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A New Geometric Average Technique to Solve Multi-Objective Linear Fractional Programming Problem and Comparison with New Arithmetic Average Technique

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Abstract : *In this paper, we have suggested a new technique to solve a MOLFPP by using new geometric averaging method. An algorithm is suggested for its solution. We have used some other techniques such as arithmetic averaging, geometric averaging, and new arithmetic averaging method to solve the same problem. New geometric averaging method gives better result than all those methods.*

Keywords: *Linear Fractional Programming Problem, Statistical average method, Arithmetic average, geometric average*

I. Introduction

Linear programming is comparatively a recent mathematical concept. A study of multi-objective linear programming problem (MOLPP) is introduced in [1] which suggests an approach to set up multi-objective function (MOF) under the limitation so that the optimum value of individual problem was greater than zero. Using mean and median the MOF was studied by solving multi-objective programming problem [2]. The multi-objective linear fractional programming problem (MOLFPP) was solved by the technique developed by Chandra Sen. The industrial production planning, financial and corporate planning, healthcare and hospital planning are important fields which use linear fraction maximum problems. So it has attracted considerable research and interest. There are several methods to solve these problems discussed in [3] where linear fractional programming is transformed to an equivalent linear program. Sing conducted a useful study about the optimality condition in fractional programming [4]. In [5], Sulaiman and Othman studied optimal transformation technique to solve MOLFPP. In [6], Hamad Amin studied MOLPP using Arithmetic average. The study in [7] suggested a new technique to transform MOLPP to the single objective linear programming problem by using harmonic mean for values of functions.

In this paper I have defined a MOLFPP and suggested a new geometric average technique to optimize the objective function where a single objective function is developed from multi-objective functions. The result is compared with that of optimization using new arithmetic average technique. The results are also compared with those of optimization which are obtained by using arithmetic mean and geometric mean. This new geometric average technique gives better result than all those results.

II. Problem Formulation

The main objective of this study is to solve multi-objective linear fractional programming problems. Before going to this problem, I would like to discuss some common definitions which will be used to understand the target of this paper.

1.1 Common Definitions:

1.1.1 Linear Programming:

Linear Programming deals with the optimization of a function of variables known as objective function, subject to set of linear equalities/inequalities known as constraints. The objective function may be profit, loss, cost, production capacity or any other measure of effectiveness which is to be obtained in the best possible or optimal manner. G.B. Dantzig in 1947 proposed the simplex algorithm as an efficient method to solve a linear programming problem [8].

A linear program is of the form:

$$\begin{aligned} &Max(\min) \quad c^t\vec{x} \\ &s/t \quad A\vec{x} = \vec{b} \\ &\qquad \vec{x} \geq 0 \end{aligned}$$

Where, ***b*** is m-dimensional vector of constants, ***x*** is n-dimensional vector of decision variables and ***A*** is $m \times n$ matrix of constants.

1.1.2 Simplex Method:

The simplex method is developed using fundamental theorem of linear programming. This method sets up an algorithm which involves repetitive application of predetermined operation [9]. There are many developed methods to solve problems relating linear programming. Among them, the simplex method provides optimum solution to the problems. To find out an optimum solution using this method, the problems should be expressed in terms of linear objective function which is subject to predefined constraints [10].

1.1.3 The Simplex Algorithm:

In constraint based solution of any linear programming problem, the optimum results of it can be predefined. Using repetitive algorithm, the results of the linear programming problems are to be found out until an optimum solution to be reached. For this purpose, the simplex algorithm is most promising. In this algorithm, an initial value of the solution is approximated. Based on this, a nearest solution point is targeted. This process is continued until an optimum solution to be obtained [10].

1.1.4 The Arithmetic Mean (A.M.):

A.M. which is sometimes referred to as simply mean, is the most commonly used central value of a distribution. The A.M. is calculated by totaling the results of all the observations and dividing this total by the number of observations when the data at hand are numerical.
If the ages of six school children are 16, 18, 17, 15, 17 and 16 years, the mean is $\frac{16+18+17+15+17+16}{6} = 16.5$ years.

1.1.5 The Geometric Mean (G.M.):

The geometric mean of n positive values $x_1, x_2, \cdots x_n$ is defined as the nth positive root of the product of the values. Symbolically,
$G.M. = (x_1 x_2 x_3 \cdots x_n)^{1/n}$; for 2 values, x_1, x_2, $G.M. = (x_1 x_2)^{1/2}$;for 3 values x_1, x_2, x_3, $G.M. = (x_1 x_2 x_3)^{1/3}$

1.2 MOLFPP:

Multi-objective function that are the ratio of two linear objective functions are said to be MOLFPP, defined as,

$$\begin{aligned} \max z_1 &= \frac{c_1^t x + \gamma_1}{d_1^t x + \beta_1} \\ \max z_2 &= \frac{c_2^t x + \gamma_2}{d_2^t x + \beta_2} \\ &\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\ \max z_r &= \frac{c_r^t x + \gamma_r}{d_r^t x + \beta_r} \\ \min z_{r+1} &= \frac{c_{r+1}^t x + \gamma_{r+1}}{d_{r+1}^t x + \beta_{r+1}} \\ &\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\ \min z_s &= \frac{c_s^t x + \gamma_s}{d_s^t x + \beta_s} \end{aligned} \qquad \text{s/t} \quad \begin{aligned} A\vec{x} &= \vec{b} \\ \vec{x} &\ge 0 \end{aligned} \qquad (1.1)$$

where, ***b*** is m-dimensional vector of constants, ***x*** is n-dimensional vector of decision variables and ***A*** is $m \times n$ matrix of constants. Both types of objective functions must be present.

III. MOLFPP Solution Techniques

We can solve MOLFPP using Chandra Sen's technique but it gives comparatively poor result of the objective function. But statistical average techniques (arithmetic and geometric) which are proposed in this paper for solving MOLFPP give better result of the objective function. These techniques are briefly described below.

1.1 Solving MOLFPP by Chandra Sen's Technique:

We obtain a single value corresponding to each of the objective functions of MOLFPP of equation (1.1). They are being optimized individually subject to the constraints of equation (1.1) as follows:

$$\begin{aligned}
&\text{Max } z_1 = \varphi_1 \\
&\text{Max } z_2 = \varphi_2 \\
&\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \\
&\text{Max } z_r = \varphi_r \\
&\text{Min } z_{r+1} = \varphi_{r+1} \\
&\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \\
&\text{Min } z_s = \varphi_s ;
\end{aligned} \tag{1.2}$$

where $\varphi_1, \varphi_2, \cdots\cdots \varphi_s$ are values of the objective functions. These values can be put in the equation (1.3), which is known as Chandra Sen's method, to find out a single objective function.

$$\max \quad z = \sum_{i=1}^{r} \frac{z_i}{|\varphi_i|} - \sum_{i=r+1}^{s} \frac{z_i}{|\varphi_i|} \tag{1.3}$$

where, $\varphi_i \neq 0, \quad i = 1, 2, \quad \cdots s$. Subject to the constraints of equation (1.1) and the optimum value of the objective functions φ_i may be positive or negative.

1.2 Proposed Technique

1.2.1 Statistical averaging method:

$$\max \quad z = \sum_{i=1}^{r} \frac{z_i}{A.M.(AA_i)} - \sum_{i=r+1}^{s} \frac{z_i}{A.M.(AL_i)} \tag{1.4}$$

$$\max \quad z = \sum_{i=1}^{r} \frac{z_i}{G.M.(AA_i)} - \sum_{i=r+1}^{s} \frac{z_i}{G.M.(AL_i)} \tag{1.5}$$

Where, $AA_i = |\varphi_i|,\ i = 1 \cdots r$ and $AL_i = |\varphi_i|,\ i = 1 + r \cdots s$

1.2.2 Solving MOLFPP by using the new arithmetic average technique:

Let $m_1 = \min\langle AA_i \rangle$, where $AL_i = |\varphi_i|, \varphi_i$ is maximum value of z_i, $i = 1 \cdots r$

$m_2 = \min\langle AL_i \rangle$, where $AA_i = |\varphi_i|, \varphi_i$ is minimum value of z_i, $i = r + 1 \cdots s$

$A.Av = \dfrac{m_1 + m_2}{2}$ so

$$\max \quad z = \left(\sum_{i=1}^{r} z_i - \sum_{i=r+1}^{s} z_i \right) / A.Av \tag{1.6}$$

1.2.3 Solving MOLFPP by using the new geometric average technique:

Using m_1 and m_2 obtained in section 1.2.2, we can find the geometric average as follows:

$$G.Av = \sqrt{m_1 m_2}$$

$$\max \quad z = \left(\sum_{i=1}^{r} z_i - \sum_{i=r+1}^{s} z_i \right) / G.Av \tag{1.7}$$

1.3 Algorithm for new Arithmetic and Geometric average technique:

Step 1: Find the value of each of individual objective functions which is to be maximized or minimized.
Step 2: Solve the first objective problem by simplex method.
Step 3: Check the feasibility of the solution in step 2. If it is feasible then go to step 4. Otherwise, use dual simplex method to remove infeasibility.
Step 4: Assign a name to the optimum value of the first objective function z_1 say φ_1 .
Step5: Repeat the step 2, i=1, 2, …s
Step 6: Select $m_1 = \min\langle AA_i \rangle,\ m_2 = \min\langle AL_i \rangle,\ i = 1\cdots s$

$$A.Av = \frac{m_1 + m_2}{2} \qquad \text{and} \qquad G.Av = \sqrt{m_1 m_2}$$

Step 7: Optimize the combined objective function with the same constraints

$$\max \quad z = \left(\sum_{i=1}^{r} z_i - \sum_{i=r+1}^{s} z_i \right) / A.Av \qquad \text{and} \qquad \max \quad z = \left(\sum_{i=1}^{r} z_i - \sum_{i=r+1}^{s} z_i \right) / G.Av$$

1.4 Program solution for new Arithmetic and Geometric average technique:

The following program can be used to solve MOLFPP by proposed method.
For this, let

φA_i = value of objective functions which is to be maximized.

φL_i = value of objective functions which is to be minimized

So

$$AA_i = |\varphi A_i|;\ \forall\ i = 1\cdots r;\ AL_i = |\varphi L_i|;\ \forall\ i = 1 + r\cdots s$$

$$SM = \sum_{i=1}^{r} z_i;\quad SN = \sum_{i=r+1}^{s} z_i$$

$$m_1 = \min\langle AA_i \rangle; \qquad m_2 = \min\langle AL_i \rangle$$

$$\max \quad z = (SM - SN) / A.Av \text{ and}$$

$$\max \quad z = (SM - SN) / G.Av$$

1.5 Flow Chart for new Arithmetic and Geometric average technique:

The flow chart, shown in Fig. 1.1, describes how the objective functions are optimized.

IV. Test Calculations

Example: Consider the following multi-objective linear fractional programming problems. But as these functions are fractional, we have to use modified simplex method first to obtain the values of these objective functions. Then we can use these values to develop a single objective function using Chandra Sen's technique.
Multi-objective fractional functions:

$$\begin{aligned}
\max \quad z_1 &= \frac{3x_1 - 2x_2}{x_1 + x_2 + 1} \\
\max \quad z_2 &= \frac{9x_1 + 3x_2}{x_1 + x_2 + 1} \\
\max \quad z_3 &= \frac{3x_1 - 5x_2}{2x_1 + 2x_2 + 2} \\
\min \quad z_4 &= \frac{-6x_1 + 2x_2}{2x_1 + 2x_2 + 2} \\
\min \quad z_5 &= \frac{-3x_1 - x_2}{x_1 + x_2 + 1}
\end{aligned}
\qquad
\text{s/t} \quad
\begin{aligned}
x_1 + x_2 &\le 2 \\
9x_1 + x_2 &\le 9 \\
x_1, x_2 &\ge 0
\end{aligned}
\qquad (1.8)$$

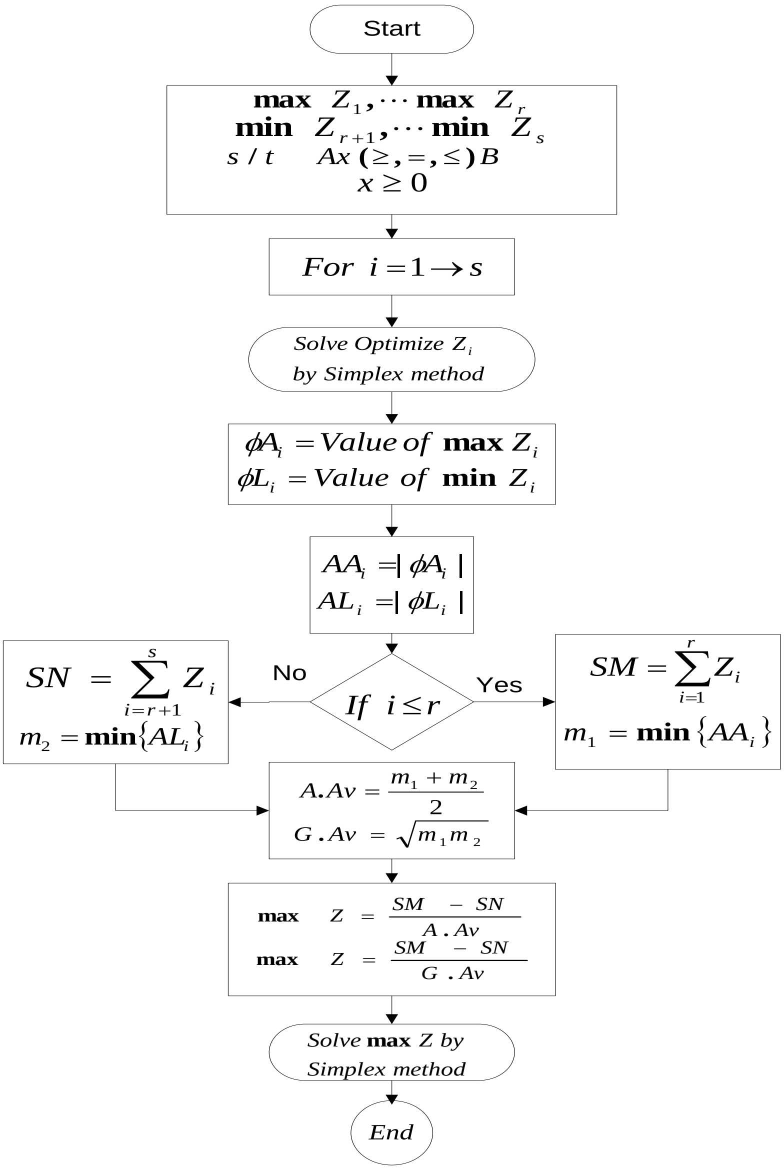

Fig. 1.1 Flow chart.

We have to solve by modified simplex method (equation (1.9)) [11] which implies

$$\max\ z = \frac{cx+\alpha}{dx+\beta} \qquad \text{s/t} \quad \begin{array}{l} Ax(\leq, =, \geq)b \\ x \geq 0 \\ b \geq 0 \end{array} \tag{1.9}$$

The objective function $H(y) = Iy + j$

Where, $I = \frac{c\beta - d\alpha}{\beta}, y = \frac{x}{dx+\beta}, j = \frac{\alpha}{\beta}$; $Ky \leq L, K = A\beta + bd, L = b$

Now from equation (1.8)

$$\max \quad z_1 = \frac{3x_1 - 2x_2}{x_1 + x_2 + 1} \qquad \text{s/t} \begin{array}{l} x_1 + x_2 \leq 2 \\ 9x_1 + x_2 \leq 9 \\ x_1, x_2 \geq 0 \end{array}$$

$$c = (3,-2), d = (1,1), \alpha = 0, \beta = -1, A_1 = (1,1), A_2 = (9,1), b_1 = 2, b_2 = 9$$

$$\max \quad H(y) = [Iy] + j = \frac{c\beta - d\alpha}{\beta}[y] = (3,-2).\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 3y_1 - 2y_2$$

For first constraint,

$$K_1 = (1,1).1 + 2(1,1) = (1,1) + (2,2) = (3,3)\text{, } Ky \le L \Rightarrow (3,3)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \le 2 \Rightarrow 3y_1 + 3y_2 \le 2$$

For second constraint,

$$K_2 = (9,1).1 + 9(1,1) = (9,1) + (9,9) = (18,10)\text{, } Ky \le L \Rightarrow (18,10)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \le 9 \Rightarrow 18y_1 + 10y_2 \le 9$$

Thus

$$\begin{array}{ll} \max, H(y) = 3y_1 - 2y_2 & \max H = 3y_1 - 2y_2 \\ s/t\; 3y_1 + 3y_2 \le 2 & s/t\; 3y_1 + 3y_2 + s_1 = 2 \\ 18y_1 + 10y_2 \le 9 \quad \Leftrightarrow & 18y_1 + 10y_2 + s_2 = 9 \\ y_1, y_2 \ge 0 & where \quad y_1, y_2, s_1, s_2 \ge 0 \end{array}$$

Table I

c_B	c_j	3	-2	0	0		
	Basis	Y_1	Y_2	S_1	S_2		
0	S_1	3	3	1	0	2	0.67
0	S_2	⑱	10	0	1	9	0.5
	C_j-E_j	3 ↑	-2	0	0	0	
0	S_1	0	4/3	1	-1/6	1/2	
3	Y_1	1	5/9	0	1/18	1/2	
	C_j-E_j	0	-11/3	0	-1/6	3/2	

Thus $y_1 = 1/2, \quad y_2 = 0$

Now $(x_1, x_2) = \dfrac{(y_1, y_2)\beta}{1 - d(y_1, y_2)} \quad = \dfrac{(1/2,0).1}{1-(11)(1/2,0)} = \dfrac{(1/2,0)}{1-1/2} = \dfrac{(1/2,0)}{1/2} = (1,0)$

Thus max Z_1 =3/2 with x_1=1, x_2=0

Second objective function in equation (1.8),

$$\max z_2 = \frac{9x_1 + 3x_2}{x_1 + x_2 + 1} \qquad \text{s/t} \begin{array}{l} x_1 + x_2 \le 2 \\ 9x_1 + x_2 \le 9 \\ x_1, x_2 \ge 0 \end{array}$$

So, $c = (9,3), d = (1,1), \alpha = 0, \beta = 1, A_1 = (1,1), A_2 = (9,1), b_1 = 2, b_2 = 9$

$$\max \quad H(y) = [Iy] + j = \frac{c\beta - d\alpha}{\beta}[y] + 0 = (9,3).\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 9y_1 + 3y_2$$

For first constraint,

$$K_1 = (1,1).1 + 2(1,1) = (1,1) + (2,2) = (3,3)\text{, } Ky \le L \Rightarrow (3,3)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \le 2 \Rightarrow 3y_1 + 3y_2 \le 2$$

For second constraint,

$$K_2 = (9,1).1 + 9(1,1) = (9,1) + (9,9) = (18,10)\text{, } Ky \le L \Rightarrow (18,10)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \le 9 \Rightarrow 18y_1 + 10y_2 \le 9$$

$$\begin{aligned} &\max,\ H(y)=9y_1+3y_2 && \max\ H=9y_1+3y_2 \\ &\quad s/t\ 3y_1+3y_2\le 2 && \quad s/t\ 3y_1+3y_2+s_1=2 \\ &\quad 18y_1+10y_2\le 9 \quad \Leftrightarrow && \quad 18y_1+10y_2+s_2=9 \\ &\quad y_1,y_2\ge 0 && where\quad y_1,y_2,s_1,s_2\ge 0 \end{aligned}$$

Table II

c_B	c_j	9	3	0	0		
	Basis	Y_1	Y_2	S_1	S_2		
0	S_1	3	3	1	0	2	0.667
0	S_2	18	10	0	1	9	0.5
	C_j-E_j	9 ↑	3	0	0	0	
0	S_1	0	4/3	1	-1/6	1/2	
9	Y_1	1	5/9	0	1/18	1/2	
	C_j-E_j	0	-2	0	-1/2	9/2	

Thus $y_1=1/2,\quad y_2=0$

Now $(x_1,x_2)=\dfrac{(y_1,y_2)\beta}{1-d(y_1,y_2)}\quad =\dfrac{(1/2,0).1}{1-(1,1)(1/2,0)}=\dfrac{(1/2,0)}{1-1/2}=\dfrac{(1/2,0)}{1/2}=(1,0)$

Thus max $Z_2=9/2$ with $x_1=1$, $x_2=0$

For third objective function from equation (1.8)

$$\max\quad z_3=\frac{3x_1-5x_2}{2x_1+2x_2+2}\qquad \text{s/t}\ \begin{aligned} &x_1+x_2\le 2\\ &9x_1+x_2\le 9\\ &x_1,x_2\ge 0\end{aligned}$$

So, $c=(3,-5),\ d=(2,2),\ \alpha=0,\ \beta=2,\ A_1=(1,1),\ A_2=(9,1),\ b_1=2,\ b_2=9$

$$\max\quad H(y)=[Iy]+j=\frac{c\beta-d\alpha}{\beta}[y]+0=(3,-5).\begin{bmatrix} y_1\\ y_2\end{bmatrix}=3y_1-5y_2$$

For first constraint,

$$K_1=(1,1).2+2(2,2)=(2,2)+(4,4)=(6,6),$$

$$Ky\le L\Rightarrow(6,6)\begin{pmatrix} y_1\\ y_2\end{pmatrix}\le 2\Rightarrow 6y_1+6y_2\le 2$$

For second constraint,

$$K_2=(9,1).2+9(2,2)=(18,2)+(18,18)=(36,20)$$

$$Ky\le L\Rightarrow(36,20)\begin{pmatrix} y_1\\ y_2\end{pmatrix}\le 9\Rightarrow 36y_1+20y_2\le 9$$

Thus

$$\begin{aligned} &\max,\ H(y)=3y_1-5y_2 && \max\ H=3y_1-5y_2 \\ &\quad s/t\ 6y_1+6y_2\le 2 && \quad s/t\ 6y_1+6y_2+s_1=2 \\ &\quad 36y_1+20y_2\le 9 \quad \Leftrightarrow && \quad 36y_1+20y_2+s_2=9 \\ &\quad y_1,y_2\ge 0 && where\quad y_1,y_2,s_1,s_2\ge 0 \end{aligned}$$

Table III

c_B	c_j	3	-5	0	0		
	Basis	Y_1	Y_2	S_1	S_2		
0	S_1	6	6	1	0	2	0.33
0	S_2	36	20	0	1	9	0.25
	C_j-E_j	3 ↑	-5	0	0	0	
0	S_1	0	8/3	1	-1/6	1/2	
3	Y_1	1	5/9	0	1/36	1/4	
	C_j-E_j	0	-20/3	0	-1/2	3/4	

Thus $y_1 = 1/4, \quad y_2 = 0$

Now $(x_1, x_2) = \dfrac{(y_1, y_2)\beta}{1 - d(y_1, y_2)} \quad = \dfrac{(1/4, 0).2}{1-(2, 2)(1/4,0)} = \dfrac{(1/2, 0)}{1-1/2} = \dfrac{(1/2, 0)}{1/2} = (1, 0)$

Thus max Z_3 =3/4 with x_1=1, x_2=0

For fourth objective function from equation (1.8)

$$\min \quad z_4 = \frac{-6x_1 + 2x_2}{2x_1 + 2x_2 + 2} \qquad \text{s/t} \quad \begin{aligned} x_1 + x_2 &\le 2 \\ 9x_1 + x_2 &\le 9 \\ x_1, x_2 &\ge 0 \end{aligned}$$

So, $c = (-6, 2), d = (2, 2), \alpha = 0, \beta = 2, A_1 = (1,1), A_2 = (9,1), b_1 = 2, b_2 = 9$

$$\min \quad H(y) = [Iy] + j = \frac{c\beta - d\alpha}{\beta}[y] + 0 = (-6, 2).\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = -6y_1 + 2y_2$$

For first constraint,

$$K_1 = (1,1).2 + 2(2, 2) = (2, 2) + (4, 4) = (6, 6)\,, \; Ky \le L \Rightarrow (6, 6)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \le 2 \Rightarrow 6y_1 + 6y_2 \le 2$$

For second constraint,

$$K_2 = (9,1).2 + 9(2, 2) = (18, 2) + (18,18) = (36, 20)$$

$$Ky \le L \Rightarrow (36, 20)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \le 9 \Rightarrow 36y_1 + 20y_2 \le 9$$

Thus

$$\begin{aligned} &\max, H(y) = 6y_1 - 2y_2 \\ &\quad s/t\; 6y_1 + 6y_2 \le 2 \\ &\quad\; 36y_1 + 20y_2 \le 9 \\ &\quad\; y_1, y_2 \ge 0 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} &\max H = 6y_1 - 2y_2 \\ &\quad s/t\; 6y_1 + 6y_2 + s_1 = 2 \\ &\quad\; 36y_1 + 20y_2 + s_2 = 9 \\ &where \quad y_1, y_2, s_1, s_2 \ge 0 \end{aligned}$$

Table IV

c_B	c_j	6	-2	0	0		
	Basis	Y_1	Y_2	S_1	S_2		
0	S_1	6	6	1	0	2	0.33
0	S_2	36	20	0	1	9	0.25
	C_j-E_j	6 ↑	-2	0	0	0	
0	S_1	0	8/3	1	-1/6	1/2	
3	Y_1	1	5/9	0	1/36	1/4	
	C_j-E_j	0	-16/3	0	-1/6	3/2	

Thus $y_1 = 1/4, \quad y_2 = 0$

Now $(x_1, x_2) = \dfrac{(y_1, y_2)\beta}{1 - d(y_1, y_2)} \quad = \dfrac{(1/4, 0).2}{1-(2, 2)(1/4,0)} = \dfrac{(1/2, 0)}{1-1/2} = \dfrac{(1/2, 0)}{1/2} = (1, 0)$

Thus max Z$_4$ = -3/2 with x$_1$=1, x$_2$=0

For fifth objective function from equation (1.8)

$$\min \quad z_5 = \frac{-3x_1 - x_2}{x_1 + x_2 + 1} \qquad \text{s/t} \begin{array}{l} x_1 + x_2 \le 2 \\ 9x_1 + x_2 \le 9 \\ x_1, x_2 \ge 0 \end{array}$$

So, $c = (-3, -1), d = (1, 1), \alpha = 0, \beta = 1, A_1 = (1, 1), A_2 = (9, 1), b_1 = 2, b_2 = 9$

$$\min \quad H(y) = [Iy] + j = \frac{c\beta - d\alpha}{\beta}[y] + 0 = (-3, -1).\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = -3y_1 - y_2$$

For first constraint,

$$K_1 = (1, 1).1 + 2(1, 1) = (1, 1) + (2, 2) = (3, 3)\text{,} \; Ky \le L \Rightarrow (3, 3)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \le 2 \Rightarrow 3y_1 + 3y_2 \le 2$$

For second constraint,

$$K_2 = (9, 1).1 + 9(1, 1) = (9, 1) + (9, 9) = (18, 10)\text{,} \; Ky \le L \Rightarrow (18, 10)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \le 9 \Rightarrow 18y_1 + 10y_2 \le 9$$

Thus
$$\begin{array}{l} \max, H(y) = 3y_1 + y_2 \\ \quad s/t\; 3y_1 + 3y_2 \le 2 \\ \quad\quad 18y_1 + 10y_2 \le 9 \\ \quad\quad y_1, y_2 \ge 0 \end{array} \Leftrightarrow \begin{array}{l} \max H = 3y_1 + y_2 \\ \quad s/t\; 3y_1 + 3y_2 + s_1 = 2 \\ \quad\quad 18y_1 + 10y_2 + s_2 = 9 \\ where \quad y_1, y_2, s_1, s_2 \ge 0 \end{array}$$

Table V

c$_B$	c$_j$	3	1	0	0		
	Basis	Y$_1$	Y$_2$	S$_1$	S$_2$		
0	S$_1$	3	3	1	0	2	0.67
0	S$_2$	18	10	0	1	9	0.5
	C$_j$-E$_j$	3 ↑	1	0	0	0	
0	S$_1$	0	4/3	1	-1/6	1/2	
3	Y$_1$	1	5/9	0	1/18	1/2	
	C$_j$-E$_j$	0	-2/3	0	-1/6	3/2	

Thus $y_1 = 1/2, \quad y_2 = 0$

Now $(x_1, x_2) = \dfrac{(y_1, y_2)\beta}{1 - d(y_1, y_2)} \quad = \dfrac{(1/2, 0).1}{1-(1,1)(1/2,0)} = \dfrac{(1/2, 0)}{1-1/2} = \dfrac{(1/2, 0)}{1/2} = (1, 0)$

Thus max Z$_5$ = -3/2 with x$_1$=1, x$_2$=0
The values obtained from the objective functions are summarized in Table VI.

Table VI

I	φ_i	x$_i$	AA$_i$=| φ_i |	AL$_i$=| φ_i |
1	3/2	(1, 0)	3/2	
2	9/2	(1, 0)	9/2	
3	¾	(1, 0)	3/4	3/2
4	-3/2	(1, 0)		3/2
5	-3/2	(1, 0)		

Now, by Chandra Sen's approach:

Max Z= $\displaystyle\sum_{k=1}^{r} \frac{z_k}{|\varphi_k|} - \sum_{k=r+1}^{s} \frac{z_k}{|\varphi_k|}$

$$=\frac{2}{3}\frac{3x_1-2x_2}{x_1+x_2+1}+\frac{2}{9}\frac{9x_1+3x_2}{x_1+x_2+1}+\frac{4}{3}\frac{3x_1-5x_2}{2x_1+2x_2+2}-[\frac{2}{3}\frac{-6x_1+2x_2}{2x_1+2x_2+2}+\frac{2}{3}\frac{-3x_1-x_2}{x_1+x_2+1}]$$

$$=\frac{1}{x_1+x_2+1}[x_1(2+2+2)+x_2(2/3-10/3-4/3)]-\frac{1}{x_1+x_2+1}[-4x_1]=\frac{10x_1-4x_2}{x_1+x_2+1}$$

So the single objective function is

$$\max,\, z=\frac{10x_1-4x_2}{x_1+x_2+1} \qquad \text{s/t} \quad \begin{aligned} x_1+x_2 &\le 2 \\ 9x_1+x_2 &\le 9 \\ x_1, x_2 &\ge 0 \end{aligned} \tag{1.10}$$

Again the equation (1.10) is fractional, so we have to use modified simplex method to find its solution. So from modified simplex method described in equation (1.9) we get

$$c=(10,4),\, d=(1,1),\, \alpha=0,\, \beta=1,\, A_1=(1,1),\, A_2=(9,1),\, b_1=2,\, b_2=9$$

$$\max \quad H(y)=[Iy]+j=\frac{c\beta-d\alpha}{\beta}[y]+0=(10,4).\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}=10y_1+4y_2$$

For first constraint,

$$K_1=(1,1).1+2(1,1)=(1,1)+(2,2)=(3,3),$$

$$Ky\le L \Rightarrow (3,3)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\le 2 \Rightarrow 3y_1+3y_2\le 2$$

For second constraint,

$$K_2=(9,1).1+9(1,1)=(9,1)+(9,9)=(18,10),$$

$$Ky\le L \Rightarrow (18,10)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\le 9 \Rightarrow 18y_1+10y_2\le 9$$

Thus

$$\begin{aligned} &\max,\, H(y)=10y_1+4y_2 \\ &s/t\; 3y_1+3y_2\le 2 \\ &18y_1+10y_2\le 9 \\ &y_1, y_2\ge 0 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} &\max H=10y_1+4y_2 \\ &s/t\; 3y_1+3y_2+s_1=2 \\ &18y_1+10y_2+s_2=9 \\ &where \quad y_1, y_2, s_1, s_2\ge 0 \end{aligned}$$

Table VII

c_B	c_j	10	4	0	0		
	Basis	Y_1	Y_2	S_1	S_2		
0	S_1	3	3	1	0	2	0.67
0	S_2	18	10	0	1	9	0.5
	C_j-E_j	10 ↑	4	0	0	0	
0	S_1	0	4/3	1	-1/6	1/2	
10	Y_1	1	5/9	0	1/18	1/2	
	C_j-E_j	0	-14/9	0	-5/9	5/2	

Thus $y_1=1/2, \quad y_2=0$

Now $(x_1,x_2)=\dfrac{(y_1,y_2)\beta}{1-d(y_1,y_2)} \quad =\dfrac{(1/2,0).1}{1-(1,1)(1/2,0)}=\dfrac{(1/2,0)}{1-1/2}=\dfrac{(1/2,0)}{1/2}=(1,0)$

Thus max Z = 5 with $x_1=1$, $x_2=0$

From Table VI, we get

A.M. (3/2, 9/2, 3/4) = 9/4, *A.M.* (3/2, 3/2) = 3/2

So, by using equation (1.4) we have

$$\text{Max Z}=\sum_{i=1}^{r}\frac{z_i}{A.M(AA_i)}-\sum_{i=r+1}^{s}\frac{z_i}{A.M(AL_i)}$$

$$Max\ z=\frac{4}{9}[\frac{3x_1-2x_2}{x_1+x_2+1}+\frac{9x_1+3x_2}{x_1+x_2+1}+\frac{3x_1-5x_2}{2x_1+2x_2+2}]-\frac{2}{3}[\frac{-6x_1+2x_2}{2x_1+2x_2+2}+\frac{-3x_1-x_2}{x_1+x_2+1}]$$

$$=\frac{10x_1-2/3x_2}{x_1+x_2+1}$$

Thus $\max,\ z=\dfrac{10x_1-2/3x_2}{x_1+x_2+1}$ s/t $\begin{array}{l} x_1+x_2\le 2 \\ 9x_1+x_2\le 9 \\ x_1,x_2\ge 0 \end{array}$

$$c=(10,-2/3), d=(1,1), \alpha=0, \beta=1, A_1=(1,1), A_2=(9,1), b_1=2, b_2=9$$

$$\max\quad H(y)=[Iy]+j=\frac{c\beta-d\alpha}{\beta}[y]+0=(10,-2/3).\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}=10y_1-2/3y_2$$

For first constraint,

$$K_1=(1,1).1+2(1,1)=(1,1)+(2,2)=(3,3),\quad Ky\le L\Rightarrow(3,3)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\le 2\Rightarrow 3y_1+3y_2\le 2$$

For second constraint,

$$K_2=(9,1).1+9(1,1)=(9,1)+(9,9)=(18,10),\quad Ky\le L\Rightarrow(18,10)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\le 9\Rightarrow 18y_1+10y_2\le 9$$

Thus
$$\begin{array}{l} \max, H(y)=10y_1-2/3y_2 \\ s/t\ 3y_1+3y_2\le 2 \\ 18y_1+10y_2\le 9 \\ y_1,y_2\ge 0 \end{array} \quad\Leftrightarrow\quad \begin{array}{l} \max\ H=10y_1-2/3y_2 \\ s/t\ 3y_1+3y_2+s_1=2 \\ 18y_1+10y_2+s_2=9 \\ where\quad y_1,y_2,s_1,s_2\ge 0 \end{array}$$

Table VIII

c_B	c_j Basis	10	-2/3	0	0		
		Y_1	Y_2	S_1	S_2		
0	S_1	3	3	1	0	2	0.67
0	S_2	18	10	0	1	9	0.5
	C_j-E_j	10 ↑	-2/3	0	0	0	
0	S_1	0	4/3	1	-1/6	1/2	
10	Y_1	1	5/9	0	1/18	1/2	
	C_j-E_j	0	-56/9	0	-5/9	-5/9	

Thus $y_1=1/2,\quad y_2=0$

Now $(x_1,x_2)=\dfrac{(y_1,y_2)\beta}{1-d(y_1,y_2)}\quad=\dfrac{(1/2,0).1}{1-(1,1)(1/2,0)}=\dfrac{(1/2,0)}{1-1/2}=\dfrac{(1/2,0)}{1/2}=(1,0)$

Thus max Z = 5 with x_1=1, x_2=0

From Table VI, we get *G.M.* (3/2, 9/2, 3/4) = 1.7171; *G.M.* (3/2, 3/2) = 1.5

So, by using equation (1.5) we have

Max Z= $\displaystyle\sum_{i=1}^{r}\frac{z_i}{G.M(AA_i)}-\sum_{i=r+1}^{s}\frac{z_i}{G.M(AL_i)}$

$$Max\ z=\frac{1}{1.7171}[\frac{3x_1-2x_2}{x_1+x_2+1}+\frac{9x_1+3x_2}{x_1+x_2+1}+\frac{3x_1-5x_2}{2x_1+2x_2+2}]-\frac{1}{1.5}[\frac{-6x_1+2x_2}{2x_1+2x_2+2}+\frac{-3x_1-x_2}{x_1+x_2+1}]$$

$$=\frac{11.862x_1-0.8736x_2}{x_1+x_2+1}$$

Thus $\max Z = \dfrac{11.862x_1 - 0.8736x_2}{x_1 + x_2 + 1}$ s/t $\begin{aligned} x_1 + x_2 &\le 2 \\ 9x_1 + x_2 &\le 9 \\ x_1, x_2 &\ge 0 \end{aligned}$

$$c = (11.862, -0.8736), d = (1,1), \alpha = 0, \beta = 1, A_1 = (1,1), A_2 = (9,1), b_1 = 2, b_2 = 9$$

$$\max \quad H(y) = [Iy] + j = \frac{c\beta - d\alpha}{\beta}[y] + 0 = (11.862, -0.8736).\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 11.862y_1 - 0.8736y_2$$

For first constraint,

$$K_1 = (1,1).1 + 2(1,1) = (1,1) + (2,2) = (3,3), \; Ky \le L \Rightarrow (3,3)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \le 2 \Rightarrow 3y_1 + 3y_2 \le 2$$

For second constraint

$$K_2 = (9,1).1 + 9(1,1) = (9,1) + (9,9) = (18,10), \; Ky \le L \Rightarrow (18,10)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} 9 \Rightarrow 18y_1 + 10y_2 \le 9$$

Thus $\begin{aligned} &\max, H(y) = 11.862y_1 - 0.8736y_2 \\ &s/t \; 3y_1 + 3y_2 \le 2 \\ &\quad 18y_1 + 10y_2 \le 9 \\ &\quad y_1, y_2 \ge 0 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} &\max H = 11.862y_1 - 0.8736y_2 \\ &s/t \; 3y_1 + 3y_2 + s_1 = 2 \\ &\quad 18y_1 + 10y_2 + s_2 = 9 \\ &where \quad y_1, y_2, s_1, s_2 \ge 0 \end{aligned}$

Table IX

c_B	c_j Basis	11.862	-0.8736	0	0		
		Y_1	Y_2	S_1	S_2		
0	S_1	3	3	1	0	2	0.67
0	S_2	18	10	0	1	9	0.5
	C_j-E_j	11.862 ↑	-0.8736	0	0	0	
0	S_1	0	4/3	1	-1/6	½	
11.862	Y_1	1	5/9	0	1/18	½	
	C_j-E_j	0	-7.4636	0	-0.659	-5.931	

Thus $y_1 = 1/2, \quad y_2 = 0$

Now $(x_1, x_2) = \dfrac{(y_1, y_2)\beta}{1 - d(y_1, y_2)} \quad = \dfrac{(1/2, 0).1}{1 - (1,1)(1/2,0)} = \dfrac{(1/2, 0)}{1 - 1/2} = \dfrac{(1/2, 0)}{1/2} = (1, 0)$

Thus max Z = 5.931 with x_1=1, x_2=0

New Arithmetic average technique:

Let m_1=0.75, m_2=1.5; $\dfrac{m_1 + m_2}{2} = \dfrac{0.75 + 1.5}{2} = 1.125$

$$\max \quad z = \left(\sum_{i=1}^{r} z_i - \sum_{i=r+1}^{s} z_i \right) / A.Av = \frac{39x_1 - 3x_2}{2(1.125)(x_1 + x_2 + 1)} = \frac{39x_1 - 3x_2}{2.25.(x_1 + x_2 + 1)} = \frac{17.33x_1 - 1.33x_2}{x_1 + x_2 + 1}$$

Thus $\max Z = \dfrac{17.33x_1 - 1.33x_2}{x_1 + x_2 + 1}$ s/t $\begin{aligned} x_1 + x_2 &\le 2 \\ 9x_1 + x_2 &\le 9 \\ x_1, x_2 &\ge 0 \end{aligned}$

$$c = (17.33, -1.33), d = (1,1), \alpha = 0, \beta = 1, A_1 = (1,1), A_2 = (9,1), b_1 = 2, b_2 = 9$$

$$\max \quad H(y) = [Iy] + j = \frac{c\beta - d\alpha}{\beta}[y] + 0 = (17.33, -1.33).\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 17.33y_1 - 1.33y_2$$

For first constraint,

$$K_1 = (1,1).1 + 2(1,1) = (1,1) + (2,2) = (3,3)\, , \; Ky \le L \Rightarrow (3,3)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \le 2 \Rightarrow 3y_1 + 3y_2 \le 2$$

For second constraint,

$$K_2 = (9,1).1 + 9(1,1) = (9,1) + (9,9) = (18,10)\, , \; Ky \le L \Rightarrow (18,10)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \le 9 \Rightarrow 18y_1 + 10y_2 \le 9$$

Thus

$$\begin{aligned} &\max, H(y) = 17.33y_1 - 1.33y_2 \\ &\quad s/t\; 3y_1 + 3y_2 \le 2 \\ &\quad\quad 18y_1 + 10y_2 \le 9 \\ &\quad\quad y_1, y_2 \ge 0 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} &\max H = 17.33y_1 - 1.33y_2 \\ &\quad s/t\; 3y_1 + 3y_2 + s_1 = 2 \\ &\quad\quad 18y_1 + 10y_2 + s_2 = 9 \\ &where \quad y_1, y_2, s_1, s_2 \ge 0 \end{aligned}$$

Table X

c_B	c_j Basis	17.33	-1.33	0	0		
		Y_1	Y_2	S_1	S_2		
0	S_1	3	3	1	0	2	0.67
0	S_2	18	10	0	1	9	0.5
	C_j-E_j	17.33 ↑	-1.33	0	0	0	
0	S_1	0	4/3	1	-1/6	½	
17.33	Y_1	1	5/9	0	1/18	½	
	C_j-E_j	0	-10.96	0	-0.963		

Thus $y_1 = 1/2, \quad y_2 = 0$

Now $(x_1, x_2) = \dfrac{(y_1, y_2)\beta}{1 - d(y_1, y_2)} \quad = \dfrac{(1/2, 0).1}{1 - (1,1)(1/2,0)} = \dfrac{(1/2, 0)}{1 - 1/2} = \dfrac{(1/2, 0)}{1/2} = (1, 0)$

Thus max Z = 8.665 with $x_1=1$, $x_2=0$

New Geometric average technique:

Let $m_1=0.75$, $m_2=1.5$; So $G.Av = \sqrt{0.75(1.5)} = 1.061$

$$\max \quad z = \left(\sum_{i=1}^{r} z_i - \sum_{i=r+1}^{s} z_i \right) / G.Av$$

$$= \frac{39x_1 - 3x_2}{2(1.061)(x_1 + x_2 + 1)} = \frac{39x_1 - 3x_2}{2.122.(x_1 + x_2 + 1)} = \frac{18.379x_1 - 1.4137x_2}{x_1 + x_2 + 1}$$

Thus $maz\; z = \dfrac{18.379x_1 - 1.4137x_2}{x_1 + x_2 + 1}$

$$\begin{aligned} & x_1 + x_2 \le 2 \\ & 9x_1 + x_2 \le 9 \\ \text{s/t}\quad & x_1, x_2 \ge 0 \end{aligned}$$

$$c = (18.379, -1.4137), d = (1,1), \alpha = 0, \beta = 1, A_1 = (1,1), A_2 = (9,1), b_1 = 2, b_2 = 9$$

$$\max \quad H(y) = [Iy] + j = \frac{c\beta - d\alpha}{\beta}[y] + 0 = (18.379, -1.4137).\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 18.379y_1 - 1.4137y_2$$

For first constraint,

$$K_1 = (1,1).1 + 2(1,1) = (1,1) + (2,2) = (3,3)\, , \; Ky \le L \Rightarrow (3,3)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \le 2 \Rightarrow 3y_1 + 3y_2 \le 2$$

For second constraint,

$$K_2 = (9,1).1 + 9(1,1) = (9,1) + (9,9) = (18,10)\, , \; Ky \le L \Rightarrow (18,10)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \le 9 \Rightarrow 18y_1 + 10y_2 \le 9$$

Thus

$$\begin{aligned} &\max, H(y) = 18.379y_1 - 1.4137y_2 \\ &\quad s/t\ 3y_1 + 3y_2 \le 2 \\ &\quad\ 18y_1 + 10y_2 \le 9 \\ &\quad\ y_1, y_2 \ge 0 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} &\max H = 18.379y_1 - 1.4137y_2 \\ &\quad s/t\ 3y_1 + 3y_2 + s_1 = 2 \\ &\quad\ 18y_1 + 10y_2 + s_2 = 9 \\ &where \quad y_1, y_2, s_1, s_2 \ge 0 \end{aligned}$$

Table XI

c_B	c_j Basis	18.379	-1.4137	0	0		
		Y_1	Y_2	S_1	S_2		
0	S_1	3	3	1	0	2	0.67
0	S_2	⑱	10	0	1	9	0.5
	C_j-E_j	18.379 ↑	-1.4137	0	0	0	
0	S_1	0	4/3	1	-1/6	½	
18.379	Y_1	1	5/9	0	1/18	½	
	C_j-E_j	0	-11.624	0	-1.021		

Thus $y_1 = 1/2, \quad y_2 = 0$

Now $(x_1, x_2) = \dfrac{(y_1, y_2)\beta}{1 - d(y_1, y_2)} \quad = \dfrac{(1/2, 0).1}{1-(1,1)(1/2,0)} = \dfrac{(1/2, 0)}{1-1/2} = \dfrac{(1/2, 0)}{1/2} = (1, 0)$

Thus max Z = 9.1895 with x_1=1, x_2=0

The solutions of the MOLFPP solved by different approaches are summarized in Table XII which shows that we get the improved value of the objective function using new statistical average method proposed in this paper. So the proposed technique is justified here.

Table XII

Chandra Sen's Approach	Statistical Average Method		New Statistical Average Method	
	Using A.M	Using G.M	New A. Av method	New G. Av method
Max Z=5 with x_1=1, x_2=0	Max Z=5 with x_1=1, x_2=0	Max Z=5.931 with x_1=1, x_2=0	Max Z=8.665 with x_1=1, x_2=0	Max Z=9.1895 with x_1=1, x_2=0

V. Conclusion

In this paper, a MOLFPP has been solved using different methods such as Chandra Sen's approach and proposed statistical average method, and the results are compared in Table XII. In statistical average method, we proposed geometric and arithmetic average approach. We also proposed a new statistical average approach to solve the problem. It is observed that statistical average method results better optimization than Chandra Sen's approach of the MOLFPP. The proposed new statistical average method optimizes the problem better than that of statistical average method. We also found that geometric average technique is suited for optimizing MOLFPP better that that of arithmetic average technique.

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