TAMKANG JOURNAL OF MATHEMATICS
Volume 28, Number 3, Autumn 1997

A NEW INEQUALITY OF OSTROWSKI'S TYPE IN L_1 NORM AND APPLICATIONS TO SOME SPECIAL MEANS AND TO SOME NUMERICAL QUADRATURE RULES

SEVER S. DRAGOMIR AND SONG WANG

Abstract. In this paper we prove a new Ostrowski's inequality in L_1-norm and apply it to the estimation of error bounds for some special means and for some numerical quadrature rules.

1. Introduction

The following theorem contains the integral inequality which is known in the literature as Ostrowski's inequality [2, p.469].

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \mapsto \mathbb{R}$ be a differentiable mapping in $\dot{I}$, the interior of I, and let $a, b \in \dot{I}$ with $a < b$. If $\|f'\|_\infty := \sup_{t\in[a,b]} |f'(t)| \leq M$, then we have the inequality:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)M \quad \forall x \in [a,b]. \tag{1.1}$$

For some applications of this theorem to the estimation of error bounds for the special means and for the numerical quadrature rules we refer to a recent paper [1]. In the rest of this paper we shall derive a new inequality of Ostrowski's type in the L_1-norm of f' and shall consider the applications of this new inequality in the theory of special means and in numerical integrations.

2. The Results

Before further discussion we let $L_1[a,b]$ denote the usual linear space of all absolutely integrable functions on $[a,b]$ with the conventional L_1-norm $\|\cdot\|_1$. Then, the following theorem establishes an inequality of Ostrowski type in the L_1-norm.

Received December 24, 1996, Revised May 20, 1997.
1991 *Mathematics Subject Classification.* 26D15, 26D20, 65D32.
Key words and phrases. Ostrowsri inequality, Special means, Adaptive quadrature rules.
This research was supported in part by a research grant from Curtin University of Technology.

Theorem 2.1. *Let $f : I \subseteq \mathbb{R} \mapsto \mathbb{R}$ be a differentiable mapping in $\dot{I}$ and $a, b \in \dot{I}$ with $a < b$. If $f' \in L_1[a, b]$, then we have the following inequality:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{2} + \frac{\left| x - \frac{a+b}{2} \right|}{b-a} \right] ||f'||_1 \tag{2.1}$$

for all $x \in [a, b]$.

Proof. Integrating by parts, we have

$$\int_a^x (t-a)f'(t)dt = (x-a)f(x) - \int_a^x f(t)dt$$

and

$$\int_x^b (t-b)f'(t)dt = (b-x)f(x) - \int_x^b f(t)dt.$$

Adding these two equalities we get

$$\int_a^x (t-a)f'(t)dt + \int_x^b (t-b)f'(t)dt = (b-a)f(x) - \int_a^b f(t)dt.$$

From this we obtain

$$f(x) - \frac{1}{b-a} \int_a^b f(t)dt = \frac{1}{b-a} \int_a^b p(x,t) f'(t)dt \tag{2.2}$$

where

$$p(x,t) := \begin{cases} t-a & \text{if } t \in [a, x] \\ t-b & \text{if } t \in (x, b] \end{cases}$$

for all $x \in [a, b]$ and $t \in [a, b]$.

Now, using (2.2) we get

$$\begin{aligned} \left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| &\leq \frac{1}{b-a} \left| \int_a^x (t-a)f'(t)dt \right| + \frac{1}{b-a} \left| \int_x^b (t-b)f'(t)dt \right| \\ &\leq \frac{1}{b-a} \int_a^x (t-a)|f'(t)|dt + \frac{1}{b-a} \int_x^b (b-t)|f'(t)dt \\ &\leq \frac{x-a}{b-a} \int_a^x |f'(t)|dt + \frac{b-x}{b-a} \int_x^b |f'(t)|dt \\ &\leq \max_{x \in [a,b]} \{x-a, b-x\} \frac{1}{b-a} \left[\int_a^x |f'(t)|dt + \int_x^b |f'(t)|dt \right] \\ &= \frac{1}{2} ||f'||_1 + \frac{\left| x - \frac{a+b}{2} \right|}{b-a} ||f'||_1, \end{aligned}$$

proving the theorem.

Corollary 2.2. *With the assumption in Theorem 2.1, we have the following inequalities*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{2}\int_a^b |f'(t)|dt \tag{2.3}$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \int_a^b |f'(t)|dt. \tag{2.4}$$

Remark 2.3. If we assume that f is convex on and $[a,b] \in \dot{I}$, then by (2.3), (2.4) and the classical Hermite-Hadamard's inequalities

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}$$

we have the following inequalities:

$$0 \leq \frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\int_a^b |f'(t)|dt$$

and

$$0 \leq \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt \leq \int_a^b |f'(t)|dt.$$

3. Applications to the Special Means

We first discuss the application of (2.1) to lower and upper bounds estimations of some important relationships between the following means:

(a) The arithmetic mean: $A = A(a,b) := (a+b)/2$, $a, b \geq 0$,
(b) The geometric mean: $G = G(a,b) := \sqrt{ab}$, $a, b \geq 0$,
(c) The harmonic mean:

$$H = H(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}}, \quad a, b > 0,$$

(d) The logarithmic mean:

$$L = L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b \\ a & \text{if } a = b \end{cases}, \quad a, b > 0,$$

(e) The identric mean:

$$I = I(a,b) := \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a \neq b \\ a & \text{if } a = b \end{cases}, \quad a, b > 0,$$

(f) The p-logarithmic mean

$$L_p = L_p(a,b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if } a \neq b \\ a & \text{if } a = b \end{cases}, \quad p \in \mathbb{R}\backslash\{-1,0\}; \quad a,b>0.$$

The following simple relationships are well known in the literature:

$$H \le G \le L \le I \le A.$$

It is also known that L_p is monotonically increasing in $p \in \mathbb{R}$ with $L_0 = I$ and $L_{-1} = L$.

Case 1. $f(x) = x^p (p \in \mathbb{R}\backslash\{-1,0\})$.
Replacing f in (2.1) by this function we get

$$|x^p - L_p^p| \le \frac{b-a}{2}|p|L_{p-1}^{p-1} + |x-A||p|L_{p-1}^{p-1} \tag{3.1}$$

for all $x \in [a,b]$ and $p \neq 1$. If we choose $x = A$, then we get

$$|L_p^p - A| \le \frac{b-a}{2}|p|L_{p-1}^{p-1}.$$

Moreover, for $p \in (-\infty,0) \cup (1,\infty)\backslash\{-1\}$, we have

$$0 \le L_p^p - A^p \le \frac{b-a}{2}|p|L_{p-1}^{p-1}$$

and for $p \in (0,1)$ we have

$$0 \le A^p - L_p^p \le \frac{b-a}{2}pL_{p-1}^{p-1}.$$

Let $x = I$ in (3.1). We get

$$|I^p - L_p^p| \le \frac{b-a}{2}|p|L_{p-1}^{p-1} + |I-A||p|L_{p-1}^{p-1}$$

for all $p \in \mathbb{R}\backslash\{-1,0\}$.

Case 2: $f(x) = 1/x$.
Substituting this function into (2.1), we get the following inequality:

$$|x-L| \le \frac{xL(b-a)}{2}L_{-2}^{-2} + |x-A|L_{-2}^{-2} \tag{3.2}$$

for all $x \in [a,b]$. Replacing x in this inequality by A and I, we get respectively the following inequalities:

$$\begin{aligned} 0 &\le A - L \le AL\frac{b-a}{2}L_{-2}^{-2}, \\ 0 &\le I - L \le IL\frac{b-a}{2}L_{-2}^{-2} + |I-A|L_{-2}^{-2}. \end{aligned}$$

Case 3. $f(x) = -\ln x$.
Now, replacing f in (2.1) by this function we obtain

$$|\ln I - \ln x| \leq \frac{b-a}{2}L^{-1} + |x - A|L^{-1} \tag{3.3}$$

for all $x \in [a, b]$. Choosing $x = A$ and $x = L$ in this inequality we get respectively the following inequalities:

$$\begin{aligned} 1 &\leq \frac{A}{L} \leq \exp\Big[\frac{b-a}{2}L^{-1}\Big] = \sqrt{\frac{b}{a}}, \\ 1 &\leq \frac{I}{L} \leq \exp\Big[\big(\frac{b-a}{2} + A - L\big)L^{-1}\Big]. \end{aligned}$$

We comment that we can also choose $x = L$, $x = G$ or $x = H$ in the above three inequalities (3.1), (3.2) and (3.3). The resulting inequalities in L and G will be similar to those obtained above. However, for brevity, we omit these discussion and leave them to the reader.

4. Application to the Numerical Quadrature Rules

We now consider the application of (2.1) to some numerical quadrature rules. In the rest of the paper we will use standard notation for function spaces and norms. The following theorem establishes the error bound for the quadrature rules of Riemann type.

Theorem 4.1. *For any $a, b \in \mathbb{R}$ with $a < b$, let $f : (a, b) \mapsto \mathbb{R}$ be a differentiable mapping. If $f' \in L_1[a, b]$, then for any partition $I_h : a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ of I and any intermediate point vector $\xi = (\xi_0, \xi_1, \cdots, \xi_{n-1})$ satisfying $\xi_i \in [x_i, x_{i+1}]\,(i = 0, 1, \ldots, n-1)$, we have*

$$\Big|\int_a^b f(x)dx - A_R(f, I_h, \xi)\Big| \leq ||f'||_1 \max_{0 \leq i \leq n-1} \Big\{\frac{h_i}{2} + \big|\xi_i - \frac{x_i + x_{i+1}}{2}\big|\Big\} \leq h||f'||_1 \tag{4.1}$$

where $h_i = x_{i+1} - x_i$, $h = \max_{0 \leq i \leq n-1} h_i$ and A_R denotes the quadrature rules of Riemann type defined by

$$A_R(f, I_h, \xi) := \sum_{i=0}^{n-1} f(\xi_i)h_i. \tag{4.2}$$

Proof. From Theorem 2.1 we get

$$\begin{aligned} \Big|f(\xi_i)h_i - \int_{x_i}^{x_{i+1}} f(x)dx\Big| &\leq \frac{1}{2}h_i \int_{x_i}^{x_{i+1}} |f'(t)|dt + \Big|\xi_i - \frac{x_i + x_{i+1}}{2}\Big| \int_{x_i}^{x_{i+1}} |f'(x)|dx \\ &\leq \max_{0 \leq i \leq n-1} \Big\{\frac{h_i}{2} + \Big|\xi_i - \frac{x_i + x_{i+1}}{2}\Big|\Big\} \int_{x_i}^{x_{i+1}} |f'(x)|dx \end{aligned}$$

for all $i = 0, 1, 2, \ldots, n-1$. Summing the above inequality we have the left inequality in (4.1). Now, observe that

$$\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right| \leq \frac{h_i}{2}, \quad i = 0, 1, 2, \ldots, n-1.$$

From this we have

$$\max_{0 \leq i \leq n-1} \left\{ \frac{h_i}{2} + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right| \right\} \leq \max_{0 \leq i \leq n-1} h_i = h.$$

This completes the proof.

Corollary 4.2. *Let the assumptions in Theorem* 4.1 *be fulfilled. If* $\xi_i^* = (x_i + x_{i+1})/2$ *then we have*

$$\left| \int_a^b f(x)dx - A_R(f, I_h, \xi^*) \right| \leq \frac{\|f'\|_1 h}{2} \tag{4.3}$$

where $A_R(f, I_h, \xi^*)$ *is the mid-point quadrature rule defined in* (4.2).

Remark 4.3 It is well known that the classical error estimate (based on the Taylor's expansion) for the mid-point quadrature rule contains $\|f''\|_\infty$, though it is second order accuarate in h_i. In the case that f'' does not exist or very large at some points in $[a, b]$, the classical estimate can not be applied, and so (4.3) provides an alternative first order error estimate for the mid-point quadrature rule.

Remark 4.4. In [1] we obtain a first order estimate similar to (4.3) for the mid-point quadrature rule which contains $\|f'\|_\infty$. The estimate (4.3) can be regarded as an improvement of that in [1] because it contains only $\|f'\|_1$. In the case that $\|f'\|_\infty$ does not exist, (4.3) gives a realistic error bound in the L_1-norm for the mid-point quadrature rule.

References

[1] S. S. Dragomir and S. Wang, "Applications of Ostrowski's inequality to the estimation of error bounds for some special means and for some numerical quadrature rules," submitted.

[2] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Inequalities for Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht-Boston-London 1994.

Department of Applied Mathematics, The University of Transkei, South Africa.

School of Mathematics and Statistics, Curtin University of Technology, Perth 6854, Australia.