

# A new graph product and its spectrum

C.D. Godsil and B.D. McKay

A new graph product is introduced, and the characteristic polynomial of a graph so-formed is given as a function of the characteristic polynomials of the factor graphs. A class of trees produced using this product is shown to be characterized by spectral properties.

## 1. Notation and preliminaries

All graphs considered in this paper are finite, and without loops and multiple or directed edges. Any undefined graph-theoretical terms will have the meanings given to them in Behzad and Chartrand [1].

If  $G$  is a graph with adjacency matrix  $A(G)$ , then we denote the characteristic polynomial  $\det(\lambda I - A(G))$  of  $A(G)$  by  $G(\lambda)$ , and refer to it as the characteristic polynomial of  $G$ . If  $G$  is a rooted graph then we denote by  $G'$  the graph obtained from  $G$  when the root vertex is removed. The characteristic polynomial of the rooted graph  $G$  is just the characteristic polynomial of the unrooted graph with the same vertex and edge sets as  $G$ .

**DEFINITION 1.1.** Let  $H$  be a labelled graph on  $n$  vertices. Let  $G$  be a sequence of  $n$  rooted graphs  $G_1, G_2, \dots, G_n$ . Then by  $H(G)$  we denote the graph obtained by identifying the root of  $G_i$  with the  $i$ th vertex of  $H$ . We call  $H(G)$  the *rooted product* of  $H$  by  $G$ .

Figure 1 illustrates this construction with  $H$  the path on three

---

Received 14 September 1977.

vertices and  $G$  consisting of three copies of the rooted path on two vertices

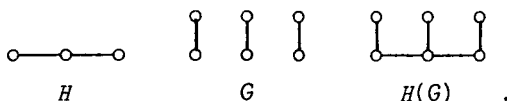


FIGURE 1

**DEFINITION 1.2.** Given a labelled graph  $H$  on  $n$  vertices and a sequence  $G$  of  $n$  rooted graphs, we define the matrix  $A_\lambda(H, G)$  as follows:

$$A_\lambda(H, G) = (a_{ij})$$

where

$$a_{ij} = \begin{cases} G_i(\lambda) & , \quad i = j , \\ -h_{ij} G_i'(\lambda) & , \quad i \neq j , \end{cases}$$

and  $A(H) = (h_{ij})$  is the adjacency matrix of  $H$ .

If, for example,  $H$  and  $G$  are represented in Figure 1, then  $A_\lambda(H, G)$  is the matrix

$$\begin{bmatrix} \lambda^2 - 1 & -\lambda & 0 \\ -\lambda & \lambda^2 - 1 & -\lambda \\ 0 & -\lambda & \lambda^2 - 1 \end{bmatrix} .$$

## 2. The polynomial of the rooted product

In this section we prove the following:

**THEOREM 2.1.**  $H(G)(\lambda) = \det A_\lambda(H, G)$ .

This result has already been proved by Schwenk [4] in the case where  $G$  consists of  $n$  isomorphic rooted graphs. The method we use to prove the result in general is quite different from his, however.

We will need the following lemma.

LEMMA 2.2. Let  $K$  and  $L$  be rooted graphs, and let  $K \cdot L$  denote the graph obtained by identifying the roots of  $K$  and  $L$ . Then

$$K \cdot L(\lambda) = K(\lambda)L'(\lambda) + K'(\lambda)L(\lambda) - \lambda K'(\lambda)L'(\lambda).$$

Proof. See Schwenk [4], or Godsil and McKay [2].  $\square$

Proof of Theorem 2.1. We will use induction on the number of vertices of  $H(G)$ . Suppose this number is  $N$ , and that the theorem holds for all labelled graphs  $H$  and sequences  $G$  such that  $H(G)$  has less than  $N$  vertices. For  $n = 1$ , the theorem follows from the definition of  $A_\lambda(H, G)$ , so we assume  $n \geq 2$ .

Let  $F$  denote the sequence of rooted graphs obtained from  $G$  by replacing the graph  $G_n$  by  $K_1$ , the graph with only one vertex.  $F'$  will be used to denote the subsequence of  $G$  consisting of the graphs  $G_1, G_2, \dots, G_{n-1}$ . Let  $H'$  denote the graph obtained from  $H$  by deleting the vertex labelled  $n$ , and let  $H(F)'$  denote the graph obtained from  $H(F)$  by deleting the vertex which was labelled  $n$  in  $H$ . Clearly  $H(F)' = H'(F')$ . The situation is represented diagrammatically in Figure 2.

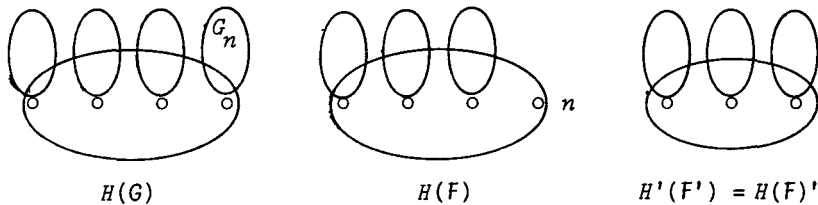


FIGURE 2

It follows at once from Lemma 2.2 that

$$(1) \quad H(G)(\lambda) = G_n(\lambda)H(F)'(\lambda) + G_n'(\lambda)H(F)(\lambda) - \lambda G_n'(\lambda)H(F)'(\lambda).$$

Now

$$(2) \quad \det A_\lambda(H, G) = \det \left[ \begin{array}{c|c} A_\lambda(H', F') & \begin{matrix} -h_{1n} G'_1(\lambda) \\ -h_{2n} G'_2(\lambda) \\ \vdots \\ -h_{n-1,n} G'_{n-1}(\lambda) \end{matrix} \\ \hline \begin{matrix} -G'_n(\lambda) \cdot h_n \end{matrix} & G_n(\lambda) \end{array} \right],$$

where  $h_n$  denotes the row vector  $(h_{n1}, h_{n2}, \dots, h_{n,n-1})$ . Since the determinant of a matrix is a linear function of any row, the right side of (2) can be expressed as

$$\det \left[ \begin{array}{c|c} A_\lambda(H', F') & \begin{matrix} -h_{1n} G'_1(\lambda) \\ \vdots \\ -h_{n-1,n} G'_{n-1}(\lambda) \end{matrix} \\ \hline \begin{matrix} -G'_n(\lambda) \cdot h_n \end{matrix} & \lambda G'_n(\lambda) \end{array} \right] + \det \left[ \begin{array}{c|c} A_\lambda(H', F') & \begin{matrix} -h_{1n} G'_1(\lambda) \\ \vdots \\ -h_{n-1,n} G'_{n-1}(\lambda) \end{matrix} \\ \hline 0 & G_n(\lambda) - \lambda G'_n(\lambda) \end{array} \right],$$

which equals

$$(3) \quad G'_n(\lambda) \det A_\lambda(H, F) + \{G_n(\lambda) - \lambda G'_n(\lambda)\} \det A_\lambda(H', F').$$

By our induction hypothesis  $\det A_\lambda(H', F') = H'(F')(\lambda)$ , and  $\det A_\lambda(H, F) = H(F)(\lambda)$ . Hence (3) may be rewritten as

$$(4) \quad G'_n(\lambda) H(F)(\lambda) + G_n(\lambda) H'(F')(\lambda) - \lambda G'_n(\lambda) H'(F')(\lambda).$$

Since  $H'(F') = H(F)'$ , a comparison of (4) with (1) shows that we have established the theorem.

We note that on dividing the  $i$ th row of  $A_\lambda(H, G)$  by  $G'_i(\lambda)$  for  $i = 1, 2, \dots, n$ , one obtains a matrix of the form  $\Lambda - A(H)$ , where

$$\Lambda = \text{diag} \left\{ \frac{G_1(\lambda)}{G'_1(\lambda)}, \frac{G_2(\lambda)}{G'_2(\lambda)}, \dots, \frac{G_n(\lambda)}{G'_n(\lambda)} \right\}.$$

Hence

$$(5) \quad \det A_\lambda(H, G) = \det(\Lambda - A(H)) \cdot \prod_{i=1}^n G'_i(\lambda).$$

In the special case where the  $G_i$  are all isomorphic,  $\Lambda = (G_1(\lambda)/G'_1(\lambda))I$  and so

$$(6) \quad H(G)(\lambda) = G'_1(\lambda)^n H\left(\frac{G_1(\lambda)}{G'_1(\lambda)}\right).$$

This is the formula given in [4].

Finally if  $G$  consists of  $n$  copies of  $P_2$ , the path on two vertices, one obtains, from (6),

$$(7) \quad H(G)(\lambda) = \lambda^n H\left(\lambda - \frac{1}{\lambda}\right),$$

since  $P_2(\lambda) = \lambda^2 - 1$ , and  $P'_2(\lambda) = \lambda$ . We will use (7) in the next section.

### 3. A spectral characterization of a class of trees

**NOTATION 3.1.** A *matching* of a graph  $T$  is a set of mutually non-adjacent edges. An *m-matching* consists of  $m$  such edges. A matching  $M$  such that every vertex of  $T$  is an end vertex of some edge in  $M$  is called a *1-factor*.

We recall, from [3] for example, that if  $T$  is a tree on  $n$  vertices, then

$$(8) \quad T(\lambda) = \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^m a_{2m} \lambda^{n-2m},$$

where  $a_{2m}$  is the number of  $m$ -matchings of  $T$ .

We will use  $T(P_2)$  to denote the rooted product of  $T$  by the collection consisting of one copy of  $P_2$  for each vertex of  $T$ . It follows from (8) that, if  $T$  is a tree on  $n$  vertices, then  $T(-\lambda) = (-1)^n T(\lambda)$ , and so from (7) above we find

$$(9) \quad \begin{aligned} T(P_2)\left(\frac{1}{\lambda}\right) &= \left(\frac{-1}{\lambda}\right)^n T\left(\lambda - \frac{1}{\lambda}\right) \\ &= (-1)^n \lambda^{-2n} T(P_2)(\lambda). \end{aligned}$$

We will call a polynomial of degree  $2n$  satisfying (9) *symmetric*.

**THEOREM 3.2.** *Let  $T$  be a tree on  $2n$  vertices. Then  $T(\lambda)$  is symmetric if and only if  $T = S(P_2)$  for some tree  $S$ .*

**Proof.** The sufficiency follows from the remarks above. We give the proof of the necessity in a number of steps.

We assume  $n \geq 2$ . Let  $a_{2m}$  denote the number of  $m$ -matchings of  $T$ .

(a)  $T$  has 1  $n$ -matching and  $2n - 1$   $(n-1)$ -matchings.

Since  $T(\lambda)$  is symmetric we have  $a_0 = a_{2n}$  and  $a_2 = a_{2n-2}$ . But  $a_0 = 1$  and  $a_2$  is just the number of edges of  $T$  and so the claim follows.

(b)  $T$  has  $n$  end vertices.

Let  $M$  be the  $n$ -matching of  $T$ . By counting  $(n-1)$ -matchings we will show that every edge in  $M$  contains an end vertex of  $T$ .

Say that an  $(n-1)$ -matching  $N$  is of type I if it is a subset of  $M$ . Clearly there are  $n$  such matchings.

Let  $v_2v_3$  be an edge of  $T$  not in  $M$ . Then there are vertices  $v_1$  and  $v_4$  of  $T$  such that both  $v_1v_2$  and  $v_3v_4$  lie in  $M$ . Let  $N$  be the  $(n-1)$ -matching obtained from  $M$  by replacing  $v_1v_2$  and  $v_3v_4$  by the edge  $v_2v_3$ . We will call  $N$  a type II  $(n-1)$ -matching. The number of type II  $(n-1)$ -matchings is just the number of edges of  $T$  not in  $M$ . This equals  $n - 1$ .

Since a type II  $(n-1)$ -matching is not a subset of  $M$  we have already found  $2n - 1$  distinct  $(n-1)$ -matchings.

Let  $v_3v_4$  be an edge in  $M$  such that neither  $v_3$  nor  $v_4$  is an end-vertex. Let  $v_2$  and  $v_5$  be vertices of  $T$  adjacent to  $v_3$  and  $v_4$  respectively. Then there exist vertices  $v_1$  and  $v_6$  in  $T$  such that  $v_1v_2$  and  $v_5v_6$  lie in  $M$ . Replacing the edges  $v_1v_2$ ,  $v_3v_4$ , and  $v_5v_6$  of  $M$  by the edges  $v_2v_3$  and  $v_4v_5$  we obtain an  $(n-1)$ -matching  $N$ .

Since  $|N \cap M| \leq n - 3$ ,  $N$  is not of type I or II.

Thus the existence of an edge  $v_3v_4$  in  $M$  such that neither  $v_3$  nor  $v_4$  is an end vertex of  $T$  implies that  $T$  has at least  $2n(n-1)$ -matchings. Hence every edge in  $M$  contains at least one end vertex of  $T$ . If some edge in  $M$  consisted of two adjacent end-vertices, then  $T$  would be disconnected. Therefore  $T$  must have exactly  $|M| = n$  end vertices.

(c)  $T = S(P_2)$  for some tree  $S$ .

Let  $S$  be the tree obtained by removing the  $n$  end vertices from  $T$ . As  $T$  has a 1-factor, it cannot have a vertex adjacent to two end vertices. Hence  $T = S(P_2)$ .  $\square$

We remark that the proof of the theorem actually shows that a tree on  $2n$  vertices with an  $n$ -matching, and  $2n - 1$   $(n-1)$ -matchings is a rooted product.

Note that Theorem 3.2 does not hold when the assumption that  $T$  is a tree is dropped. For example the graph shown in Figure 3 is obviously not a rooted product, although its characteristic polynomial is  $\lambda^8 - 9\lambda^6 + 16\lambda^4 - 9\lambda^2 + 1$ , which is symmetric.

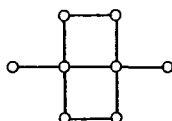


FIGURE 3

## References

- [1] Mehdi Behzad, Gary Chartrand, *Introduction to the theory of graphs* (Allyn and Bacon, Boston, 1971).
- [2] C. Godsil and B. McKay, "Some computational results on the spectra of graphs", *Combinatorial Mathematics IV*, 73-92 (Proc. Fourth Austral. Conf., University of Adelaide, 1975. Lecture Notes in Mathematics, 560. Springer-Verlag, Berlin, Heidelberg, New York, 1976).

- [3] Horst Sachs, "Beziehungen zwischen den in einem Graphen enthaltenen Kreisen und seinem charakteristischen Polynom", *Publ. Math. Debrecen* 11 (1964), 119–134.
- [4] Allen J. Schwenk, "Computing the characteristic polynomial of a graph", *Graphs and combinatorics*, 153–172 (Proc. Capital Conf. Graph Theory and Combinatorics, George Washington University, 1973. Lecture Notes in Mathematics, 406. Springer-Verlag, Berlin, Heidelberg, New York, 1974).

Department of Mathematics,  
Syracuse University,  
Syracuse,  
New York,  
USA;

Department of Mathematics,  
University of Melbourne,  
Parkville,  
Victoria.