alpha) > 0, \Re(\beta) > 0, \Re(p) > 0, \Re(x) > 0, \Re(m) > 0) \quad (18)$$

$$B_p^{(\alpha,\beta;m)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha;\beta;-\frac{p}{t^m(1-t)^m}\right)dt$$
$$(\Re(p) > 0, \Re(x) > 0, \Re(y) > 0, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(m) > 0) \quad (19)$$

It is obvious,

$$\Gamma_p^{(\alpha,\beta;1)}(x) = \Gamma_p^{(\alpha,\beta)}(x), \qquad \Gamma_p^{(\alpha,\alpha;1)}(x) = \Gamma_p(x), \qquad \Gamma_0^{(\alpha,\alpha)}(x) = \Gamma(x),$$
$$B_p^{(\alpha,\beta;1)}(x,y) = B_p^{(\alpha,\beta)}(x,y), \quad B_p^{(\alpha,\alpha;m)}(x,y) = B_p^m(x,y), \quad B_0^{(\alpha,\alpha;1)}(x,y) = B(x,y)$$

2.1. Some properties of generalized gamma and beta function

In this section some properties of new generalized gamma and beta functions are obtained.

Theorem 2.1. *For the new generalized gamma function, we have*

$$\Gamma_p^{(\alpha,\beta;m)}(s) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1 \Gamma_{p\mu^{m+1}}(s)\mu^{\alpha-s-1}(1-\mu)^{\beta-\alpha-1}d\mu$$

Proof. Using the definition of gamma function (18) and integral representation of confluent hypergeometric function, we have

$$\Gamma_p^{(\alpha,\beta;m)}(s) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^\infty\int_0^1 u^{s-1}e^{-ut-\frac{pt}{u^m}}t^{\alpha-1}(1-t)^{\beta-\alpha-1}dtdu$$

Now using a one-to-one transformation (except possibly at the boundaries and maps the region onto itself) $\nu = ut$, $\mu = t$ in the above equality and considering that the Jacobian of the transformation is J = $\frac{1}{\mu}$, we get

$$\Gamma_p^{(\alpha,\beta;m)}(s) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^\infty\int_0^1 \nu^{s-1}e^{-\nu-\frac{p\mu^{m+1}}{\nu}}d\nu\,\mu^{\alpha-s-1}(1-\mu)^{\beta-\alpha-1}d\mu$$

From the uniform convergence of the integrals, the order of integration can be interchanged to yield that

$$\Gamma_p^{(\alpha,\beta;m)}(s) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1\left[\int_0^\infty \nu^{s-1}e^{-\nu-\frac{p\mu^{m+1}}{\nu}}d\nu\right]\mu^{\alpha-s-1}(1-\mu)^{\beta-\alpha-1}d\mu,$$

$$\Gamma_p^{(\alpha,\beta;m)}(s) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^1 \Gamma_{p\mu^{m+1}} \mu^{\alpha-s-1}(1-\mu)^{\beta-\alpha-1} d\mu.$$

This completes the proof. □

Remark 2.2. In theorem 2.1, the case $m = 1$ and for $p = 0$ gives (see [9, page 192])

$$\Gamma_0^{(\alpha,\beta;1)}(s) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^1 \Gamma(s)\mu^{\alpha-s-1}(1-\mu)^{\beta-\alpha-1} d\mu = \frac{\Gamma(\beta)\Gamma(\alpha-s)\Gamma(s)}{\Gamma(\alpha)\Gamma(\beta-s)}$$

Theorem 2.3. *For the new generalized beta function, we have the following functional relation:*

$$B_p^{(\alpha,\beta;m)}(x,y+1) + B_p^{(\alpha,\beta;m)}(x+1,y) = B_p^{(\alpha,\beta;m)}(x,y)$$

Proof. Direct calculations yield

$$\begin{aligned}
&B_p^{(\alpha,\beta;m)}(x,y+1) + B_p^{(\alpha,\beta;m)}(x+1,y) \\
&= \int_0^1 t^x(1-t)^{y-1} {}_1F_1\left(\alpha;\beta;\frac{-p}{t^m(1-t)^m}\right) dt \\
&+ \int_0^1 t^{x-1}(1-t)^y {}_1F_1\left(\alpha;\beta;\frac{-p}{t^m(1-t)^m}\right) dt \\
&= \int_0^1 \left[t^x(1-t)^{y-1} + t^{x-1}(1-t)^y\right] {}_1F_1\left(\alpha;\beta;\frac{-p}{t^m(1-t)^m}\right) dt \\
&= \int_0^1 t^{x-1}(1-t)^{y-1} {}_1F_1\left(\alpha;\beta;\frac{-p}{t^m(1-t)^m}\right) dt \\
&= B_p^{(\alpha,\beta;m)}(x,y)
\end{aligned}$$

which completes the proof.

□

Theorem 2.4. *For the new generalized beta function, we have the following summation relation:*

$$B_p^{(\alpha,\beta;m)}(x,1-y) = \sum_{n=0}^{\infty} \frac{(y)_n}{n!} B_p^{(\alpha,\beta;m)}(x+n,1) \qquad (\Re(p) > 0, \Re(m) > 0)$$

Proof. From the definition of the generalized beta function (19), we have

$$B_p^{(\alpha,\beta;m)}(x,1-y)=\int_0^1 t^{x-1}(1-t)^{-y}{}_1F_1\left(\alpha;\beta;\tfrac{-p}{t^m(1-t)^m}\right)dt$$

Using the following binomial series expansion

$$(1-t)^{-y}=\sum_{n=0}^{\infty}(y)_n\tfrac{t^n}{n!}\qquad (|\,t\,|<1)$$

we obtain

$$B_p^{(\alpha,\beta;m)}(x,1-y)=\int_0^1\sum_{n=0}^{\infty}\tfrac{(y)_n}{n!}t^{x+n-1}{}_1F_1\left(\alpha;\beta;\tfrac{-p}{t^m(1-t)^m}\right)dt.$$

Therefore, interchanging the order of integration and summation and then using definition (19), we obtain

$$\begin{aligned}B_p^{(\alpha,\beta;m)}(x,1-y)&=\sum_{n=0}^{\infty}\tfrac{(y)_n}{n!}\int_0^1 t^{x+n-1}{}_1F_1\left(\alpha;\beta;\tfrac{-p}{t^m(1-t)^m}\right)dt\\&=\sum_{n=0}^{\infty}\tfrac{(y)_n}{n!}B_p^{(\alpha,\beta;m)}(x+n,1)\end{aligned}$$

which completes the proof. □

Theorem 2.5. *For the new generalized beta function, we have the following summation relation:*

$$B_p^{(\alpha,\beta;m)}(x,y)=\sum_{n=0}^{\infty}B_p^{(\alpha,\beta;m)}(x+n,y+1)\qquad (\Re(p)>0,\Re(m)>0)$$

Proof. From the definition of generalized beta function(19), we have

$$B_p^{(\alpha,\beta;m)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha;\beta;\frac{-p}{t^m(1-t)^m}\right)dt$$

Using $(1-t)^{y-1}=(1-t)^y\sum_{n=0}^{\infty}t^n\qquad (|\,t\,|<1)$, we get

$$B_p^{(\alpha,\beta;m)}(x,y)=\int_0^1 t^{x-1}(1-t)^y\sum_{n=0}^{\infty}t^n{}_1F_1\left(\alpha;\beta;\tfrac{-p}{t^m(1-t)^m}\right)dt$$

Interchanging the order of integration and summation, we get

$$\begin{aligned}B_p^{(\alpha,\beta;m)}(x,y)&=\sum_{n=0}^{\infty}\int_0^1 t^{x+n-1}(1-t)^y{}_1F_1\left(\alpha;\beta;\tfrac{-p}{t^m(1-t)^m}\right)dt\\&=\sum_{n=0}^{\infty}B_p^{(\alpha,\beta;m)}(x+n,y+1)\end{aligned}$$

which completes the proof. □

2.2. Mellin transform representation of generalized beta function

In this section the Mellin transform representation of the generalized beta function $B_p^{(\alpha,\beta;m)}(x,y)$ is obtained in terms of the classical beta function and the gamma function $\Gamma^{(\alpha,\beta)}(s)$ for the case when $p=0$.

Theorem 2.6. *Mellin transform representation of the new generalized beta function is given by*

$$\mathcal{M}\left\{B_p^{(\alpha,\beta;m)}(x,y):s\right\}=B(x+ms,y+ms)\Gamma^{(\alpha,\beta)}(s)$$

$$(\Re(s)>0,\Re(x+ms)>0,\Re(y+ms)>0,\Re(p)>0,\Re(\alpha)>0,\Re(\beta)>0)$$

Proof. Taking Mellin transform of $B_p^{(\alpha,\beta;m)}(x,y)$, we get

$$\mathcal{M}\left\{B_p^{(\alpha,\beta;m)}(x,y):s\right\}=\int\limits_0^\infty p^{s-1}\int\limits_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha;\beta,-\tfrac{p}{t^m(1-t)^m}\right)dtdp$$

From the uniform convergence of the integral, the order of integration can be interchanged. Therefore, we have

$$\mathcal{M}\left\{B_p^{(\alpha,\beta;m)}(x,y):s\right\}=\int\limits_0^1 t^{x-1}(1-t)^{y-1}\int\limits_0^\infty p^{s-1}{}_1F_1\left(\alpha;\beta,-\tfrac{p}{t^m(1-t)^m}\right)dpdt$$

Now using the one-to-one transformation(except possibly at the boundaries and maps the region onto itself)$\nu=\frac{p}{t^m(1-t)^m}$,$t=\mu$, we get,

$$\mathcal{M}\left\{B_p^{(\alpha,\beta;m)}(x,y):s\right\}=\int\limits_0^1 \mu^{(ms+x)-1}(1-\mu)^{(ms+y)-1}d\mu$$

$$\times\int\limits_0^\infty \nu^{s-1}{}_1F_1(\alpha;\beta,-\nu)\,d\nu$$

$$=\int\limits_0^1 \mu^{(ms+x)-1}(1-\mu)^{(ms+y)-1}d\mu\Gamma^{(\alpha,\beta)}(s)$$

$$=B(ms+x,y+ms)\Gamma^{(\alpha,\beta)}(s)$$

which completes the proof. □

Corollary 2.7. *By the Mellin inversion formula, we have the following complex integral representation for $B_p^{(\alpha,\beta;m)}(x,y)$:*

$$B_p^{(\alpha,\beta;m)}(x,y)=\frac{1}{2\pi i}\int\limits_{-i\infty}^{+i\infty}B(ms+x,y+ms)\Gamma^{(\alpha,\beta)}(s)p^{-s}ds$$

Proof. Taking Mellin inversion of Theorem 2.6, we get the result. □

Remark 2.8. Applying definition of Mellin transform and putting $s=1$ and considering that $\Gamma^{(\alpha,\beta)}(1)=\frac{\Gamma(\beta)\Gamma(\alpha-1)}{\Gamma(\alpha)\Gamma(\beta-1)}$ in Theorem 2.6, we get

$$\int_0^\infty B_p^{(\alpha,\beta;m)}(x,y)dp = B(x+m,y+m)\frac{\Gamma(\beta)\Gamma(\alpha-1)}{\Gamma(\alpha)\Gamma(\beta-1)}.$$

Letting $\alpha = \beta$, it reduces to Lee et al. [11] relation

$$\int_0^\infty B_p^m(x,y)dp = B(x+m,y+m) \quad (\Re(x+m) > -1, \Re(y+m) > -1)$$

and for $m = 1$, it reduces to Chaudhry et al. [2] interesting relations between the classical and the extended beta functions.

$$\int_0^\infty B_p(x,y)dp = B(x+1,y+1) \quad (\Re(x) > -1, \Re(y) > -1)$$

2.3. Integral representation of generalized gamma and beta function

Theorem 2.9. *For the product of two new generalized gamma function, we have the following integral representation:*

$$\Gamma_p^{(\alpha,\beta;m)}(x)\Gamma_p^{(\alpha,\beta;m)}(y) = 4\int_0^{\frac{\pi}{2}}\int_0^\infty r^{2(x+y)-1}\cos^{2x-1}\theta\sin^{2y-1}\theta$$

$${}_1F_1\left(\alpha;\beta;-r^2\cos^2\theta - \frac{p}{r^{2m}\cos^{2m}\theta}\right){}_1F_1\left(\alpha;\beta;-r^2\sin^2\theta - \frac{p}{r^{2m}\sin^{2m}\theta}\right)drd\theta$$

Proof. Substituting $t = \eta^2$ and $t = \xi^2$ in (18), we get

$\Gamma_p^{(\alpha,\beta;m)}(x) = 2\int_0^\infty \eta^{2x-1}{}_1F_1\left(\alpha;\beta;-\eta^2 - \frac{p}{\eta^{2m}}\right)d\eta$

and

$\Gamma_p^{(\alpha,\beta;m)}(x) = 2\int_0^\infty \xi^{2x-1}{}_1F_1\left(\alpha;\beta;-\xi^2 - \frac{p}{\xi^{2m}}\right)d\xi$

Therefore

$$\Gamma_p^{(\alpha,\beta;m)}(x)\Gamma_p^{(\alpha,\beta;m)}(y)$$

$$= 4\int_0^\infty\int_0^\infty \eta^{2x-1}\xi^{2y-1}{}_1F_1\left(\alpha;\beta;-\eta^2 - \frac{p}{\eta^{2m}}\right){}_1F_1\left(\alpha;\beta;-\xi^2 - \frac{p}{\xi^{2m}}\right)d\eta d\xi$$

Letting $\eta = r\cos\theta$ and $\xi = r\sin\theta$ in the above equality,we get

$$\Gamma_p^{(\alpha,\beta;m)}(x)\Gamma_p^{(\alpha,\beta;m)}(y) = 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} r^{2(x+y)-1}\cos^{2x-1}\theta\sin^{2y-1}\theta$$

$${}_1F_1\left(\alpha;\beta;-r^2\cos^2\theta - \frac{p}{r^{2m}\cos^{2m}\theta}\right){}_1F_1\left(\alpha;\beta;-r^2\sin^2\theta - \frac{p}{r^{2m}\sin^{2m}\theta}\right)drd\theta$$

This completes the proof. □

Remark 2.10. Putting $\alpha = \beta$ and $p = 0$, we get the classical relation between the gamma and beta function:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

Theorem 2.11. *For the new generalized beta function, we have the following integral representations:*

$$B_p^{(\alpha,\beta;m)}(x,y) = 2\int_0^{\frac{\pi}{2}}\cos^{2x-1}\theta\sin^{2y-1}\theta\,{}_1F_1(\alpha;\beta;-psec^{2m}\theta csc^{2m}\theta)d\theta$$

$$B_p^{(\alpha,\beta;m)}(x,y) = \int_0^{\infty}\frac{u^{x-1}}{(1+u)^{x+y}}{}_1F_1(\alpha;\beta;-p\left(2+u+\tfrac{1}{u}\right)^m)du$$

$$B_p^{(\alpha,\beta;m)}(x,y) = \int_0^{1}\frac{u^{x-1}+u^{y-1}}{(1+u)^{x+y}}{}_1F_1(\alpha;\beta;-p\left(2+u+\tfrac{1}{u}\right)^m)du$$

$$(\Re(p)>0,\Re(x)>0,\Re(y)>0,\Re(\alpha)>0,\Re(\beta)>0,\Re(m)>0)$$

Proof. Letting $t = \cos^2\theta$ in (19), we get

$$B_p^{(\alpha,\beta;m)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha;\beta;\frac{-p}{t^m(1-t)^m}\right)dt$$

$$= 2\int_0^{\frac{\pi}{2}}\cos^{2x-1}\theta\sin^{2y-1}\theta\,{}_1F_1\left(\alpha;\beta;-psec^{2m}\theta csc^{2m}\theta\right)d\theta$$

On the other hand, letting $t = \frac{u}{1+u}$ in (19), we get

$$B_p^{(\alpha,\beta;m)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha;\beta;\frac{-p}{t^m(1-t)^m}\right)dt$$

$$= \int_0^{\infty}\frac{u^{x-1}}{(1+u)^{x+y}}{}_1F_1\left(\alpha;\beta;-p\left(2+u+\frac{1}{u}\right)^m\right)du$$

Again, we have by above integral

$$B_p^{(\alpha,\beta;m)}(x,y) = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\, {}_1F_1(\alpha;\beta;-p\left(2+u+\frac{1}{u}\right)^m)du$$
$$= \int_0^1 \frac{u^{x-1}}{(1+u)^{x+y}}\, {}_1F_1\left(\alpha;\beta;-p\left(2+u+\frac{1}{u}\right)^m\right)du$$
$$+ \int_1^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\, {}_1F_1\left(\alpha;\beta;-p\left(2+u+\frac{1}{u}\right)^m\right)du$$

Substituting $u = \frac{1}{t}$ in second integral and simplifying , we get the result. $\square$

2.4. The beta distribution for new generalized beta function

As an one of the application of this new generalized beta function is in statistics.For example, generalizing conventional beta distribution to variables a and b with an infinite range.

Let the new generalized beta distribution be

$$f(t) = \begin{cases} \frac{1}{B_p^{(\alpha,\beta;m)}(a,b)} t^{a-1}(1-t)^{b-1}\, {}_1F_1(\alpha;\beta;-\frac{p}{t^m(1-t)^m}) & 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

A random variable X with probability density function given by $f(t)$ will be said to have the generalized beta distribution with parameter a and b such that $-\infty < a < \infty, -\infty < b < \infty, p > 0, \Re(\alpha) > 0, \Re(\beta) > 0$ and $\Re(m) > 0$.

If ν is any real number, then

$$E(X^\nu) = \frac{B_p^{(\alpha,\beta;m)}(a+\nu,b)}{B_p^{(\alpha,\beta)}(a,b)}; \quad \text{and for } \nu = 1,\ \mu = E(X) = \frac{B_p^{(\alpha,\beta;m)}(a+1,b)}{B_p^{(\alpha,\beta)}(a,b)}$$

represents the mean of the distribution and

$$\sigma^2 = E(X^2) - \{E(X)\}^2$$
$$= \frac{B_p^{(\alpha,\beta;m)}(a,b)B_p^{(\alpha,\beta;m)}(a+2,b) - \left(B_p^{(\alpha,\beta;m)}(a+1,b)\right)^2}{\left(B_p^{(\alpha,\beta;m)}(a,b)\right)^2}$$

is the variance of the distribution.

The moment generating function of the distribution is

$$\mathbb{M}(t)=\sum_{n=0}^{\infty}\frac{t^n}{n!}E(X^n)=\frac{1}{B_p^{(\alpha,\beta;m)}(a,b)}\sum_{n=0}^{\infty}B_p^{(\alpha,\beta;m)}(a+n,b)\frac{t^n}{n!}$$

The cumulative distribution of $f(t)$ can be written as

$$F(x)=\frac{B_x^{(\alpha,\beta;m)}(a,b;p)}{B_p^{(\alpha,\beta;m)}(a,b)}$$

where

$$B_x^{(\alpha,\beta;m)}(a,b;p)=\int_0^x t^{a-1}(1-t)^{b-1}{}_1F_1(\alpha;\beta;-\frac{p}{t^m(1-t)^m})dt$$

is called new generalized incomplete beta function.

3. Generalized Gauss's hypergeometric and confluent hypergeometric functions

In this section, the new generalization of beta function (19) is used to define the new generalized hypergeometric and confluent hypergeometric functions as

$$F_p^{(\alpha,\beta;m)}(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\alpha,\beta;m)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}$$

$$(p\geq 0;|\,z\,|<1;\Re(c)>\Re(b)>0,\Re(\alpha)>0,\Re(\beta)>0,\Re(m)>0)$$

and

$$\Phi_p^{(\alpha,\beta;m)}(b;c;z)=\sum_{n=0}^{\infty}\frac{B_p^{(\alpha,\beta;m)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}$$

$$(p\geq 0;\Re(c)>\Re(b)>0,\Re(\alpha)>0,\Re(\beta)>0,\Re(m)>0)$$

respectively.

Clearly,

$F_p^{(\alpha,\alpha;m)}(a,b;c;z)=F_p^{m}(a,b;c;z),\quad F_p^{(\alpha,\beta;1)}(a,b;c;z)=F_p^{(\alpha,\beta)}(a,b;c;z),$

$F_0^{(\alpha,\alpha;1)}(a,b;c;z)={}_2F_1(a,b;c;z)$

and

$\Phi_p^{(\alpha,\alpha;m)}(b;c;z)=\Phi_p^{m}(b;c;z),\quad \Phi_p^{(\alpha,\beta;1)}(b;c;z)=\Phi_p^{(\alpha,\beta)}(b;c;z),$

$\Phi_0^{(\alpha,\alpha;1)}(b;c;z)={}_1F_1(b;c;z)$

3.1. Integral Representations

In this section, integral representation of the new generalized hypergeometric and confluent hypergeometric function is obtained by using definition of generalized beta function (19).

Theorem 3.1. *For the new generalized hypergeometric function , we have the following integral representations:*

$$F_p^{(\alpha,\beta;m)}(a,b;c;z)$$
$$= \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} {}_1F_1\left(\alpha;\beta;\frac{-p}{t^m(1-t)^m}\right) dt$$
$$(p>0; p=0 \text{ and } \mid arg(1-z) \mid < \pi; \Re(c) > \Re(b) > 0, \Re(m) > 0) \tag{20}$$

$$F_p^{(\alpha,\beta;m)}(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^\infty u^{b-1}(1+u)^{a-c}[1+u(1-z)]^{-a}$$
$${}_1F_1\left(\alpha;\beta;-p\left(2+u+\frac{1}{u}\right)^m\right) du$$

$$F_p^{(\alpha,\beta;m)}(a,b;c;z) = \frac{2}{B(b,c-b)} \int_0^{\frac{\pi}{2}} \sin^{2b-1}\nu \cos^{2c-2b-1}\nu (1-z\sin^2\nu)^{-a}$$
$${}_1F_1\left(\alpha;\beta;-p\sec^{2m}\nu\csc^{2m}\nu\right) d\nu$$

Proof. Direct calculations yield

$$F_p^{(\alpha,\beta;m)}(a,b;c;z) = \sum_{n=0}^\infty (a)_n \frac{B_p^{(\alpha,\beta;m)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!}$$
$$= \frac{1}{B(b,c-b)} \sum_{n=0}^\infty (a)_n \int_0^1 t^{b+n-1}(1-t)^{c-b-1} {}_1F_1\left(\alpha;\beta;\frac{-p}{t^m(1-t)^m}\right) \frac{z^n}{n!} dt$$
$$= \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} {}_1F_1\left(\alpha;\beta;\frac{-p}{t^m(1t)^m}\right) \sum_{n=0}^\infty (a)_n \frac{(zt)^n}{n!} dt$$
$$= \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} {}_1F_1\left(\alpha;\beta;\frac{-p}{t^m(1-t)^m}\right) (1-zt)^{-a} dt.$$

Setting $u=\frac{t}{1-t}$ in (20), we get

$$F_p^{(\alpha,\beta;m)}(a,b;c;z)=\frac{1}{B(b,c-b)}\int_0^\infty u^{b-1}(1+u)^{a-c}[1+u(1-z)]^{-a}$$
$$ {}_1F_1\left(\alpha;\beta;-p\left(2+u+\frac{1}{u}\right)^m\right)du$$

On the other hand, substituting $t=\sin^2 v$ in (20), we have

$$F_p^{(\alpha,\beta;m)}(a,b;c;z)=\frac{2}{B(b,c-b)}\int_0^{\frac{\pi}{2}} \sin^{2b-1} v\cos^{2c-2b-1} v(1-z\sin^2 v)^{-a}$$
$$ {}_1F_1\left(\alpha;\beta;-p\sec^{2m} v\csc^{2m} v\right)dv$$

□

Theorem 3.2. *For the new generalized confluent hypergeometric function, we have the following integral representations:*

$$\Phi_p^{(\alpha,\beta;m)}(b;c;z)=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}e^{zt}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t^m(1-t)^m}\right)dt$$
$$\Phi_p^{(\alpha,\beta;m)}(b;c;z)=\int_0^1 \frac{(1-u)^{b-1}u^{c-b-1}}{B(b,c-b)}e^{z(1-u)}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{u^m(1-u)^m}\right)dt$$
$$(p\geq 0;\Re(c)>\Re(b)>0 \textit{ and } \Re(m)>0) \quad (21)$$

Proof. A similar procedure yields an integral representation of the new generalized confluent hypergeometric function by using the definition of the new generalized beta function (19). □

Remark 3.3. Putting $\alpha=\beta$ and $p=0$ in (20) and (21), we get the integral representations of the classical GHF and CHF.

3.2. Differentiation formulas

In this section, new formulas including derivatives of generalized hypergeometric and confluent hypergeometric function with respect to the variable z in terms of a shift of the operator is obtained by using the formulas :

$B(b,c-b)=\frac{c}{b}B(b+1,c-b)$ and $(a)_{n+1}=a(a+1)_n$,

Theorem 3.4. *For the new generalized hypergeometric function we have the following differentiation formula:*

$$\frac{d^n}{dz^n}\left[F_p^{(\alpha,\beta;m)}(a,b;c;z)\right] = \frac{(b)_n(a)_n}{(c)_n} F_p^{(\alpha,\beta;m)}(a+n,b+n;c+n;z).$$

Proof. Taking the derivative of $F_p^{(\alpha;\beta;m)}(a,b;c;z)$ with respect to z, we obtain

$$\begin{aligned}\frac{d}{dz}\left[F_p^{(\alpha,\beta;m)}(a,b;c;z)\right] &= \frac{d}{dz}\left[\sum_{n=0}^{\infty}(a)_n \frac{B_p^{(\alpha,\beta;m)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}\right] \\ &= \sum_{n=1}^{\infty}(a)_n \frac{B_p^{(\alpha,\beta;m)}(b+n,c-b)}{B(b,c-b)}\frac{z^{n-1}}{(n-1)!}\end{aligned}$$

Replacing $n \to n+1$, we get

$$\begin{aligned}\frac{d}{dz}\left[F_p^{(\alpha,\beta;m)}(a,b;c;z)\right] &= \frac{ba}{c}\sum_{n=0}^{\infty}(a+1)_n \frac{B_p^{(\alpha,\beta;m)}(b+n+1,c-b)}{B(b+1,c-b)}\frac{z^n}{n!} \\ &= \frac{ba}{c}F_p^{(\alpha,\beta;m)}(a+1,b+1;c+1;z)\end{aligned}$$

Recursive application of this procedure gives us the general form

$$\frac{d^n}{dz^n}\left[F_p^{(\alpha,\beta;m)}(a,b;c;z)\right] = \frac{(b)_n(a)_n}{(c)_n} F_p^{(\alpha,\beta;m)}(a+n,b+n;c+n;z).$$

□

Theorem 3.5. *For the new generalized confluent hypergeometric function, we have the following differentiation formula:*

$$\frac{d^n}{dz^n}\left[\Phi_p^{(\alpha,\beta;m)}(b;c;z)\right] = \frac{(b)_n}{(c)_n}\Phi_p^{(\alpha,\beta;m)}(b+n;c+n;z)$$

Proof. A similar procedure as Theorem 3.4 gives the result. □

3.3. Mellin transform representation

In this section, the Mellin transform representations of the generalized hypergeometric and confluent hypergeometric function is obtained.

Theorem 3.6. *For the new generalized hypergeometric function, we have the following Mellin transform representation:*

$$\mathcal{M}\left[F_p^{(\alpha,\beta;m)}(a,b;c;z):s\right] = \frac{\Gamma^{(\alpha,\beta)}(s)B(b+ms,c+ms-b)}{B(b,c-b)}\,{}_2F_1(a,b+ms;c+2ms;z).$$

Proof. To obtain the Mellin transform, multiply both sides of (20) by p^{s-1} and integrate with respect to p over the interval $[0,\infty)$. Thus we get

$$\mathcal{M}\left[F_p^{(\alpha,\beta;m)}(a,b;c;z):s\right]=\int_0^\infty p^{s-1}F_p^{(\alpha,\beta;m)}(a,b;c;z)dp=\frac{1}{B(b,c-b)}$$
$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\left[\int_0^\infty p^{s-1}{}_1F_1\left(\alpha,\beta;\frac{-p}{t^m(1-t)^m}\right)dp\right]dt$$

Substitution of $u=\frac{p}{t^m(1-t)^m}$ in the integral then leads to

$$\begin{aligned}\int_0^\infty p^{s-1}\quad{}_1F_1\left(\alpha;\beta;\frac{-p}{t^m(1-t)^m}\right)dp&=\int_0^\infty u^{s-1}t^{ms}(1-t)^{ms}{}_1F_1(\alpha;\beta;-u)du\\&=t^{ms}(1-t)^{ms}\int_0^\infty u^{s-1}{}_1F_1(\alpha;\beta;-u)du\\&=t^{ms}(1-t)^{ms}\Gamma^{(\alpha,\beta)}(s)\end{aligned}$$

Thus we get

$$\begin{aligned}&\mathcal{M}\left[F_p^{(\alpha,\beta;m)}(a,b;c;z):s\right]\\&=\frac{1}{B(b,c-b)}\int_0^1 t^{b+ms-1}(1-t)^{c+ms-b-1}(1-zt)^{-a}\Gamma^{(\alpha,\beta)}(s)dt\\&=\frac{\Gamma^{(\alpha,\beta)}(s)}{B(b,c-b)}\int_0^1 t^{b+ms-1}(1-t)^{c+2ms-(b+ms)-1}(1-zt)^{-a}dt\\&=\frac{\Gamma^{(\alpha,\beta)}(s)B(b+ms,c+ms-b)}{B(b,c-b)}{}_2F_1(a,b+ms;c+2ms;z).\end{aligned}$$

□

Corollary 3.7. *By the Mellin inversion formula, we have the following complex integral representation for $F_p^{(\alpha,\beta;m)}(a,b;c;z)$:*

$$\begin{aligned}&F_p^{(\alpha,\beta;m)}(a,b;c;z)\\&=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\Gamma^{(\alpha,\beta)}(s)B(b+ms,c+ms-b)}{B(b,c-b)}{}_2F_1(a,b+ms;c+2ms;z)p^{-s}ds\end{aligned}$$

Proof. Taking Mellin inversion of Theorem 3.6, we get the result. □

Theorem 3.8. *For the new generalized confluent hypergeometric function, we have the following Mellin transform representation:*

$$\mathcal{M}\left[\Phi_p^{(\alpha,\beta;m)}(b;c;z):s\right]=\frac{\Gamma^{(\alpha,\beta)}(s)B(b+ms,c+ms-b)}{B(b,c-b)}\,{}_1F_1(b+ms;c+2ms;z)$$

Proof. A similar argument as Theorem 3.6 gives the Mellin transform of new generalized confluent hypergeometric function □

Corollary 3.9. *By the Mellin inversion formula, we have the following complex integral representation for* $\Phi_p^{(\alpha,\beta;m)}(b;c;z)$*:*

$$\Phi_p^{(\alpha,\beta;m)}(b;c;z)=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\Gamma^{(\alpha,\beta)}(s)B(b+ms,c+ms-b)}{B(b,c-b)}\,{}_1F_1(b+ms;c+2ms;z)p^{-s}ds$$

Proof. Taking Mellin inversion of Theorem 3.8, we get the result. □

3.4. Transformation formulas

Theorem 3.10. *For the new generalized hypergeometric function, we have the following transformation formula:*

$$F_p^{(\alpha,\beta;m)}(a,b;c;z)=(1-z)^{-a}F_p^{(\alpha,\beta;m)}\left(a,c-b;c;\frac{z}{z-1}\right),(\mid arg(1-z)\mid<\pi)$$

Proof. By writing

$$[1-z(1-t)]^{-a}=(1-z)^{-a}\left(1+\frac{z}{1-z}t\right)^{-a}$$

and replacing $t\to 1-t$ in (20), we obtain

$$\begin{aligned}&F_p^{(\alpha,\beta;m)}(a,b;c;z)\\&=\frac{(1-z)^{-a}}{B(b,c-b)}\int_0^1(1-t)^{b-1}t^{c-b-1}\left(1-\frac{z}{z-1}t\right)^{-a}{}_1F_1\left(\alpha;\beta;\frac{-p}{t^m(1-t)^m}\right)dt\end{aligned}$$

Hence,

$$F_p^{(\alpha,\beta;m)}(a,b;c;z)=(1-z)^{-a}F_p^{(\alpha,\beta;m)}\left(a,c-b;c;\frac{z}{z-1}\right)$$

□

Remark 3.11. Note that, replacing z by $1-\frac{1}{z}$ in Theorem 3.10, we get the following transformation formula

$$F_p^{(\alpha,\beta;m)}\left(a,b;c;1-\frac{1}{z}\right)=z^{\alpha}F_p^{(\alpha,\beta;m)}(a,c-b;b;1-z),(|\,arg(z)\,|<\pi).$$

Furthermore, replacing z by $\frac{z}{1+z}$ in Theorem 3.10, we get the following transformation formula

$$F_p^{(\alpha,\beta;m)}\left(a,b;c;\frac{z}{1+z}\right)=(1+z)^{\alpha}F_p^{(\alpha,\beta;m)}(a,c-b;b;z),(|\,arg(1+z)\,|<\pi)$$

Theorem 3.12. *For the new generalized confluent hypergeometric function, we have the following transformation formula:*

$$\Phi_p^{(\alpha,\beta;m)}(b;c;z)=\exp(z)\Phi_p^{(\alpha,\beta;m)}(c-b;c;-z)$$

Proof. Using the definition of new generalized confluent hypergeometric function (21), we have

$$\Phi_p^{(\alpha,\beta;m)}(b;c;z)=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}e^{zt}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t^m(1-t)^m}\right)dt$$

replacing $t\to t-1$, we get the result. □

Remark 3.13. For $\alpha=\beta$ and $p=0$ we get Kummer's first transformation formula.

3.5. Summation Formula

Theorem 3.14. *For new generalized hypergeometric function we have the following summation formula*

$$F_p^{(\alpha,\beta;m)}(a,b;c;1)=\frac{B_p^{(\alpha,\beta;m)}(b,c-a-b)}{B(b,c-b)}$$

Proof. Setting $z=1$ in (20), we have the following relation between new generalized hypergeometric and beta function:

$F_p^{(\alpha,\beta;m)}(a,b;c;1)=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-a-b-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t^m(1-t)^m}\right)dt,$

$F_p^{(\alpha,\beta;m)}(a,b;c;1)=\frac{B_p^{(\alpha,\beta;m)}(b,c-a-b)}{B(b,c-b)};$

when $\alpha=\beta$ and $p=0$, we get Gauss's summation formula. □

4. Concluding remarks

In this present paper, generalizations of gamma, beta, hypergeometric and confluent hypergeometric functions are introduced.The special cases of these generalizations include the extension of gamma, beta, hypergeometric and confluent hypergeometric functions which were proposed in [1, 2, 8, 10, 11], respectively.Some properties of these generalized functions are investigated in this paper and most of which are analogous with their original functions. Most of the special functions of mathematical physics and engineering can be expressed in terms of these generalizations. Therefore, the corresponding generalizations of several other familiar special functions are expected to be useful and need to be investigated.

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