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A New Integrable Equation with Cuspons and W/M-Shape-Peaks Solitons

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A new integrable equation with cuspons and W/M-shape-peaks solitons

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In this paper, we propose a new completely integrable wave equation: $m_t+m_x(u^2-u_x^2)+2m^2u_x=0$, $m=u-u_{xx}$. The equation is derived from the two dimensional Euler equation and is proven to have Lax pair and bi-Hamiltonian structures. This equation possesses new cusp solitons—cuspons, instead of regular peakons $ce^{-|x-ct|}$ with speed c. Through investigating the equation, we develop a new kind of soliton solutions—"W/M"-shape-peaks solitons. There exist no smooth solitons for this integrable water wave equation. © *2006 American Institute of Physics.*
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I. INTRODUCTION

Solitons and integrable models are very attractive objects in nonlinear sciences. Originally found in experiments on shallow water wave propagations one and a half centuries ago, they have since then become an abundant and fascinating area of theoretical and mathematical study. Today, solitons and integrable systems are playing an increasingly important role in dynamical systems, harmonic analysis, and string theories.

There are well-known constructions of integrable systems. The idea of compatibility is the crucial of integrable systems theory. One is already at the very definition of the complete integrability of a Hamiltonian flow in the Liouville-Arnold sense, which means that the flow is able to be included into a complete family of commuting Hamiltonian flows.[2] Analogically, it is a symbolic feature of soliton (integrable) partial differential equations that they take on not separately but are always assigned in hierarchies of compatible systems. It is impossible to list all applications or adoptions of this idea. We mention only some that are relevant for our present purpose. A condition of the existence of a number of commuting systems may be taken as the basis of the bi-Hamiltonian structure and symmetry approach.[1,7–9,11] However, a key procedure is to figure out bi-Hamiltonian operators. In general, no universal method is available, and we have to work on concrete equations.

In the present paper, we use Hamiltonian methods to present a new completely integrable water wave equation:

$$u_t - u_{xxt} + 3u^2u_x - u_x^3 = (4u - 2u_{xx})u_xu_{xx} + (u^2 - u_x^2)u_{xxx}, \tag{1}$$

namely,

$$m_t + m_x(u^2 - u_x^2) + 2m^2u_x = 0, \quad m = u - u_{xx}, \tag{2}$$

where u is the fluid velocity and subscripts denote the partial derivatives. Actually, this equation can be reduced from the two-dimensional Euler equation by using the approximation procedure. In two dimensional Euler equations, $\mathbf{v}_t+\mathbf{v}\cdot\nabla\mathbf{v}=-\nabla p$, div $\mathbf{v}=0$ (p is a pressure), we take the velocity $\mathbf{v}=(-\psi_y,\psi_x)^T$, where ψ is a streamfunction. Then, the following equation:

$$r_t + \psi_x r_y - \psi_y r_x = 0, \quad r := \psi_{xx} + \psi_{yy},$$

is generated from the Euler equations, where r stands for the vorticity. Employing $\psi = \phi(\xi, y, \tau)$, $\xi = \epsilon(x - c_0 t)$, $\tau = \epsilon^3 t$, imposing $\phi(\xi, y, \tau) = \epsilon\phi_1(\xi, y, \tau) + \epsilon^2\phi_2(\xi, y, \tau) + \epsilon^3\phi_3(\xi, y, \tau)$ and $\phi_1(\xi, y, \tau) = B_1(y)F(\xi, \tau)$, $\phi_2(\xi, y, \tau) = B_2(y)F(\xi, \tau) + B_3(y)F(\xi, \tau)^2$, and picking up the coefficient of ϵ^4 term in the approximation expansion of the equation, we will eventually arrive at

$$F_\tau - a_1 F_{\xi\xi\tau} + (3a_2 F^2 - a_3 F_\xi^2)F_\xi - (2a_4 F - 2a_5 F_{\xi\xi})F_\xi F_{\xi\xi} - ((a_4 - a_3)F^2 - a_5 F_\xi^2)F_{\xi\xi\xi} = 0, \tag{3}$$

where $a_1, \ldots, a_5$ are constants. If we take $a_1 = a_2 = a_3 = a_5 = 1$ and $a_4 = 2$, Eq. (3) exactly gives the new equation (1). So, Eq. (1) is a new nonlinear water wave equation.

In the paper, Eq. (1) is shown to have the bi-Hamiltonian structure and the Lax pair, which implies the integrability of the equation so that the initial value problem may be solved by the Inverse Scattering Transform (IST) method.[10,1] Our equation is proven to have new cusp solitons—cuspons, which are not peakons in the regular type of $ce^{-|x-ct|}$ (c is a wave speed)[3] and whose first order derivative is discontinuous at some point (see more mathematical studies about the Camassa-Holm equation in Refs. 5, 6, and 12). Furthermore, we develop a new kind of soliton solution, named "W-shape-peaks" or "M-shape-peaks" soliton, which is given in an explicit form for this water wave equation. We will take some graphs to show how these W/M-shape-peaks solitons look.

II. HAMILTONIAN STRUCTURE AND INTEGRABILITY

By using $m = u - u_{xx}$, $(1 - \partial^2)(\delta H_1 / \delta m) = \delta H_1 / \delta u$, the wave equation (2) can be rewritten as

$$m_t = -(m(u^2 - u_x^2))_x = J\frac{\delta H_0}{\delta m} = K\frac{\delta H_1}{\delta m}, \tag{4}$$

where

$$J = -\partial m \partial^{-1} m \, \partial, \tag{5}$$

$$K = \partial^3 - \partial, \tag{6}$$

$$H_0 = 2\int_\Omega mu \, dx,$$

$$H_1 = \frac{1}{4}\int_\Omega (u^4 + 2u^2 u_x^2) dx,$$

$\Omega = (x_0, x_0 + T)$ or $\Omega = (-\infty, +\infty)$ is the domain of u that needs to be periodic with T or to approach a constant, and H_0, H_1 are two Hamiltonian functions. Apparently, both operator J and operator K are Hamiltonian. So, our equation is bi-Hamiltonian.

In order to show the integrability of this equation, let us consider the following spectral problem:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\lambda m \\ -\frac{1}{2}\lambda m & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \equiv U(m, \lambda)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{7}$$

where λ is a spectral parameter, m is a scalar potential function periodic or approaching the same constant at both infinities, and $\psi = (\psi_1, \psi_2)^T$ is the spectral function corresponding to the spectral parameter λ. Then, we have

$$K\,\nabla\lambda = \lambda^2 J\,\nabla\lambda, \tag{8}$$

where $\nabla\lambda=(\lambda/2)(\psi_1^2+\psi_2^2)$.

Remark 1: Equation (8) plays a very important role in the discussions of the periodic solutions of the wave equation (2), which we will deal with in a subsequent paper.[14] *Actually, on the basis of these two operators, following our earlier method*[13] *we are able to generate a new integrable hierarchy.*

By a careful calculation, we can verify the following statement.

Equation (2) has the following Lax pair:

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x = U(m,\lambda)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{9}$$

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_t = V(m,u,\lambda)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{10}$$

where

$$U(m,\lambda)=\begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\lambda m \\ -\frac{1}{2}\lambda m & \frac{1}{2}\end{pmatrix}, \quad m=u-u_{xx},$$

$$V(m,u,\lambda)=\begin{pmatrix} \lambda^{-2}+\frac{1}{2}(u^2-u_x^2) & -\lambda^{-1}(u-u_x)-\frac{1}{2}\lambda m(u^2-u_x^2) \\ \lambda^{-1}(u+u_x)+\frac{1}{2}\lambda m(u^2-u_x^2) & -\lambda^{-2}-\frac{1}{2}(u^2-u_x^2)\end{pmatrix}.$$

In fact, one just checks that the compatibility condition

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_{xt} = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_{tx},$$

namely,

$$U_t - V_x + [U,V] = 0$$

generates Eq. (2). A direct substitution of $U(m,\lambda)$ and $V(m,u,\lambda)$ is able to guarantee the truth of the above statement. One can also use maple to verify this procedure.

So the wave equation (2) is accordingly completely integrable by the Inverse Scattering Transformation.[1]

III. W/M-SHAPE-PEAKS SOLITONS

A. Traveling wave setting

Let us consider the traveling wave solution of our equation (2) through a generic setting $u(x,t)=U(x-ct)$, where c is the wave speed. Let $\xi=x-ct$, then $u(x,t)=U(\xi)$. Substituting it into Eq. (2) yields

$$(U^2-U'^2-c)(U-U'')'+2U'(U-U'')^2=0, \tag{11}$$

where $U'=U_\xi$, $U''=U_{\xi\xi}$, $U'''=U_{\xi\xi\xi}$.

Remark 2: (1) Our ODE (11), derived from our new PDE (2) through a traveling wave setting, *is different from the Camassa-Holm (CH) case. In the CH case (see Ref. 15), the traveling wave ODE reads as*

$$(U-c)(U-U'')+2U'(U-U'')=0.$$

Here our new equation is more high order nonlinear terms than the CH case.

(2) The integrable CH equation has both peakon and smooth soliton solutions.[3,15] *However, our new integrable equation (2) has no smooth soliton solution, only cusp solitons and M/W-shape-peaks solitons solutions (see the following sections). This means that an integrable equation may have no smooth soliton, which may clue on a good idea to classify the integrable equations by the mathematical features of solitons.*

Generally, we have the following trivial facts.

(1) *Any constant function is a solution of Eq. (2) and the ODE (11).*
(2) *Any translation $U(\xi-\xi_0)$ of a solution $U(\xi)$ of ODE (11) is still a solution.*
(3) *If $u(x,t)$ is a solution of Eq. (2), then any translation $u(x-x_0,t-t_0)$ in space x and time t is a solution, too.*

Because of the translation invariance of the differential equation (11), without any loss of generality, we choose ξ_0 as vanishing, namely, $\xi_0=0$. If $U-U''=0$, then Eq. (11) has general solutions $U(\xi)=c_1e^{\xi}+c_2e^{-\xi}$ with any real constants c_1, c_2. Of course, they are the solutions of Eq. (2). This result is not interesting for us. Actually, there are general weak solutions to Eq. (2):

$$u(x,t)=c\cosh(|x-ct|)\pm\sqrt{c(c-1)}\sinh(|x-ct|), \tag{12}$$

where c ($c\geq 1$ or $c\leq 0$) is the wave speed.

Because the solution $u(x,t)$ defined by (12) approaches infinity as x goes to infinity, this solution is not a soliton, which is not so interesting, either. Let us now assume that U is NEITHER a constant solution of Eq. (11) NOR satisfies $U^2-U'^2=0$. Taking the integration on both sides of Eq. (11) leads to

$$(U-U'')[c-(U^2-U'^2)]=C_2, \tag{13}$$

where $C_2\neq 0\in\mathbb{R}$ is an integration constant.

Multiplying both sides of Eq. (13) by $-4U'$ and then taking another integration, we obtain

$$[c-(U^2-U'^2)]^2=-4C_2U+C_1, \tag{14}$$

namely,

$$U'^2=U^2-c\pm\sqrt{C_1-4C_2U}, \tag{15}$$

where $C_1\in\mathbb{R}$ is another integration constant.

Let us now impose the boundary condition

$$\lim_{\xi\to\pm\infty}U=A, \tag{16}$$

to figure out the two constants C_1, C_2. Substituting the boundary condition (16) into the ODEs (13) and (14) generates the following two constants:

$$C_1=(c-A^2)(c+3A^2), \tag{17}$$

$$C_2=A(c-A^2). \tag{18}$$

So the ODE (14) becomes

$$[c-(U^2-U'^2)]^2=(c-A^2)(c+3A^2-4AU). \tag{19}$$

Remark 3: There is no shock wave (or kink) solution for our equation (2) because we cannot *be allowed to set* $\lim_{\xi\to+\infty} U \neq \lim_{\xi\to-\infty} U$. *In other words, the traveling wave solution of our equation, satisfying the boundary condition* (16), *is definitely a soliton!*

Let us solve Eq. (19) in the coming subsection.

B. W/M-shape-peaks solitons

Let us assume that $A\neq 0$ and $c\neq A^2$ (if $A=0$ or $c=A^2$, then $C_2=0$, which corresponds to $U-U''=0$. We already discussed it in Sec. III A. Actually, no solitons are found in this case). The fact that both sides of Eq. (19) are non-negative implies

$$(c-A^2)(c+3A^2-4AU) > 0. \tag{20}$$

Let $V=\sqrt{(c-A^2)(c+3A^2-4AU)}$ and $s=c-A^2$, then $V>0, V=\sqrt{s(s+4A^2-4AU)}$, $U=A+(1/4sA)(s^2-V^2)$ and Eq. (19) becomes

$$\frac{VdV}{|s-\epsilon V|\sqrt{(s+\epsilon V)^2-8sA^2}} = \mp\frac{d\xi}{2}, \quad \epsilon=\pm 1. \tag{21}$$

Let us discuss the cases of $\epsilon=1$ and $\epsilon=-1$ separately.

(1) *Case* $\epsilon=1$. If we chose $s<0$, then Eq. (21) can be integrated as

$$(V+s+\sqrt{(V+s)^2-8sA^2})\left[\frac{-V-s+4A^2+a\sqrt{(V+s)^2-8sA^2}}{V-s}\right]^{1/2a} = e^{-|\xi|/2-\ln(-4s)/2a}, \tag{22}$$

where $a=\sqrt{(s-2A^2)/s}=\sqrt{(3A^2-c)/(A^2-c)}$, $s-2A^2<0$, and $a>1$.

Because $V \nrightarrow \pm s$ as $\xi\to\infty$ whereas the right hand side of Eq. (22) goes to 0 as $\xi\to\infty$, the solution V determined by Eq. (22) does not exist. So, $s<0$ is not available for case $\epsilon=1$.

Therefore, s must be positive, namely, $s>0$, which assures that Eq. (21) can be integrated as

$$(V+s+\sqrt{(V+s)^2-8sA^2})\left[\frac{V-s}{V+s-4A^2+a\sqrt{(V-s)^2-8sA^2}}\right]^{1/2a} = e^{|\xi|/2+\ln(4s)/2a}, \tag{23}$$

where $a=\sqrt{(s-2A^2)/s}=\sqrt{(c-3A^2)/(c-A^2)}$, $s-2A^2>0$, and $0<a<1$.

In general, we cannot solve the equation (23) for V in an explicit form. But, for some very special a's, we do have the explicit solution V. For example, let us take $a=\frac{1}{2}$; then $c=\frac{11}{3}A^2$ and $s=\frac{8}{3}A^2$. Substituting it into Eq. (23) generates

$$V^2-bV+sb=0, \tag{24}$$

where

$$b = 3se^{-\frac{|\xi|}{2}} + \frac{3s}{4}e^{\frac{|\xi|}{2}} + s = 3s\cosh\left(\frac{|\xi|}{2}-\ln(2)\right)+s = s\left(3\cosh\left(\frac{|\xi|}{2}-\ln(2)\right)+1\right).$$

Therefore, solving Eq. (24) for V gives

$$V=\frac{1}{2}(b-\sqrt{b^2-4sb}) = \frac{3s}{2}\left(\frac{2}{3}+z-\sqrt{z^2-\frac{4}{9}}\right) = s+\frac{3s}{2}\left(z-\sqrt{z^2-\frac{4}{9}}\right),$$

where $z=\cosh(|\xi|/2-\ln(2))-\frac{1}{3}$. So, we arrive at explicit solutions $U(\xi)$ of Eq. (2),

$$U(\xi)=A\left(\frac{5}{3}-(3z+2)\left(z-\sqrt{z^2-\frac{4}{9}}\right)\right),$$

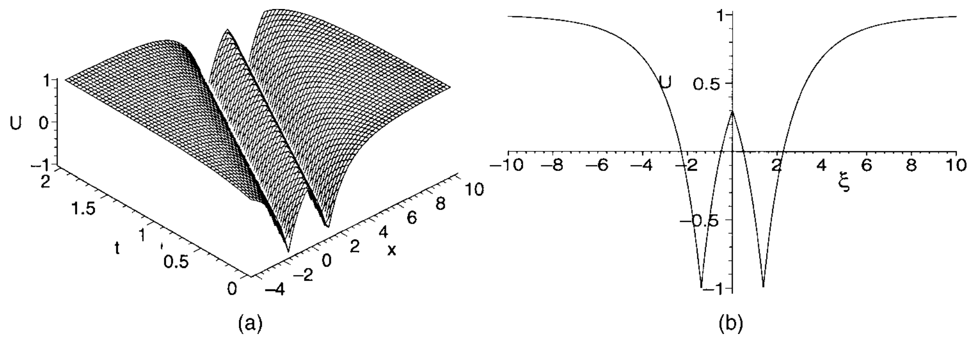

FIG. 1. (a) A three-dimensional (3-D) graph of the explicit solution $U(\xi)$ defined by (25), when $A=1$, $s=8/3$, wave speed $c=11/3$, and the range of x,t,u: $-4\leq x\leq 10$, $0\leq x\leq 2$, $-1\leq u\leq 1$. (b) A 2D graph of the explicit solution $U(\xi)$ defined by (25) when $A=1$, $s=8/3$, the wave speed $c=11/3$, and the range of ξ: $-10\leq\xi\leq 10$.

$$z=\cosh\left(\frac{|\xi|}{2}-\ln(2)\right)-\frac{1}{3},$$

$$\xi=x-\frac{11}{3}A^2t. \tag{25}$$

Apparently, $V(\xi)\to s$ and $U(\xi)\to A$ as $\xi\to\infty$. Since $A\neq 0$, there is no peaked soliton for a homogeneous boundary condition. Let us select a special $A=1$, then the solution $U(\xi)$ reads as

$$U(X)=2-3\cosh^2 X+\left(\cosh X+\frac{1}{3}\right)\sqrt{3(3\cosh X+1)(\cosh X-1)}.$$

$$X=\frac{\left|x-\frac{11}{3}t\right|}{2}-\ln 2.$$

Apparently, this solution has three peaks and looks like a "W" wave. So, we called it a "W-shape peaks" soliton. Three peaks occur at $x=\frac{11}{3}t_0$, $x=\frac{11}{3}t_0-2\ln(2)$, $x=\frac{11}{3}t_0+2\ln(2)$, for each time t_0. See Fig. 1 for more details.

If we select the boundary constant $A=-1$, we are able to get the anti-"W-shape-peaks" soliton, called an "M-shape-peaks" soliton. See a 3D and a 2D graph in Fig. 2 for more details.

For other a's, in a similar way we can also obtain the corresponding peaked soliton solutions, which are left for the readers' practice.

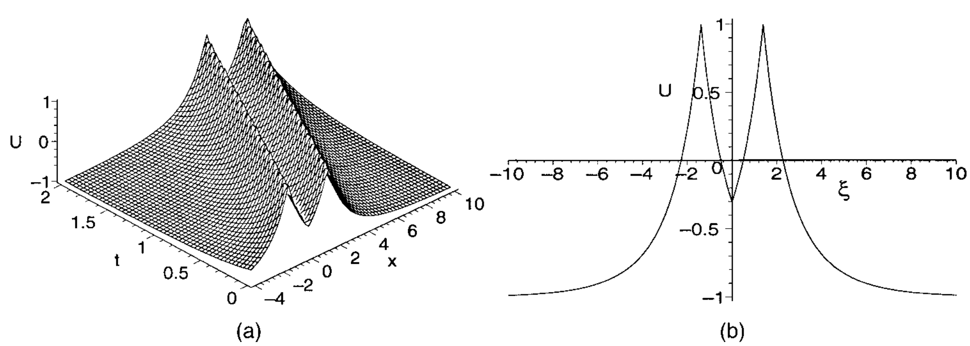

FIG. 2. (a) A 3D graph of the "M-shape-peaks" solution $U(\xi)$ defined by (25) with $A=-1$ and $c=11/3$. (b) 2D graph of the "M-shape-peaks" solution $U(\xi)$ defined by (25) with $A=-1$ and $c=11/3$.

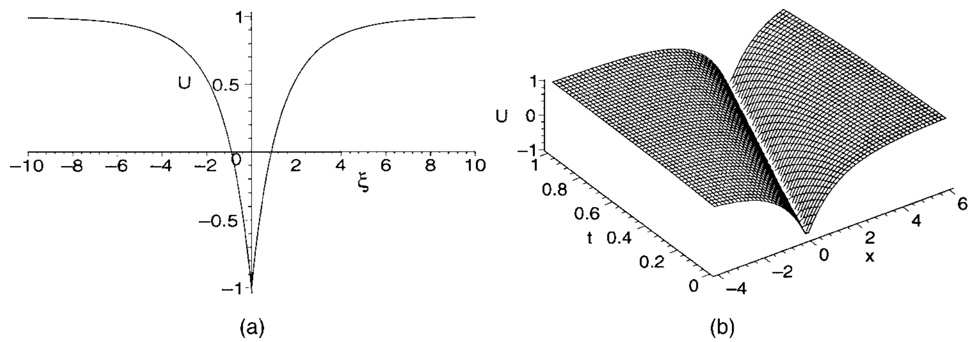

FIG. 3. (a) 2D graph of new cuspon solution $U(\xi)$ defined by (26) with amplitude $A=1$ and wave speed $c=11/3$. (b) 3D graph of new cuspon solution $U(\xi)$ defined by (26) with $A=1$ and $c=11/3$.

New cusp solitons-cuspons. In Eq. (22), we consider the solution V without the absolute value of ξ. Following the above procedure, in case of $a=\frac{1}{2}$ we may obtain

$$U(\xi)=A\left(\frac{5}{3}-(3z+2)\left(z-\sqrt{z^2-\frac{4}{9}}\right)\right),$$

$$z=\cosh\left(\frac{x}{2}-\frac{11}{6}A^2t\right)-\frac{1}{3}. \tag{26}$$

A direct verification shows us the following: (26) is indeed another explicit solutions of Eq. (2). Let us take $A=\pm 1$. Then the corresponding solutions read as

$$U(X)=\pm\left(2-3\cosh^2 X+\left(\cosh X+\frac{1}{3}\right)\sqrt{3(3\cosh X+1)(\cosh X-1)}\right),$$

$$X=\frac{x}{2}-\frac{11}{6}t,$$

which have the following characteristic features:

$$U(0)=\mp 1,\quad U'(0+)=\pm\infty,\quad U'(0-)=\mp\infty.$$

Apparently, they differ from the regular peakons.[3] So, both are new peaked solitons for our equation (2) (see Figs. 3, 4 for more details).

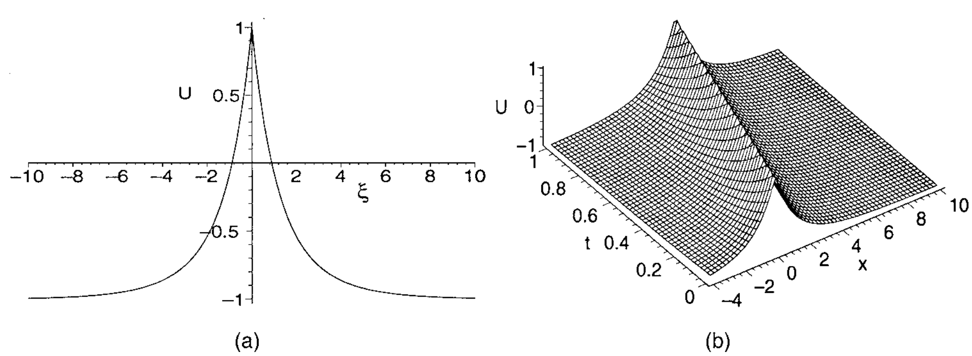

FIG. 4. (a) A 2D graph of new cuspon solution $U(\xi)$ defined by (25) when $A=-1$ and $c=11/3$. (b) A 3D graph of a new cuspon solution $U(\xi)$ defined by (25) when $A=-1$ and $c=11/3$.

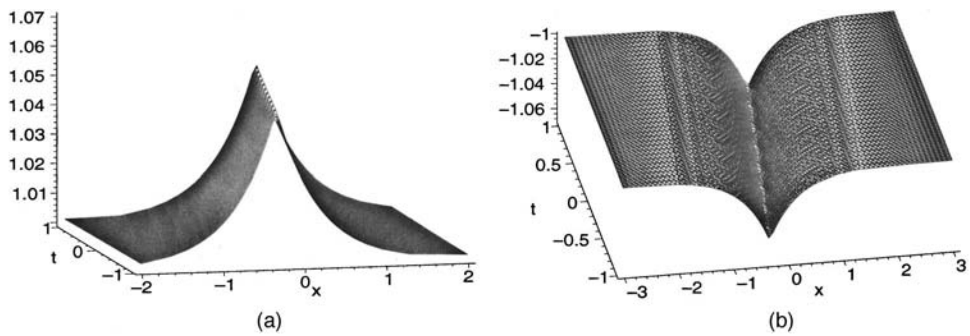

FIG. 5. (a) A 3D numerical graph of the cuspon solution $U(\xi)$ of Eq. (2) based on (28) with amplitude $A=1$ and wave speed $c=1/3$. (b) A 3D numerical graph of the cuspon solution $U(\xi)$ of Eq. (2) based on (28) with amplitude $A=-1$ and wave speed $c=1/3$.

(2) *Case* $\epsilon=-1$. If we chose $s>0$, then Eq. (21) can be integrated as

$$(V-s+\sqrt{(V-s)^2-8sA^2})\left[\frac{s-V-4A^2+a\sqrt{(V-s)^2-8sA^2}}{V+s}\right]^{1/2a}=e^{-|\xi|/2-\ln(4s)/2a}, \tag{27}$$

where $a=\sqrt{(s-2A^2)/s}=\sqrt{(c-3A^2)/(c-A^2)}$, $s-2A^2>0$, and $0<a<1$.

Because $V\nrightarrow\pm s$ as $\xi\to\infty$ whereas the right hand side of Eq. (27) goes to 0 as $\xi\to\infty$, the solution V determined by Eq. (27) does not exist. So, $s>0$ is not available for case $\epsilon=-1$. Therefore, s must be negative, namely, $s<0$, which gurantees that Eq. (21) can be integrated as

$$(V-s+\sqrt{(V-s)^2-8sA^2})\left[\frac{V+s}{V-s+4A^2+a\sqrt{(V-s)^2-8sA^2}}\right]^{1/2a}=e^{-|\xi|/2+\ln(-4s)/2a}, \tag{28}$$

where $\xi=x-ct$, $a=\sqrt{(s-2A^2)/s}=\sqrt{(3A^2-c)/(A^2-c)}$, $s-2A^2<0$, and $1<a$.

In general, we cannot solve Eq. (28) for V in an explicit form. But, we can numerically determine V from Eq. (28), and then according to the equation, $U=A+(1/4sA)(s^2-V^2)$ to figure out the solution $U(\xi)$ for our equation (2). For instance, in the case of $A=\pm 1$, $c=1/3$, we have $a=2$ and $s=-2/3$, and numerically solve our equation (2) (see Fig. 5 for details).

For other a's (for instance, $a=3,4$), in a similar way we can also numerically obtain the corresponding cuspon solutions of Eq. (2).

IV. CONCLUSIONS AND OPEN PROBLEMS

In the paper, we present a new integrable water wave equation (2). Through studying the equation, we develop a new kind of soliton solution—"W-shape-peaks"/"M-shape-peaks" solutions (see Fig. 1, 2). Our equation is shown to possess not only "W-shape-peaks"/"M-shape-peaks" solitons, but new cuspon solution as well (see Figs. 3, 4 which are different from regular peakons).

Our new equation (2) naturally has a physical meaning since it is derived from the two dimensional Euler equation (see the Introduction). It can be cast into the following Newton equation $U'^2=P(U)-P(A^2)$ of a particle with a new potential $P(U)=U^2+\mathrm{sign}(s)\sqrt{s(s+4A^2-4AU)}$, $s=c-A^2$, or $V'^2=Q(V)-Q(A)$ with $U=A+(1/4sA)(s^2-V^2)$, $Q(V)=V^2/4+4s|s|A^2/V+s^3(s-8A^2)/4V^2$. In the paper, we successfully solve this Newton equation with new cuspons and M-shape/W-shape-peaks solitons. Those peaked and cusped solutions may be applied to neuroscience for providing a mathematical model and explaining electrophysiological responses of visceral nociceptive neurons and sensitization of dorsal root reflexes.[4]

No smooth solitons are found for our equation, but our equation is completely integrable. Furthermore, we suggest a more general partial differential equation: $m_t+m_x(u^2-u_x^2)+km^2u_x=0$, $m=u-u_{xx}$ with any constant $k\in\mathbb{R}$. When $k=2$, the equation is integrable, which is already discussed in this paper. Any other integrable cases? We do not know yet.

ACKNOWLEDGMENTS

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