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A New Hilbert-type Integral Inequality with Some Parameters and Its Reverse

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Abstract. In this paper, by introducing some parameters and estimating the weight function, we give a new Hilbert-type integral inequality with a best constant factor. The equivalent inequality and the reverse forms are considered.

1. Introduction

If $f(x),\ g(x) \geq 0$, such that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$, then (see [1])

$$\int_0^\infty \int_0^\infty \frac{f(x)g(x)}{x+y} dxdy < \pi \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{1/2}, \tag{1.1}$$

where the constant factor π is the best possible. Inequality (1.1) is well-known as Hilbert's integral inequality, which had been extended by Hardy-Riesz as (see [2]):

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $f(x),\ g(x) \geq 0$, such that $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then we have the following Hardy-Hilbert's integral inequality:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} \left\{ \int_0^\infty f^p(x)dx \right\}^{1/p} \left\{ \int_0^\infty g^q(x)dx \right\}^{1/q}, \tag{1.2}$$

where the constant factor $\dfrac{\pi}{\sin(\pi/p)}$ is the best possible.

Hardy-Hilbert's integral inequality is important in analysis and its applications (see [3]). In recent years, Yang [4], [5] gave two different best generalizations of (1.2) by introducing a parameter $\lambda > 0$, and Yang et al. [6] gave an extension of the above

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results by introducing the index of conjugate parameter (r,s) $(r>1, \frac{1}{r}+\frac{1}{s}=1)$ as follows (see [6],(21) for n=2):

If $f(x), g(x) \geq 0$, and $0 < \int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x)dx < \infty, 0 < \int_0^\infty x^{q(1-\frac{\lambda}{s})-1} g^q(x)dx < \infty$, then

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy \\ < \; & B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right) \left\{ \int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{q(1-\frac{\lambda}{s})-1} g^q(x)dx \right\}^{\frac{1}{q}}, \end{aligned} \tag{1.3}$$

where the constant factor $B\left(\frac{\lambda}{r}, \frac{\lambda}{s}\right)$ is the best possible (B(u,v) is the Beta function). In particular, for $\lambda = 1, r = q, s = p$, inequality (1.3) reduces to (1.2); for $\lambda = 4, r = s = 2$, (1.3) reduces to:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^4} dxdy < \frac{1}{6} \left\{ \int_0^\infty \frac{1}{x^{p+1}} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \frac{1}{x^{q+1}} g^q(x)dx \right\}^{\frac{1}{q}}. \tag{1.4}$$

In this paper, by introducing some parameters and estimating the weight function, we prove a new Hilbert-type integral inequality with a best constant factor similar to (1.4). The equivalent inequality and the reverse forms are considered.

2. Some lemmas

Lemma 2.1. *If $a, b > 0$, $a \neq b$, define the weight functions $\varpi(x)$ and $\omega(y)$ as*

$$\varpi(x) = \int_0^\infty \frac{x^2 y}{(x+ay)^2(x+by)^2} dy \; (x \in (0,\infty)); \tag{2.1}$$

$$\omega(y) = \int_0^\infty \frac{xy^2}{(x+ay)^2(x+by)^2} dx \; (y \in (0,\infty)), \tag{2.2}$$

then we have

$$\varpi(x) = \omega(y) = K := \frac{a+b}{(b-a)^2}\left[\frac{ln(b/a)}{b-a} - \frac{2}{a+b}\right]. \tag{2.3}$$

Proof. For fixed x, setting $u = y/x$ in (2.1), we obtain

$$\begin{aligned} \varpi(x) &= \int_0^\infty \frac{u}{(1+au)^2(1+bu)^2} du \\ &= \frac{1}{(b-a)^2}\left[\frac{1}{1+au} + \frac{1}{1+bu}\right]_0^\infty - \left[\frac{a+b}{(b-a)^3} ln\left(\frac{1+au}{1+bu}\right)\right]_0^\infty \\ &= \frac{1}{(b-a)^2}\left[-2 + (a+b)\frac{ln(b/a)}{b-a}\right]. \end{aligned}$$

Hence we obtain $\varpi(x) = K$. In the same way, we obtain $\omega(y) = K$. The lemma is proved. $\square$

Lemma 2.2. *If* $a, b > 0$ *and* $a \neq b$*, then for* $0 < \varepsilon < p$*, we have*

$$\int_0^\infty \frac{u^{1-\varepsilon/p}}{(1+au)^2(1+bu)^2}du = K + o(1) \ \ (\varepsilon \to 0^+). \tag{2.4}$$

Proof. Since for $a, b > 0$ and $a \neq b$, we find

$$\begin{aligned}
&\left|\int_0^\infty \frac{u^{1-\varepsilon/p}}{(1+au)^2(1+bu)^2}du - K\right| = \left|\int_0^\infty \frac{u^{1-\varepsilon/p} - u}{(1+au)^2(1+bu)^2}du\right| \\
\leq & \int_0^1 \frac{|\, u^{1-\varepsilon/p} - u \,|}{(1+au)^2(1+bu)^2}du + \int_1^\infty \frac{|\, u^{1-\varepsilon/p} - u \,|}{(1+au)^2(1+bu)^2}du. \\
\leq & \int_0^1 (u^{1-\varepsilon/p} - u)du + \frac{1}{(ab)^2}\int_1^\infty \frac{u - u^{1-\varepsilon/p}}{u^4}du \\
= & \left(\frac{1}{2-\varepsilon/p} - \frac{1}{2}\right) + \frac{1}{(ab)^2}\left(\frac{1}{2} - \frac{1}{2+\varepsilon/p}\right) \to 0, \qquad for \ \ \varepsilon \to 0^+.
\end{aligned}$$

Then (2.4) is valid, and the lemma is proved. $\square$

Lemma 2.3. *If* $p > 1$ *(or* $0 < p < 1$*),* $\frac{1}{p} + \frac{1}{q} = 1$*,* $a, b > 0, a \neq b$*, and* $0 < \varepsilon < p$*, setting*

$$I := \int_1^\infty \left[\int_1^\infty \frac{y^{1-\varepsilon/p}}{(x+ay)^2(x+by)^2}dy\right] x^{1-\varepsilon/q}dx,$$

then we have

$$\frac{1}{\varepsilon}(K + o(1)) - O(1) \leq I \leq \frac{1}{\varepsilon}(K + o(1)), \ \varepsilon \to 0^+. \tag{2.5}$$

Proof. For fixed x, setting $y = xu$, then by (2.4), we obtain

$$\begin{aligned}
I &= \int_1^\infty x^{-1-\varepsilon}\left[\int_{x^{-1}}^\infty \frac{u^{1-\varepsilon/p}}{(1+au)^2(1+bu)^2}du\right]dx \\
&= \int_1^\infty x^{-1-\varepsilon}\left[\int_0^\infty \frac{u^{1-\varepsilon/p}}{(1+au)^2(1+bu)^2}du\right]dx \\
&\quad - \int_1^\infty x^{-1-\varepsilon}\left[\int_0^{x^{-1}} \frac{u^{1-\varepsilon/p}}{(1+au)^2(1+bu)^2}du\right]dx \\
&\geq \frac{1}{\varepsilon}(K + o(1)) - \frac{1}{a+b}\int_1^\infty x^{-1}\left(\int_0^{x^{-1}} u^{-\frac{\varepsilon}{p}}du\right)dx \quad ((1+au)^2(1+bu)^2 > (a+b)u) \\
&= \frac{1}{\varepsilon}(K + o(1)) - \frac{1}{a+b}\left(1 - \frac{\varepsilon}{p}\right)^{-2} = \frac{1}{\varepsilon}(K + o(1)) - O(1).
\end{aligned}$$

By the same way, we have

$$I \le \int_1^\infty [\int_0^\infty \frac{y^{1-\varepsilon/p}}{(x+ay)^2(x+by)^2} dy] x^{1-\varepsilon/q} dx = \frac{1}{\varepsilon}(K + o(1)).$$

The lemma is proved. □

3. Main results

Theorem 3.1. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a, b > 0, a \ne b, f(x), g(x) \ge 0$, *such that* $0 < \int_0^\infty \frac{1}{x^{p+1}} f^p(x) dx < \infty$ *and* $0 < \int_0^\infty \frac{1}{x^{q+1}} g^q(x) dx < \infty$, *then*

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+ay)^2(x+by)^2} dx dy \\ < \;& K \left\{ \int_0^\infty \frac{1}{x^{p+1}} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \frac{1}{x^{q+1}} g^q(x) dx \right\}^{\frac{1}{q}}, \end{aligned} \tag{3.1}$$

where the constant factor K *is the best possible and* K *is defined by* (2.3).

Theorem 3.2. *If* $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a, b > 0, a \ne b, f(x), g(x) \ge 0$, *such that* $0 < \int_0^\infty \frac{1}{x^{p+1}} f^p(x) dx < \infty$ *and* $0 < \int_0^\infty \frac{1}{x^{q+1}} g^q(x) dx < \infty$, *then*

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+ay)^2(x+by)^2} dx dy \\ > \;& K \left\{ \int_0^\infty \frac{1}{x^{p+1}} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \frac{1}{x^{q+1}} g^q(x) dx \right\}^{\frac{1}{q}}, \end{aligned} \tag{3.2}$$

where the constant factor K *is the best possible.*

Proof of Theorem 3.1. By Hölder's inequality with weight (see [7]) and (2.1)-(2.3), we have,

$$\begin{aligned} J :&= \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+ay)^2(x+by)^2} dx dy \\ &= \int_0^\infty \int_0^\infty \frac{1}{(x+ay)^2(x+by)^2} [(\frac{y^{1/p}}{x^{1/q}}) f(x)] [(\frac{x^{1/q}}{y^{1/p}}) g(y)] dx dy \\ &\le \left\{ \int_0^\infty \int_0^\infty \frac{1}{(x+ay)^2(x+by)^2} (\frac{y}{x^{p-1}}) f^p(x) dy dx \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \int_0^\infty \int_0^\infty \frac{1}{(x+ay)^2(x+by)^2} (\frac{x}{y^{q-1}}) g^q(y) dx dy \right\}^{\frac{1}{q}} \\ &= \left\{ \int_0^\infty \varpi(x) \frac{1}{x^{p+1}} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \omega(y) \frac{1}{y^{q+1}} g^q(y) dx \right\}^{\frac{1}{q}} \\ &= K \left\{ \int_0^\infty \frac{1}{x^{p+1}} f^p(x) dx \right\}^{1/p} \left\{ \int_0^\infty \frac{1}{x^{q+1}} g^q(x) dx \right\}^{1/q}. \end{aligned} \tag{3.3}$$

If (3.3) takes the form of equality, then there exists constants M and N, such that they are not all zero, and (see [7])

$$M(\frac{y}{x^{p-1}})f^p(x) = N(\frac{x}{y^{q-1}})g^q(y) \quad \text{a.e. in } (0,\infty)\times(0,\infty).$$

Hence, there exists a constant C, such that

$$Mx^{-p}f^p(x) = Ny^{-q}g^q(y) = C \qquad \text{a.e. in} \quad (0,\infty).$$

We claim that $M = 0$. In fact, if $M \neq 0$, then $x^{-1-p}f^p(x) = C/(Mx)$ a.e. in $(0,\infty)$, which contradicts the fact that $0 < \int_0^\infty x^{-1-p}f^p(x)dx < \infty$. In the same way, we claim that $N = 0$. This is a contradiction. Hence by (3.3), we have (3.1).

If the constant factor K in (3.1) is not the best possible, then there exists a positive constant H (with $H < K$), such that (3.1) is still valid if we replace K by H. For $0 < \varepsilon < p$ small enough, setting f_ε and g_ε as $f_\varepsilon(x) = g_\varepsilon(x) = 0$, for $x \in (0,1)$; $f_\varepsilon(x) = x^{1-\varepsilon/p}$; $g_\varepsilon(x) = x^{1-\varepsilon/q}$, for $x \in [1,\infty)$, then we obtain

$$\begin{aligned} & H\left\{\int_0^\infty \frac{1}{x^{p+1}}f_\varepsilon^p(x)dx\right\}^{1/p}\left\{\int_0^\infty \frac{1}{x^{q+1}}g_\varepsilon^q(x)dx\right\}^{1/q} \\ = & H\left\{\int_1^\infty x^{-\varepsilon-1}dx\right\}^{1/p}\left\{\int_1^\infty x^{-\varepsilon-1}dx\right\}^{1/q} = \frac{H}{\varepsilon}. \end{aligned}$$

By (2.5), we have

$$\begin{aligned} & \int_0^\infty\int_0^\infty \frac{f_\varepsilon(x)g_\varepsilon(y)}{(x+ay)^2(x+by)^2}dxdy \\ = & \int_1^\infty\left[\int_1^\infty \frac{y^{1-\frac{\varepsilon}{q}}}{(x+ay)^2(x+by)^2}dy\right]x^{1-\frac{\varepsilon}{p}}dx = I \\ \geq & \frac{1}{\varepsilon}(K+o(1)) - O(1). \end{aligned}$$

Hence we find

$$\frac{1}{\varepsilon}(K+o(1)) - O(1) < \frac{H}{\varepsilon} \quad \textit{or} \quad (K+o(1)) - \varepsilon O(1) < H.$$

For $\varepsilon \to 0^+$, it follows that $K \leq H$. This contradicts the fact that $H < K$. Hence the constant factor K in (3.1) is the best possible. The theorem is proved. □

Proof of Theorem 3.2. By the reverse Hölder's inequality with weight (see [7]) and the same way of giving (3.3), we obtain (3.2).

If the constant factor K in (3.2) is not the best possible, then there exists a positive constant H (with $H > K$), such that (3.2) is still valid if we replace K by H. For $0 < \varepsilon < p$ small enough, setting f_ε and g_ε as: $f_\varepsilon(x) = g_\varepsilon(x) = 0$, for

$x \in (0,1)$; $f_\varepsilon(x) = x^{1-\varepsilon/p}$; $g_\varepsilon(x) = x^{1-\varepsilon/q}$, for $x \in [1,\infty)$, then we obtain

$$
\begin{aligned}
(3.4) \qquad & H\left\{\int_0^\infty \frac{1}{x^{p+1}} f_\varepsilon^p(x)dx\right\}^{1/p} \left\{\int_0^\infty \frac{1}{x^{q+1}} g_\varepsilon^q(x)dx\right\}^{1/q} \\
= \; & H\left\{\int_1^\infty x^{-\varepsilon-1}dx\right\}^{1/p} \left\{\int_1^\infty x^{-\varepsilon-1}dx\right\}^{1/q} = \frac{H}{\varepsilon}.
\end{aligned}
$$

By (2.5), we have

$$
\begin{aligned}
& \int_0^\infty \int_0^\infty \frac{f_\varepsilon(x)g_\varepsilon(y)}{(x+ay)^2(x+by)^2}dxdy \\
= \; & \int_1^\infty \left[\int_1^\infty \frac{y^{1-\frac{\varepsilon}{p}}}{(x+ay)^2(x+by)^2}dy\right] x^{1-\frac{\varepsilon}{q}}dx = I \\
\leq \; & \frac{1}{\varepsilon}(K + o(1)).
\end{aligned}
$$

Hence we find

$$
\frac{1}{\varepsilon}(K + o(1)) > \frac{H}{\varepsilon} \quad or \quad (K + o(1)) > H.
$$

For $\varepsilon \to 0^+$, it follows that $K \geq H$. This contradicts the fact that $H > K$. Hence the constant K in (3.2) is the best possible. The theorem is proved. □

Theorem 3.3. *Under the same assumption of Theorem 3.1. we have*

$$
(3.5) \qquad \int_0^\infty y^{2p-1}\left[\int_0^\infty \frac{f(x)}{(x+ay)^2(x+by)^2}dx\right]^p dy < K^p \int_0^\infty \frac{1}{x^{p+1}} f^p(x)dx,
$$

where the constant factor K^p is the best possible. Inequalities (3.5) *and* (3.1) *are equivalent.*

Theorem 3.4. *Under the same assumption of Theorem 3.2. we have*

$$
(3.6) \qquad \int_0^\infty y^{2p-1}\left[\int_0^\infty \frac{f(x)}{(x+ay)^2(x+by)^2}dx\right]^p dy > K^p \int_0^\infty \frac{1}{x^{p+1}} f^p(x)dx,
$$

where the constant factor K^p is the best possible. Inequalities (3.6) *and* (3.2) *are equivalent.*

We prove only Theorem 3.3, since the proof of Theorem 3.4 is the similar.

Proof. Setting $g(y) = y^{2p-1}\left[\int_0^\infty \frac{f(x)}{(x+ay)^2(x+by)^2}dx\right]^{p-1}$, by (3.1), we have

$$
\begin{aligned}
(3.7) \qquad & \int_0^\infty y^{-1-q}g^q(y)dy = \int_0^\infty y^{2p-1}\left[\int_0^\infty \frac{f(x)}{(x+ay)^2(x+by)^2}dx\right]^p dy \\
= \; & J \leq K\left\{\int_0^\infty \frac{1}{x^{p+1}} f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty \frac{1}{x^{q+1}} g^q(x)dx\right\}^{\frac{1}{q}};
\end{aligned}
$$

$$0 < \left\{ \int_0^\infty \frac{1}{y^{q+1}} g^q(y) dy \right\}^{1/p} \leq K \left\{ \int_0^\infty \frac{1}{x^{p+1}} f^p(x) dx \right\}^{1/p} < \infty. \tag{3.8}$$

Hence by (3.1), both (3.7) and (3.8) preserve the form of strict inequalities, and we have (3.5).

By Hölder's inequality, we have

$$\begin{aligned} J &= \int_0^\infty \left[y^{1+1/q} \int_0^\infty \frac{f(x)}{(x+ay)^2(x+by)^2} dx \right] \left(y^{-1-1/q} g(y) \right) dy \\ &\leq \left\{ \int_0^\infty y^{2p-1} \left[\int_0^\infty \frac{f(x)}{(x+ay)^2(x+by)^2} dx \right]^p dy \right\}^{\frac{1}{p}} \\ &\quad \times \left(\int_0^\infty \frac{1}{y^{q+1}} g^q(y) dy \right)^{\frac{1}{q}}. \end{aligned} \tag{3.9}$$

Then by (3.5), we have (3.1). Hence inequalities (3.1) and (3.5) are equivalent.

If the constant factor in (3.5) is not the best possible, then by (3.9), we can get a contradiction that the constant factor in (3.1) is not the best possible. The theorem is proved. □

Remark. Since we obtain

$$\lim_{x \to a} \left\{ \frac{x+a}{(x-a)^2} \left[\frac{ln(\frac{x}{a})}{x-a} - \frac{2}{x+a} \right] \right\} = \lim_{x \to a} \left[\frac{(x+a)ln(\frac{x}{a}) - 2(x-a)}{(x-a)^3} \right] = \frac{1}{6a^2}, \tag{3.10}$$

then by (3.1), setting $b \to a$. we have the following new inequality:

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+ay)^4} dx dy \\ &< \frac{1}{6a^2} \left\{ \int_0^\infty \frac{1}{x^{p+1}} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \frac{1}{x^{q+1}} g^q(x) dx \right\}^{\frac{1}{q}}. \end{aligned} \tag{3.11}$$

In particular, for $a = 1$, (3.11) reduces to (1.4). Hence we can get a conclusion that inequality (3.1) is a best extension of (1.4).

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