A new implicit surface tension implementation for interfacial flows

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Modeling of Interfacial Flows

- Interfacial or two-phase flow where capillary surface tension forces are dominant poses some challenging problems

- Surface tension effects are generally modeled both explicitly in time and space leading to the capillary time step restriction

$$\Delta t_{num}^{(ca)} < \sqrt{\frac{\langle \rho \rangle \, h^3}{2\pi\sigma}}$$

Goal

Remove the capillary time step constraint
while retaining a fully Eulerian interface description

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Modeling of Interfacial Flows

- The Navier-Stokes equations govern incompressible fluid flow

$$\begin{aligned} \rho(\mathbf{x})\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) &= -\nabla p + \nabla\cdot\left(\mu(\mathbf{x})(\nabla\mathbf{u} + \nabla\mathbf{u}^T)\right) + \rho(\mathbf{x})\mathbf{g} \\ \nabla\cdot\mathbf{u} &= 0 \end{aligned}$$

with varying density $\rho(\mathbf{x})$ and viscosity $\mu(\mathbf{x})$ fields

Interfacial Boundary Conditions

- Direct interface conditions

$$[\mathbf{u}]|_\Gamma = 0, \quad -\left[-p\mathbf{I} + \mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T)\right]\Big|_\Gamma \cdot \hat{\mathbf{n}} = \sigma\kappa\hat{\mathbf{n}}$$

- Implicit conditions by weighted volume forces

$$\mathbf{f}_{st}|_\Gamma = \sigma\kappa\hat{\mathbf{n}}$$

Definitions

Definition (Tangential gradient)

The tangential gradient *of a function f, which is differentiable in an open neighborhood of Γ, is defined by*

$$\underline{\nabla} f(x) = \nabla f(x) - (\hat{\mathbf{n}}(x) \cdot \nabla f(x))\hat{\mathbf{n}}(x), \quad x \in \Gamma$$

where ∇ denotes the usual gradient in $\mathbb{R}^d$

Definition (Laplace-Beltrami operator)

If f is two times differentiable in a neighborhood of Γ, then we define the Laplace-Beltrami operator *of f as*

$$\underline{\Delta} f(x) = \underline{\nabla} \cdot (\underline{\nabla} f(x)), \quad x \in \Gamma$$

Definitions and Derivation

Theorem

A theorem of differential geometry states that

$$\underline{\Delta}\mathrm{id}_\Gamma = \kappa\hat{\mathbf{n}}$$

where κ is the mean curvature and id_Γ is the identity mapping on Γ

Derivation

First use the volume force formulation of the surface tension forces, multiply with the test function space $\mathbf{v}$, and apply partial integration

$$\mathbf{f}_{st} = \int_\Gamma \sigma\kappa\hat{\mathbf{n}} \cdot \mathbf{v}\; d\Gamma = \int_\Gamma \sigma(\underline{\Delta}\mathrm{id}_\Gamma) \cdot \mathbf{v}\; d\Gamma =$$

$$= -\int_\Gamma \sigma\underline{\nabla}\mathrm{id}_\Gamma \cdot \underline{\nabla}\mathbf{v}\; d\Gamma + \int_\gamma \sigma\partial_\gamma\mathrm{id}_\Gamma \cdot \mathbf{v}\; d\gamma$$

Time Integration

- Explicit time integration

$$\mathbf{f}_{st} = \int_{\Gamma^n} \sigma\kappa^n \hat{\mathbf{n}}^n \cdot \mathbf{v}\ d\Gamma = -\int_{\Gamma^n} \sigma \underline{\nabla}(\mathrm{id}_\Gamma)^n \cdot \underline{\nabla}\mathbf{v}\ d\Gamma$$

- Semi-implicit time integration

$$(\mathrm{id}_\Gamma)^{n+1} = (\mathrm{id}_\Gamma)^n + \Delta t^{n+1}\ \mathbf{u}^{n+1}$$

$$\mathbf{f}_{st} = -\int_{\Gamma^n} \sigma \underline{\nabla}(\mathrm{id}_\Gamma)^n \cdot \underline{\nabla}\mathbf{v}\ d\Gamma - \Delta t^{n+1} \int_{\Gamma^n} \sigma \underline{\nabla}\mathbf{u}^{n+1} \cdot \underline{\nabla}\mathbf{v}\ d\Gamma$$

Fully Implicit Evaluation in Space

- Boundary integrals can be transformed to volume integrals with the help of a Dirac delta function $\delta(\Gamma, \mathbf{x})$

$$\begin{aligned} \mathbf{f}_{st} &= \int_\Omega \sigma\kappa\hat{\mathbf{n}}\cdot\mathbf{v}\;\delta(\Gamma,\mathbf{x})\;d\mathbf{x} = \int_\Omega \sigma(\underline{\Delta}\mathrm{id}_\Gamma)\cdot(\mathbf{v}\delta(\Gamma,\mathbf{x}))\;d\mathbf{x} \\ &= -\int_\Omega \sigma\underline{\nabla}\mathrm{id}_\Gamma\cdot\underline{\nabla}(\mathbf{v}\delta(\Gamma,\mathbf{x}))\;d\mathbf{x} = -\int_\Omega \sigma\underline{\nabla}\mathrm{id}_\Gamma\cdot\underline{\nabla}\mathbf{v}\;\delta(\Gamma,\mathbf{x})\;d\mathbf{x} \end{aligned}$$

- Application of the semi-implicit time integration

$$\begin{aligned} \mathbf{f}_{st} &= -\int_\Omega \sigma\underline{\nabla}(\mathrm{id}_\Gamma)^n\cdot\underline{\nabla}\mathbf{v}\;\delta(\Gamma^n,\mathbf{x})\;d\mathbf{x} \\ &\quad -\Delta t^{n+1}\int_\Omega \sigma\underline{\nabla}\mathbf{u}^{n+1}\cdot\underline{\nabla}\mathbf{v}\;\delta(\Gamma^n,\mathbf{x})\;d\mathbf{x} \end{aligned}$$

Regularization

- Regularization of $\delta(\Gamma, \mathbf{x})$ can easily be accomplished with the help of a distance function

$$\delta_{\boldsymbol{\epsilon}}(\Gamma, \mathbf{x}) = \delta_\epsilon(dist(\Gamma, \mathbf{x})),$$

where $dist(\Gamma, \mathbf{x})$ gives the minimum distance from $\mathbf{x}$ to Γ

- The regularized continuous delta function δ_ϵ is defined as

$$\delta_\epsilon(x) = \begin{cases} \frac{1}{\epsilon}\varphi(x/\epsilon) & |x| \leq \epsilon \quad = \quad mh, \\ 0 & |x| > \epsilon \quad = \quad mh, \end{cases}$$

where h is the mesh spacing which together with the constant m defines the support ϵ of the regularized delta function, φ is a characteristic function determining the kernel shape

Implicit Surface Tension Force Expression

The surface tension forces are finally given by...

Implicit Surface Tension Force Expression

$$\mathbf{f}_{st} = \int_\Omega \sigma \, \delta_\epsilon(dist(\Gamma^n, \mathbf{x})) \, \underline{\nabla}(\widetilde{\mathrm{id}}_\Gamma)^n \cdot \underline{\nabla}\mathbf{v} \, d\mathbf{x} + \Delta t^{n+1} \int_\Omega \sigma \, \delta_\epsilon(dist(\Gamma^n, \mathbf{x})) \, \underline{\nabla}\mathbf{u}^{n+1} \cdot \underline{\nabla}\mathbf{v} \, d\mathbf{x}$$

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$$\begin{aligned} \mathbf{f}_{st} &= \int_\Omega \sigma\ \delta_\epsilon(dist(\Gamma^n, \mathbf{x}))\ \underline{\nabla}(\widetilde{\mathrm{id}}_\Gamma)^n \cdot \underline{\nabla}\mathbf{v}\ d\mathbf{x} \\ &+ \Delta t^{n+1} \int_\Omega \sigma\ \delta_\epsilon(dist(\Gamma^n, \mathbf{x}))\ \underline{\nabla}\mathbf{u}^{n+1} \cdot \underline{\nabla}\mathbf{v}\ d\mathbf{x} \end{aligned}$$

Level Set Method

A number of key points makes the *level set method* an ideal candidate for interface tracking algorithm when implementing the proposed surface tension force expressions

- Distance functions are in general readily available allowing for simple construction of the regularized Dirac delta functions
- Geometrical quantities such as normal and tangent vectors can be reconstructed globally, eliminating the need to extend these quantities from the interface separately
- The level set method can be coupled with the finite element method giving access to the variational form of the equations

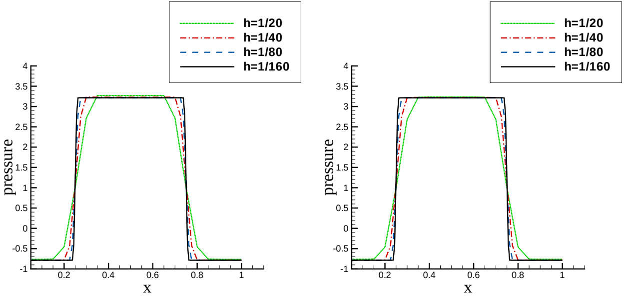
Method Validation

Numerical Examples

Validation, Laplace-Young Law

A perfect circular static bubble should follow the Laplace-Young law

$$p_{inside} = p_{outside} + \frac{\sigma}{r}$$



(a) CSF (b) CSF-LBI

Figure: Pressure cut-line for four different mesh sizes.

Example, Oscillating Bubble (CSF)

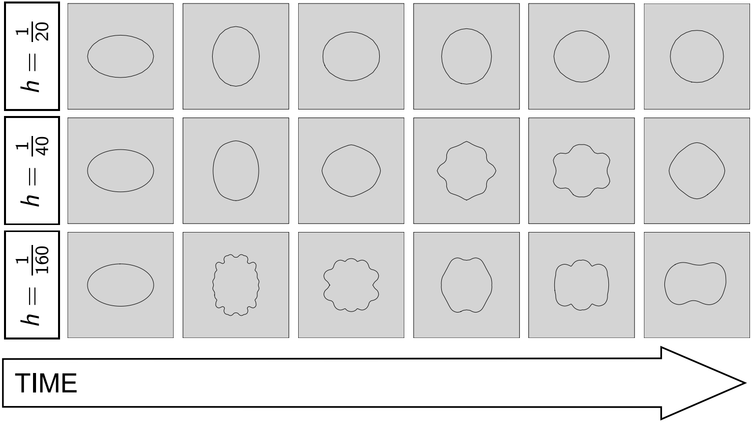


Figure: Evolution of an oscillating bubble; standard explicit CSF method.

Example, Oscillating Bubble (CSF-LBI)

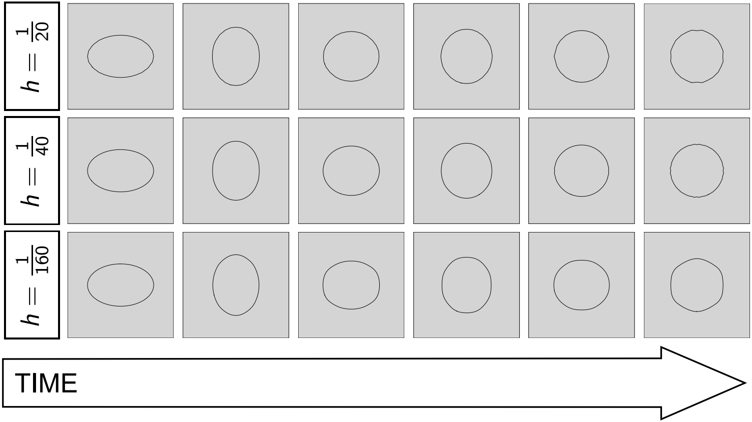


Figure: Evolution of an oscillating bubble; semi-implicit CSF-LBI method.

Example, Rising Bubble (CSF)

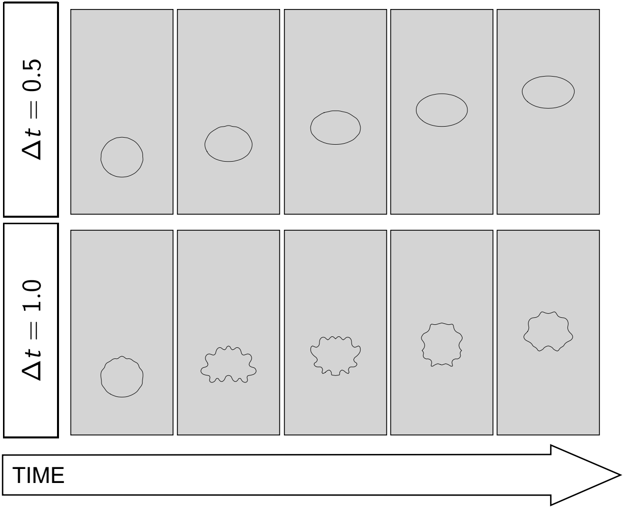


Figure: Evolution of a rising bubble with the standard explicit CSF method.

Example, Rising Bubble (CSF-LBI)

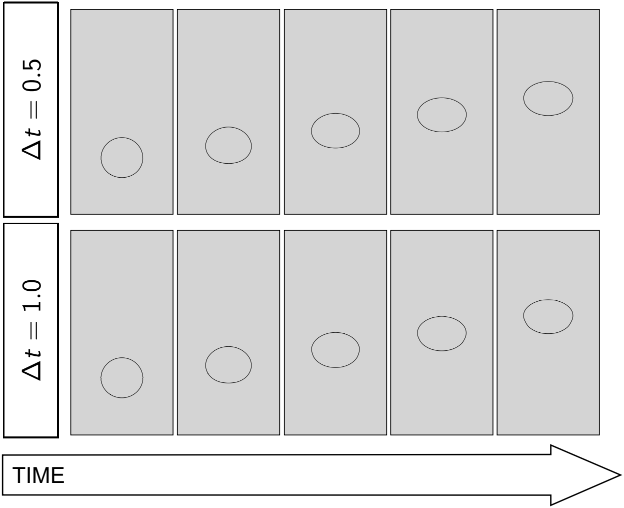


Figure: Evolution of a rising bubble with the semi-implicit CSF-LBI method.

Summary

A new implicit surface tension variant has been proposed which relaxes the capillary time step restriction imposed on explicit implementations

Additional advantages

- Fully implicit in space
- Is easily implemented when using the level set method together with finite elements
- Explicit computation of curvature not necessary
- Conceptually identical algorithm in 3D