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A NEW GENERALIZATION OF THE OSTROWSKI INEQUALITY AND APPLICATIONS

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Abstract

A new generalization of the Ostrowski inequality for functions in L^p-spaces is introduced and then applied to provide some estimates for the error value of numerical quadrature rules of equal coefficients type.

1 Introduction

Denote by $L^p\ [a,b]$ $(1 \leq p \leq \infty)$ the space of p-power integrable functions on the interval $[a,b]$ with the corresponding norm:

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt \right)^{\frac{1}{p}}. \tag{1}$$

Also let $L^\infty[a,b]$ be the space of all essentially bounded functions on $[a,b]$ with the related norm: $\|f\|_\infty = ess \sup_{x\in[a,b]} |f(x)|$.

For two absolutely continuous functions $f, g : [a,b] \longrightarrow \mathbb{R}$ such that $f,\ g,\ fg \in L^1[a,b]$, the chebyshev functional [1,6] is defined by

$$\begin{aligned} T(f,g) &= \frac{1}{b-a}\int_a^b \left(f(x) - \frac{1}{b-a}\int_a^b f(x)dx \right)\left(g(x) - \frac{1}{b-a}\int_a^b g(x)dx \right) dx \\ &= \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right). \end{aligned} \tag{2}$$

A result related to Chebyshev functional is the Ostrowski inequality [8], i.e. if $f : [a,b] \longrightarrow \mathbb{R}$ is a differentiable function with bounded derivative, then

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left(\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2} \right)(b-a)\|f'\|_\infty \ for\ all\ x \in [a,b]. \tag{3}$$

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This inequality plays a key role in numerical quadrature rules, see e.g. [3,4]. The following theorem [7] is probably the most recent work about finding appropriate bounds for the Chebyshev functional:

1.1 Theorem A Let $f,\ \alpha,\ \beta \in L^p[a,b]$ and $g \in L^q[a,b]$ $(1/p+1/q=1,\ 1 \le p \le \infty)$ be functions such that $\alpha(t)+\beta(t)$ is a constant function and $\alpha(t) \le f(t) \le \beta(t)$ for all $t \in [a,b]$. Then we have the inequality

$$|\,\mathbf{T}\,(f,g)\,| \leqslant \frac{1}{2(b-a)}\,\|\,\beta-\alpha\,\|_p \left\| g - \frac{1}{b-a}\int_a^b g(t)\,dt \right\|_q . \tag{4}$$

The main aim of this paper is to introduce a new generalization of the Ostrowski inequality and apply it to find appropriate error bounds for numerical quadrature rules of equal coefficients type. For this purpose, we first define the following kernel on $[a,b]$:

$$K_w(x;t) = \begin{cases} t - \dfrac{(b-w)f(b)-af(a)}{f(b)-f(a)} = t-\theta_1 & t \in [a,x], \\ t - \dfrac{bf(b)-(a+w)f(a)}{f(b)-f(a)} = t-\theta_2 & t \in (x,b], \end{cases} \tag{5}$$

in which $w \in \mathbb{R},\ f(b) \neq f(a)$ and $\theta_2 - \theta_1 = w$. After some computations, one can directly conclude

$$\left| \int_a^b f'(t)K_w(x;t)dt \right| = \left| wf(t) - \int_a^b f(x)dx \right| . \tag{6}$$

Thus, for $w = b-a$ in (6), the left hand side of Ostrowski inequality is generated. On the other hand, we have

$$\int_a^b K_w(x;t)dt = wx - \frac{\big((b-a)^2+2aw\big)\,f(b) + \big((b-a)^2-2bw\big)\,f(a)}{2\,(f(b)-f(a))} . \tag{7}$$

Consequently

$$\begin{aligned} (b-a)\ |\,\mathbf{T}\,(K_w\,(x;t)\,,f'(t))\,| \ = & \left| w\,f(x) - \int_a^b f(x)\,dx \right. \\ & \left. -\frac{f(b)-f(a)}{b-a} w\,x + \frac{\big((b-a)^2+2aw\big)\,f(b) + \big((b-a)^2-2bw\big)\,f(a)}{2(b-a)} \right| . \end{aligned} \tag{8}$$

The result (8) is important for us because for $w = b-a$ it changes to [2,9]:

$$|T(K_{b-a}(x;t), f'(t))| = \left| f(x) - \frac{1}{b-a}\int_a^b f(x)dx - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) \right| . \tag{9}$$

To present our main theorem we still need to consider some further corollaries.
1.2 Corollary 1. we have

$$\max_{t\in[a,b]}|K_w(x;t)| = \max\left(\max_{t\in[a,b]}|t-\theta_1|;\max_{t\in[a,b]}|t-\theta_2|\right) =$$

$$\max_{\forall x\in[a,b]}(|x-\theta_1|,|a-\theta_1|,|x-\theta_2|,|b-\theta_2|) = \frac{1}{|f(b)-f(a)|}\max_{\forall x\in[a,b]}\{|(a-b+w)f(b)|,$$
$$|(a-b+w)f(a)|,|(x-b+w)f(b)+(a-x)f(a)|,$$
$$|(x-b)f(b)+(a+w-x)f(a)|\}, \tag{10}$$

in which θ_1 and θ_2 are the same values as defined in the kernel (5).
For instance, if $w=0$ in (10) then

$$\max_{t\in[a,b]}|K_0(x;t)| = \max_{t\in[a,b]}\left(\left|t-\frac{bf(b)-af(a)}{f(b)-f(a)}\right|\right) =$$
$$\frac{b-a}{|f(b)-f(a)|}\max\left(|f(b)|,|f(a)|\right). \tag{11}$$

By using this result one can immediately conclude that if $f(b)\neq f(a)$ then

$$\left|\int_a^b f(x)dx\right| = \left|\int_a^b f'(t)K_0(x;t)dt\right| \leq \frac{b-a}{|f(b)-f(a)|}\max\left(|f(b)|,|f(a)|\right)\|f'\|_1. \tag{12}$$

Similarly, since

$$\int_a^b K_w^2(x;t)dt = wx^2 - w\frac{(2b-w)f(b)-(2a+w)f(a)}{f(b)-f(a)}x + \frac{1}{3}w^3 +$$
$$w\frac{((b-w)f(b)-af(a))(bf(b)-(a+w)f(a))}{(f(b)-f(a))^2}, \tag{13}$$

and for $w=0$,

$$\int_a^b K_0^2(x;t)dt = \int_a^b\left(t-\frac{bf(b)-af(a)}{f(b)-f(a)}\right)^2 dt =$$
$$\frac{(b-a)^3}{3}\frac{f^2(b)+f^2(a)+f(a)f(b)}{(f(b)-f(a))^2}, \tag{14}$$

therefore:
1.3. Corollary 2. If $f(b)\neq f(a)$ then

$$\left|\int_a^b f(x)dx\right| \leq \frac{(b-a)^3}{3}\frac{f^2(b)+f^2(a)+f(a)f(b)}{(f(b)-f(a))^2}\|f'\|_2. \tag{15}$$

2 Main Theorem

Let $f : I \to \mathbb{R}$, where I is an interval, be a function differentiable in the interior I^0 of I, and let $[a, b] \subset I^0$. Suppose that $f',\ \alpha,\ \beta \in L^p[a, b]$ are functions such that $\alpha(t) + \beta(t)$ is a constant function and $\alpha(t) \le f'(t) \le \beta$ for all $t \in [a, b]$. Then we respectively have the following inequalities

$$\left| wf(x) - \int_a^b f(x)dx \right| \le$$

$$\begin{cases} \left(\int_a^x |t-\theta_1|^q dt + \int_x^b |t-\theta_2|^q dt \right)^{\frac{1}{q}} \|f'\|_p, & (1 \le p < \infty; \frac{1}{p} + \frac{1}{q} = 1), \\ \left(\int_a^x |t-\theta_1| dt + \int_x^b |t-\theta_2| dt \right) \|f'\|_\infty, & (p = \infty,\ q = 1), \\ \max\limits_{\forall x \in [a,b]} \left(|x-\theta_1|, |a-\theta_1|, |x-\theta_2|, |b-\theta_2| \right) \|f'\|_1, & (q = \infty;\ p = 1), \end{cases} \tag{16}$$

and

$$\left| w\, f(x) - \int_a^b f(x)\, dx - \frac{f(b)-f(a)}{b-a} w\, x + \frac{\big((b-a)^2 + 2aw\big) f(b) + \big((b-a)^2 - 2bw\big) f(a)}{2(b-a)} \right|$$

$$\leqslant \frac{1}{2} \parallel \beta(t) - \alpha(t) \parallel_p \times$$

$$\left(\int_a^x \left| t - \theta_1 - \frac{1}{b-a} \left(wx - \frac{\big((b-a)^2 + 2aw\big) f(b) + \big((b-a)^2 - 2bw\big) f(a)}{2\left(f(b) - f(a)\right)} \right) \right|^q dt \right.$$

$$\left. + \int_x^b \left| t - \theta_2 - \frac{1}{b-a} \left(wx - \frac{\big((b-a)^2 + 2aw\big) f(b) + \big((b-a)^2 - 2bw\big) f(a)}{2\left(f(b) - f(a)\right)} \right) \right|^q dt \right)^{1/q}. \tag{17}$$

Note that to compute the integrals of the right hand side of inequalities (16) and (17) we can use the following general identity:

$$\int_c^d |t-\theta|^q dt = \begin{cases} \dfrac{(d-\theta)^{q+1} + (c-\theta)^{q+1}}{q+1}, & if\ c < \theta < d, \\ \dfrac{(d-\theta)^{q+1} - (c-\theta)^{q+1}}{q+1}, & if\ \theta < c < d, \\ \dfrac{-(d-\theta)^{q+1} + (c-\theta)^{q+1}}{q+1}, & if\ c < d < \theta, \end{cases} \tag{18}$$

in which $c < d,\ q \ge 1$ and $\theta \in \mathbb{R}$.

Proof. The proof of (16) is straightforward if we apply the well-known Hölder's inequality [6]:

$$\|fg\|_1 \leq \|f\|_p \|g\|_q \qquad (\frac{1}{p}+\frac{1}{q}=1), \tag{19}$$

for identity (6) and then refers to the defined kernel (5). For instance, by noting (13), we can apply the Cauchy-Schwarz inequality for $p=q=2$ in (19) to obtain

$$\begin{aligned}\left| w\,f(x) - \int_a^b f(x)\,dx \right| &\leqslant \Bigg(wx^2 - w\frac{(2b-w)f(b)-(2a+w)f(a)}{f(b)-f(a)}x + \frac{1}{3}w^3 \\ &+w\frac{((b-w)f(b)-af(a))\ (bf(b)-(a+w)f(a))}{(f(b)-f(a))^2}\Bigg)^{1/2} \ \ \|f'\|_2\ .\end{aligned} \tag{20}$$

Note that the inequality (20) can still be optimized for

$$x_{min} = \frac{(2b-w)f(b)-(2a+w)f(a)}{2(f(b)-f(a))}, \tag{21}$$

if and only if $w>0$.
Similarly, to prove (17) one should refer to (8) and then use theorem A and relation (7).
Remark 1. Although $\alpha(t) \leq f'(t) \leq \beta(t)$ is a general condition in the main theorem, however sometimes one might not be able to easily obtain both bounds of $\alpha(t)$ and $\beta(t)$ for f'. In this case, we should consider two analogues for the main theorem 1. In the first case, the corresponding theorem would be useful when f' is unbounded above (i.e. $\alpha(t)=\alpha^* \leq f'(t)$) and in the second case, the corresponding theorem is useful when f' is unbounded below i.e. $f'(t) \leq \beta(t) = \beta^*$.

2.1. Theorem 2. Let $f: I \to \mathbb{R}$, where I is an interval, be a differentiable function in the interior I^0 of I, and let $[a,b] \subset I^0$. If f' is a function such that $\alpha^* \leq f'(t)$ for any $t \in [a,b]$ and the constant number α^*, then we have

$$\begin{aligned}&\left| wf(x) - \int_a^b f(x)dx - \alpha^*\left(wx - \frac{((b-a)^2+2aw)f(b)+((b-a)^2-2bw)f(a)}{2(f(b)-f(a))}\right)\right| \\ &\leq \frac{f(b)-f(a)-\alpha^*(b-a)}{|f(b)-f(a)|} \max_{\forall x\in[a,b]} \{|(a-b+w)f(b)|, |(a-b+w)f(a)|, \\ &\qquad |(x-b+w)f(b)+(a-x)f(a)|, |(x-b)f(b)+(a+w-x)f(a)|\}.\end{aligned} \tag{22}$$

Proof. By noting relations (6) and (7) we have

$$\begin{aligned}&\left| wf(x) - \int_a^b f(x)dx - \alpha^*\left(wx - \frac{((b-a)^2+2aw)f(b)+((b-a)^2-2bw)f(a)}{2(f(b)-f(a))}\right)\right| \\ &= \left| \int_a^b K_w(x;t)(f'(t)-\alpha^*)dt \right| \leq \max_{t\in[a,b]} |K_w(x;t)| \int_a^b (f'(t)-\alpha^*)dt\end{aligned}$$

$$= \max_{\forall x\in[a,b]} (|x-\theta_1|, |a-\theta_1|, |x-\theta_2|, |b-\theta_2|)(f(a)-f(a)-\alpha^*(b-a)). \qquad (23)$$

Now, substituting (10) in (23) proves the theorem.

2.2. Theorem 3. Let $f : I \to \mathbb{R}$, where I is an interval, be a differentiable function in the interior I^0 of I, and let $[a,b] \subset I^0$. If f' is a function such that $f'(t) \le \beta^*$ for any $t \in [a,b]$ and the constant number β^*, then we have

$$\left| wf(x) - \int_a^b f(x)dx - \beta^* \left(wx - \frac{((b-a)^2+2aw)f(b) + ((b-a)^2-2bw)f(a)}{2(f(b)-f(a))} \right) \right|$$

$$\le \frac{\beta^*(b-a)-f(b)+f(a)}{|f(b)-f(a)|} \max_{\forall x\in[a,b]} \{ |(a-b+w)f(b)|, |(a-b+w)f(a)|$$

$$|(x-b+w)f(b)+(a-x)f(a)|, |(x-b)f(b)+(a+w-x)f(a)| \}. \qquad (24)$$

3 Application to numerical quadrature rules of equal coefficients type

Consider the following kind of weighted quadrature rules [5,10]:

$$\int_a^b w(x)f(x)dx = \sum_{i=1}^{n} w_i f(x_i) + \sum_{k=1}^{m} v_k f(z_k) + R_{n,m}[f], \qquad (25)$$

where $w(x)$ is a positive function on $[a,b]$; $\{w_i\}_{i=1}^n$, $\{v_k\}_{k=1}^m$ are unknown coefficients; $\{x_i\}_{i=1}^n$ are unknown nodes; $\{z_k\}_{k=1}^m$ are pre-determined nodes and $R_{n,m}[f]$ is the error value denoted by

$$R_{n,m}[f] = \frac{f^{(2n+m)}(\zeta)}{(2n+m)!} \int_a^b w(x) \prod_{k=1}^{m}(x-z_k) \prod_{i=1}^{n}(x-x_i)^2 dx \quad (a<\zeta<b). \qquad (26)$$

For $m = 0$, the formula (25) is reduced to an n-point Gauss quadrature rule as:

$$\int_a^b w(x)f(x)dx = \sum_{i=1}^{n} w_i f(x_i) + \frac{f^{(2n)}(\xi)}{(2n)!} \int_a^b w(x) \prod_{i=1}^{n}(x-x_i)^2 dx. \qquad (27)$$

According to definition, the precision degree of formula (27) is $2n-1$, because for any polynomial of degree $2n-1$ we have $f^{(2n)}(\xi) = 0$.
One of the special cases of Gauss quadrature formula is when $w_1 = w_2 = \cdots = w_n = C_n$. This case is called weighted quadrature rules of equal coefficients and is represented as

$$\int_a^b w(x)f(x)dx \cong C_n \sum_{i=1}^{n} f(x_i), \qquad (28)$$

where C_n is a constant number in terms of n and $\{x_i\}_{i=1}^n$ are quadrature nodes. For instance, when $w(x)=(1-x^2)^{\frac{-1}{2}}$ and $[a,b]=[-1,1]$, the rule (28) is reduced to

$$\int_{-1}^{1}(1-x^2)^{\frac{-1}{2}}f(x)dx \cong \frac{\pi}{n}\sum_{j=1}^{n} f\left(\cos\left(\frac{2j-1}{2n}\pi\right)\right), \tag{29}$$

which is known in the literature as the first kind of GaussChebyshev quadrature rule and has the highest precision degree $2n-1$ [5,10].
Another type of formula (28) is the well known trapezoidal rule for $n=2$, $w(x)=1$, $x_1=a$, $x_2=b$ and $C_n=\frac{b-a}{2}$, i.e.

$$\int_a^b f(x)dx \cong \frac{b-a}{2}(f(a)+f(b)). \tag{30}$$

And the third sample can be any random formula of type (28). For instance, the following approximation

$$\int_0^1 f(x)dx \cong f(0)+f(\frac{1}{2})+f(\frac{\sqrt{2}}{2})+f(0.86), \tag{31}$$

obeys the rule (28) for $n=4$, $[a,b]=[0,1]$, $w(x)=1$, $x_1=0$, $x_2=\frac{1}{2}$, $x_3=\frac{\sqrt{2}}{2}$, $x_4=0.86$ and $C_n=1$.
In this section, we apply inequality (16) (for $p=q=2$ as a sample) to obtain error bounds for quadrature rules of equal coefficients type in which $w(x)=1$, $f\in C^1[a,b]$ and $f'\in L^2[a,b]$ respectively.
For this purpose, let $I_n=\{a=x_0<x_1<\cdots<x_n=b\}$ be a given subdivision of the interval $[a,b]$ such that $f(x_{i+1})\neq f(x_i)$. By applying the introduced inequality (20) to the subinterval $[x_i,x_{i+1}]$ and taking $x=p_i\in[x_i,x_{i+1}]$ we get

$$\left|\int_{x_i}^{x_{i+1}} f(t)\,dt - w\,f(p_i)\right| \leqslant \Bigg(wp_i^2 - w\frac{(2x_{i+1}-w)f(x_{i+1})-(2x_i+w)f(x_i)}{f(x_{i+1})-f(x_i)}p_i+\frac{1}{3}w^3 \\ +w\frac{((x_{i+1}-w)f(x_{i+1})-x_if(x_i))\,(x_{i+1}f(x_{i+1})-(x_i+w)f(x_i))}{(f(x_{i+1})-f(x_i))^2}\Bigg)^{1/2}\left(\int_{x_i}^{x_{i+1}}(f'(t))^2dt\right)^{1/2}. \tag{32}$$

Now, summing the above inequality over i from 0 to $n-1$ and using the well known triangle inequality yield

$$\left|\int_a^b f(t)\,dt - w\sum_{i=0}^{n-1} f(p_i)\right| \leqslant \|f'\|_2 \times \\ \left(\sum_{i=0}^{n-1}\left(\begin{array}{l} wp_i^2 - w\dfrac{(2x_{i+1}-w)f(x_{i+1})-(2x_i+w)f(x_i)}{f(x_{i+1})-f(x_i)}p_i+\dfrac{1}{3}w^3 \\ +w\dfrac{((x_{i+1}-w)f(x_{i+1})-x_if(x_i))\,(x_{i+1}f(x_{i+1})-(x_i+w)f(x_i))}{(f(x_{i+1})-f(x_i))^2}\end{array}\right)^{1/2}\right), \tag{33}$$

because for any $i = 0, 1, \cdots, n-1$ we have

$$0 \leqslant \left(\int_{x_i}^{x_{i+1}} (f'(t))^2 dt \right)^{1/2} \leqslant \left(\int_a^b (f'(t))^2 dt \right)^{1/2}.$$

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