Progress of Theoretical Physics, Vol. 63, No. 3, March 1980

A New Integrable Nonlinear Evolution Equation

Toru SHIMIZU and Miki WADATI*

Institute of Physics, University of Tsukuba, Ibaraki 305
**Institute of Applied Physics, University of Tsukuba, Ibaraki 305*

(Received November 1, 1979)

It is shown that a new integrable nonlinear evolution equation

$$iq_t+\left(\frac{q}{\sqrt{1+|q|^2}}\right)_{xx}=0$$

is solved exactly by the inverse scattering method. From the Gelfand-Levitan equation, one soliton solution is obtained. It is found that one-soliton solution has two interesting limits, a small amplitude soliton and a bursting soliton.

§ 1. Introduction

In this paper, we shall study a newly found integrable nonlinear evolution equation[1)]

$$iq_t+(q/\sqrt{1+|q|^2})_{xx}=0\,, \tag{1·1}$$

under the boundary condition

$$q(x,t)\to 0 \ \text{as} \ |x|\to\infty\,. \tag{1·2}$$

Equation (1·1) has been obtained as an example of a generalization of the inverse scattering scheme.[2)] Compared with the already known systems such as the nonlinear Schrödinger equation[3)] and the derivative nonlinear Schrödinger equation,[4)] Eq. (1·1) is highly nonlinear and even has the saturation effect. Therefore its analysis is quite interesting mathematically and physically.

The main purpose of the present paper is the application of the inverse scattering method.[5),6)] In § 2, we shall introduce fundamental equations of the inverse scattering method and discuss the scattering problem, in particular, the asymptotic expansions of the Jost functions. Using the results, we shall derive the Gelfand-Levitan equation for our system in § 3. In § 4, we shall obtain one-soliton solution from the Gelfand-Levitan equation. We find that the Gelfand-Levitan equation does not give one-soliton solution explicitly in a usual sense, but provides us a sufficient information to describe it. The last section is devoted to discussion. There, we point out two interesting cases of one soliton solution. One is the small amplitude soliton which has the same properties as that of the nonlinear Schrödinger equation, and the other is a bursting soliton whose amplitude diverges under some condition.

§ 2. Scattering problem

We consider the following eigenvalue problem:

$$\psi_{1x}+i\lambda\psi_1=\lambda q\psi_2\,, \tag{2·1a}$$

$$\psi_{2x}-i\lambda\psi_2=-\lambda q^*\psi_1\,. \tag{2·1b}$$

The time dependence of the eigenfunctions is chosen to be

$$\psi_{1t}=A\psi_1+B\psi_2\,, \tag{2·2a}$$

$$\psi_{2t}=C\psi_1-A\psi_2\,, \tag{2·2b}$$

where

$$A=-(2i/\Phi)\lambda^2, \tag{2·3a}$$

$$B=(2q/\Phi)\lambda^2+i(q/\Phi)_x\lambda\,, \tag{2·3b}$$

$$C=-(2q^*/\Phi)\lambda^2+i(q^*/\Phi)_x\lambda\,, \tag{2·3c}$$

$$\Phi\equiv\sqrt{1+|q|^2}. \tag{2·3d}$$

Assuming $\partial\lambda/\partial t=0$, we find that the integrability condition for Eqs. (2·1) and (2·2) yields Eq. (1·1).

We introduce the Jost functions by

$$\left.\begin{aligned}\phi&\to\begin{pmatrix}1\\0\end{pmatrix}\exp(-i\lambda x)\\ \bar{\phi}&\to\begin{pmatrix}0\\-1\end{pmatrix}\exp(i\lambda x)\end{aligned}\right\}\quad \text{as}\quad x\to-\infty\,, \tag{2·4}$$

$$\left.\begin{aligned}\psi&\to\begin{pmatrix}0\\1\end{pmatrix}\exp(i\lambda x)\\ \bar{\psi}&\to\begin{pmatrix}1\\0\end{pmatrix}\exp(-i\lambda x)\end{aligned}\right\}\quad \text{as}\quad x\to\infty\,, \tag{2·5}$$

and the scattering coefficients by

$$\phi=a\bar{\psi}+b\psi\,, \tag{2·6a}$$

$$\bar{\phi}=-\bar{a}\psi+\bar{b}\bar{\psi}\,, \tag{2·6b}$$

where

$$a\bar{a}+b\bar{b}=1\,. \tag{2·7}$$

We note that

$$\begin{pmatrix}\bar{\phi}_1(\lambda)\\ \bar{\phi}_2(\lambda)\end{pmatrix}=\begin{pmatrix}\phi_2^*(\lambda^*)\\ -\phi_1^*(\lambda^*)\end{pmatrix}, \tag{2·8a}$$

$$\begin{pmatrix} \bar{\psi}_1(\lambda) \\ \bar{\psi}_2(\lambda) \end{pmatrix} = \begin{pmatrix} \phi_2^*(\lambda^*) \\ -\phi_1^*(\lambda^*) \end{pmatrix}, \tag{2·8b}$$

from whence it follows that

$$\bar{a}(\lambda) = a^*(\lambda^*), \quad \bar{b}(\lambda) = b^*(\lambda^*). \tag{2·9}$$

We investigate the analytic properties of $a(\lambda)$ and the Jost functions for large $|\lambda|$. From Eqs. (2·6), (2·4) and (2·5), we have

$$\log a = \int_{-\infty}^{\infty} \sigma dx, \tag{2·10}$$

where

$$\sigma = \frac{\partial}{\partial x} \log\{\phi_1 \exp(i\lambda x)\}. \tag{2·11}$$

Define

$$g_1 = \phi_1 \exp(i\lambda x), \tag{2·12a}$$

$$g_2 = \phi_2 \exp(i\lambda x). \tag{2·12b}$$

Then, Eqs. (2·1) become

$$g_{1x} = \lambda q g_2, \tag{2·13a}$$

$$g_{2x} - 2i\lambda g_2 = -\lambda q^* g_1. \tag{2·13b}$$

Substitution of Eq. (2·11) with Eqs. (2·12) into Eqs. (2·13) yields

$$\sigma_x = (q_x/q)\sigma + 2i\lambda\sigma - \lambda^2|q|^2 - \sigma^2. \tag{2·14}$$

We expand σ in the power series of λ:

$$\sigma = \sum_{n=-1}^{\infty} \sigma_n/(i\lambda)^n. \tag{2·15}$$

Inserting this into Eq. (2·14) and equating the terms of the same powers of λ, we obtain the conserved densities $\sigma_n (n=-1, 0, 1, \cdots)$. The first two conserved densities which vanishe for $q=0$ are

$$\sigma_{-1} = 1 - \Phi \equiv 1 - \sqrt{1+|q|^2}, \tag{2·16a}$$

$$\sigma_0 = -\frac{q_x}{2q}\left(\frac{1}{\Phi} - 1\right) - \frac{1}{2}\frac{\partial}{\partial x} \log \Phi. \tag{2·16b}$$

From Eqs. (2·10) and (2·15), we see that

$$\log a = i\lambda\varepsilon + \mu + O\left(\frac{1}{\lambda}\right), \tag{2·17}$$

where

$$\varepsilon = \int_{-\infty}^{\infty} \sigma_{-1} dx\,, \tag{2·18}$$

$$\mu = \int_{-\infty}^{\infty} \sigma_0 dx\,. \tag{2·19}$$

Using Eqs. (2·11) ~ (2·13), we have

$$\log[\phi_1 \exp(i\lambda x)] = i\lambda \int_{-\infty}^{x} \sigma_{-1} dx + \int_{-\infty}^{x} \sigma_0 dx + O\left(\frac{1}{\lambda}\right), \tag{2·20a}$$

$$\log[\phi_2 \exp(i\lambda x)] = \log\left[i(1-\Phi)/q + O\left(\frac{1}{\lambda}\right)\right]$$

$$+\left[i\lambda \int_{-\infty}^{x} \sigma_{-1} dx + \int_{-\infty}^{x} \sigma_0 dx + O\left(\frac{1}{\lambda}\right)\right]. \tag{2·20b}$$

Similar analysis is possible for other Jost functions. Summing up the results, as $|\lambda| \to \infty$, we have

$$a \exp(-i\lambda\varepsilon) = \exp(\mu) + O\left(\frac{1}{\lambda}\right), \tag{2·21a}$$

$$\phi \exp[i\lambda(x-\varepsilon_-)] = \begin{pmatrix} 1 \\ i(1-\Phi)/q \end{pmatrix} \exp(\mu_-) + O\left(\frac{1}{\lambda}\right), \tag{2·21b}$$

$$\psi \exp[-i\lambda(x+\varepsilon_+)] = \begin{pmatrix} i(1-\Phi)/q^* \\ 1 \end{pmatrix} \exp(-\mu_+^*) + O\left(\frac{1}{\lambda}\right), \tag{2·21c}$$

$$\bar{\psi} \exp[i\lambda(x+\varepsilon_+)] = \begin{pmatrix} 1 \\ i(1-\Phi)/q \end{pmatrix} \exp(-\mu_+) + O\left(\frac{1}{\lambda}\right), \tag{2·21d}$$

$$\bar{\phi} \exp[-i\lambda(x-\varepsilon_-)] = \begin{pmatrix} -i(1-\Phi)/q^* \\ -1 \end{pmatrix} \exp(\mu_-^*) + O\left(\frac{1}{\lambda}\right), \tag{2·21e}$$

where

$$\varepsilon_-(x) = \int_{-\infty}^{x} \sigma_{-1} dx\,, \tag{2·22a}$$

$$\varepsilon_+(x) = \int_{x}^{\infty} \sigma_{-1} dx\,, \tag{2·22b}$$

$$\mu_-(x) = \int_{-\infty}^{x} \sigma_0 dx\,, \tag{2·23a}$$

$$\mu_+(x) = \int_{x}^{\infty} \sigma_0 dx\,. \tag{2·23b}$$

§ 3. Gelfand-Levitan equation

In this section we shall derive the Gelfand-Levitan equation for a system

(2·1).

We assume that q is on compact support. Then, $a(\lambda)\exp(-i\lambda\varepsilon)$, $\phi\exp[i\lambda \times(x-\varepsilon_-)]$, $\psi\exp[-i\lambda(x+\varepsilon_+)]$, $\bar{\psi}\exp[i\lambda(x+\varepsilon_+)]$ and $\bar{\phi}\exp[-i\lambda(x-\varepsilon_-)]$ are entire functions of λ.

From Eq. (2·6a), we have

$$\frac{1}{a}\begin{pmatrix}\phi_1\\ \phi_2/\lambda\end{pmatrix}=\begin{pmatrix}\bar{\psi}_1\\ \bar{\psi}_2/\lambda\end{pmatrix}+\frac{b}{a}\begin{pmatrix}\psi_1\\ \psi_2/\lambda\end{pmatrix}. \tag{3·1}$$

We define an integral path C to be the contour in the complex λ plane, starting from $\lambda=-\infty+i0^+$, passing over all zeros of $a(\lambda)$, and ending at $\lambda=+\infty+i0^+$. Similarly, $\bar{C}$ is the contour starting from $\lambda=-\infty+i0^-$, passing under all zeros of $\bar{a}(\lambda)$, and ending at $\lambda=+\infty+i0^-$.

Consider the integral:

$$\begin{aligned}\int_c\frac{d\lambda'}{\lambda'-\lambda}&\frac{1}{a(\lambda')\exp(-i\lambda'\varepsilon)}\begin{pmatrix}\phi_1(\lambda')\\ \phi_2(\lambda')/\lambda'\end{pmatrix}\exp[i\lambda'(x-\varepsilon_-)]\\ &=\int_c\frac{d\lambda'}{\lambda'-\lambda}\begin{pmatrix}\bar{\psi}_1(\lambda')\\ \bar{\psi}_2(\lambda')/\lambda'\end{pmatrix}\exp[i\lambda'(x+\varepsilon_+)]\\ &\quad+\int_c\frac{d\lambda'}{\lambda'-\lambda}\frac{b(\lambda')}{a(\lambda')}\begin{pmatrix}\psi_1(\lambda')\\ \psi_2(\lambda')/\lambda'\end{pmatrix}\exp[i\lambda'(x+\varepsilon_+)].\end{aligned} \tag{3·2}$$

As the contour C becomes far away, then from Eqs. (2·21a) and (2·21b), we have

$$\text{L.H.S. of Eq. (3·2)}=-i\pi\begin{pmatrix}1\\0\end{pmatrix}\exp(-\mu_+).$$

From Eq. (2·21d), similarly, we have

$$\begin{aligned}&\text{R.H.S. of Eq. (3·2)}\\ &=-2i\pi\begin{pmatrix}\bar{\psi}_1(\lambda)\\ \bar{\psi}_2(\lambda)/\lambda\end{pmatrix}\exp[i\lambda(x+\varepsilon_+)]\\ &\quad+\int_{\bar{c}}\frac{d\lambda'}{\lambda'-\lambda}\begin{pmatrix}\bar{\psi}_1(\lambda')\\ \bar{\psi}_2(\lambda')/\lambda'\end{pmatrix}\exp[i\lambda'(x+\varepsilon_+)]\\ &\quad+\int_c\frac{d\lambda'}{\lambda'-\lambda}\frac{b(\lambda')}{a(\lambda')}\begin{pmatrix}\psi_1(\lambda')\\ \psi_2(\lambda')/\lambda'\end{pmatrix}\exp[i\lambda'(x+\varepsilon_+)]\\ &=-2i\pi\begin{pmatrix}\bar{\psi}_1(\lambda)\\ \bar{\psi}_2(\lambda)/\lambda\end{pmatrix}\exp[i\lambda(x+\varepsilon_+)]+i\pi\begin{pmatrix}1\\0\end{pmatrix}\exp(-\mu_+)\\ &\quad+\int_c\frac{d\lambda'}{\lambda'-\lambda}\frac{b(\lambda')}{a(\lambda')}\begin{pmatrix}\psi_1(\lambda')\\ \psi_2(\lambda')/\lambda'\end{pmatrix}\exp[i\lambda'(x+\varepsilon_+)].\end{aligned}$$

Therefore, we obtain

$$\begin{pmatrix} \bar{\psi}_1(\lambda) \\ \bar{\psi}_2(\lambda)/\lambda \end{pmatrix} \exp[i\lambda(x+\varepsilon_+)] = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp(-\mu_+)$$

$$+\frac{1}{2\pi i}\int_c \frac{d\lambda'}{\lambda'-\lambda}\frac{b(\lambda')}{a(\lambda')}\begin{pmatrix} \psi_1(\lambda') \\ \psi_2(\lambda')/\lambda' \end{pmatrix}\exp[i\lambda'(x+\varepsilon_+)]. \tag{3·3}$$

We introduce a kernel K by

$$\begin{pmatrix} \psi_1/\lambda \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp[i\lambda(x+\varepsilon_+)-\mu_+^*]$$

$$+\exp(i\lambda\varepsilon_+)\int_x^\infty \begin{pmatrix} K_1(x,s)\exp(-\mu_+) \\ K_2(x,s)\exp(-\mu_+^*) \end{pmatrix}\exp(i\lambda s)\,ds\,. \tag{3·4}$$

The kernel K is assumed to satisfy

$$\lim_{s\to\infty} K_1(x,s)=0\,,$$
$$\lim_{s\to\infty} K_2(x,s)=0\,. \tag{3·5}$$

The relation between $K_1(x,x)$ and $q(x)$ is obtained as follows. By a partial integration, we can show from Eq. (3·4) that

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \exp[-i\lambda(x+\varepsilon_+)] = \begin{pmatrix} iK_1(x,x)\exp(-\mu_+) \\ \exp(-\mu_+^*) \end{pmatrix} + O\left(\frac{1}{\lambda}\right). \tag{3·6}$$

Comparing Eq. (3·6) with Eq. (2·21c), we have

$$K_1(x,x)=\frac{1-\sqrt{1+|q|^2}}{q^*}\exp(\mu_+-\mu_+^*)\,. \tag{3·7}$$

Now we are in a position to derive the Gelfand-Levitan equation. From Eqs. (2·8b) and (3·4), we have

$$\begin{pmatrix} \bar{\psi}_1(\lambda) \\ \bar{\psi}_2(\lambda)/\lambda \end{pmatrix} \exp[i\lambda(x+\varepsilon_+)]$$

$$=\begin{pmatrix} \psi_2^*(\lambda^*) \\ -\psi_1^*(\lambda^*)/\lambda \end{pmatrix} \exp[i\lambda(x+\varepsilon_+)]$$

$$=\begin{pmatrix} 1 \\ 0 \end{pmatrix}\exp(-\mu_+)+\exp(i\lambda x)\int_x^\infty \begin{pmatrix} K_2^*(x,s)\exp(-\mu_+) \\ -K_1^*(x,s)\exp(-\mu_+^*) \end{pmatrix}$$

$$\times\exp(-i\lambda s)\,ds\,. \tag{3·8}$$

Substitution of Eqs. (3·4) and (3·8) into Eq. (3·3) gives

$$\int_x^\infty \begin{pmatrix} K_2^*(x,s) \\ -K_1^*(x,s) \end{pmatrix}\exp(-i\lambda s)\,ds$$

$$
-\frac{\exp(-i\lambda x)}{2\pi i}\cdot\int_c \frac{d\lambda'}{\lambda'-\lambda}\frac{b}{a}\begin{pmatrix}0\\1/\lambda'\end{pmatrix}\exp[2i\lambda'(x+\varepsilon_+)]
$$

$$
-\frac{\exp(-i\lambda x)}{2\pi i}\cdot\int_c \frac{d\lambda'}{\lambda'-\lambda}\frac{b}{a}\int_x^\infty\begin{pmatrix}\lambda' K_1(x,s)\\K_2(x,s)/\lambda'\end{pmatrix}
$$

$$
\times\exp[i\lambda'(s+x+2\varepsilon_+)]ds=0\,. \tag{3·9}
$$

Taking a Fourier transform of Eq. (3·9), we arrive at the Gelfand-Levitan equation:

$$
K_1(x,y)-F^*(x+y)-\int_x^\infty K_2^*(x,s)F^*(s+y)ds=0\,, \tag{3·10a}
$$

$$
K_2^*(x,y)-\int_x^\infty K_1(x,s)F''(s+y)ds=0 \tag{3·10b}
$$

for $x\leq y$. Here $F(z)$ and $F''(z)$ are defined by

$$
F(z)=\frac{1}{2\pi}\int_c\frac{b(\lambda)}{a(\lambda)}\frac{1}{\lambda}\exp[i\lambda(z+2\varepsilon_+(x))]d\lambda\,, \tag{3·11a}
$$

$$
F''(z)=\frac{\partial^2 F}{\partial z^2}=-\frac{1}{2\pi}\int_c\frac{b(\lambda)}{a(\lambda)}\lambda\exp[i\lambda(z+2\varepsilon_+(x))]d\lambda\,. \tag{3·11b}
$$

The time-dependence of the scattering coefficients are determined from Eqs. (2·2). The result is

$$
a(\lambda,t)=a(\lambda,0)\,, \tag{3·12a}
$$

$$
b(\lambda,t)=b(\lambda,0)\exp(4i\lambda^2 t)\,. \tag{3·12b}
$$

The zeros of $a(\lambda)$ in the upper half λ-plane are the bound state eigenvalues, which we shall designate by λ_k $(k=1,2,\cdots,N)$. When all the zeros of $a(\lambda)$ are simple, $F(z)$ can be represented by

$$
F(z,t)=\sum_{k=1}^N C_k(t)\frac{\exp[i\lambda_k(z+2\varepsilon_+)]}{\lambda_k}
$$

$$
+\frac{1}{2\pi}\int_{-\infty}^\infty\rho(\lambda,t)\frac{\exp[i\lambda(z+2\varepsilon_+)]}{\lambda}d\lambda\,, \tag{3·13}
$$

where the time dependence of $C_k(t)$ and $\rho(\lambda,t)$ are

$$
C_k(t)=C_k(0)\exp(4i\lambda_k^2 t)\,, \tag{3·14a}
$$

$$
\rho(\lambda,t)=\rho(\lambda,0)\exp(4i\lambda^2 t)\,. \tag{3·14b}
$$

The set of Eqs. (3·7), (3·10), (3·13) and (3·14) determines a sought function $q(x,t)$. Given the scattering data $\{\rho(\lambda,0),\lambda\text{ real};\lambda_k,C_k(0),k=1,2,\cdots,N\}$, we construct $F(z,t)$ by Eq. (3·13) with Eqs. (3·14), then we solve Eqs. (3·10) for $K_1(x, y; t)$, and by Eq. (3·7) we can obtain $q(x, t)$. However, as we shall see in the next section, the analysis is quite complicated due to the presence of $\varepsilon_+(x)$ and $\mu_+(x)$.

§ 4. One-soliton solution

We shall analyze a one-soliton solution. For the purpose, we restrict ourselves to the case that $a(\lambda)$ has only one simple zero in the upper half λ-plane and $\rho(\lambda, 0)=0$ for real λ. Then, Eqs. (3·11) become

$$F(z)=\frac{1}{\lambda_1}C_1(t)\exp[i\lambda_1(z+2\varepsilon_+)], \tag{4·1a}$$

$$F''(z)=-\lambda_1 C_1(t)\exp[i\lambda_1(z+2\varepsilon_+)]. \tag{4·1b}$$

The bound state eigenvalue λ_1 is denoted by

$$\lambda_1=\xi+i\eta,\ \eta>0. \tag{4·2}$$

Substitution of Eqs. (4·1) into the Gelfand-Levitan equation (3·10) yields

$$K_1(x, y)=\frac{C_1{}^*(t)}{\lambda_1{}^*}\frac{\exp[-i\lambda_1{}^*(x+y+2\varepsilon_+(x))]}{1+\dfrac{|C_1(t)|^2}{4\eta^2}\dfrac{\lambda_1}{\lambda_1{}^*}\exp[-4\eta(x+\varepsilon_+(x))]}. \tag{4·3}$$

Combining Eq. (3·7) and Eq. (4·3), we obtain

$$K_1(x, x)=\frac{1-\sqrt{1+|q|^2}}{q^*}\exp(\mu_+-\mu_+{}^*)$$

$$=\frac{C_1{}^*(t)}{\lambda_1{}^*}\frac{\exp[-2i\lambda_1{}^*(x+\varepsilon_+(x))]}{1+\dfrac{|C_1(t)|^2}{4\eta^2}\dfrac{\lambda_1}{\lambda_1{}^*}\exp[-4\eta(x+\varepsilon_+(x))]}. \tag{4·4}$$

From Eq. (3·14a), we have

$$\frac{1}{2\eta}C_1(t)=\exp[-8\xi\eta t+4i(\xi^2-\eta^2)t+\delta_0], \tag{4·5}$$

where the constant δ_0 is defined by

$$\exp\delta_0\equiv C_1(0)/2\eta. \tag{4·6}$$

With Eq. (4·5), Eq. (4·4) yields

$$|q|^2=\frac{4\eta^2}{\xi^2+\eta^2}\frac{\cosh^2[2\eta(x+4\xi t)+2\eta\varepsilon_+(x)]-\eta^2/(\xi^2+\eta^2)}{\{\cosh^2[2\eta(x+4\xi t)+2\eta\varepsilon_+(x)]-2\eta^2/(\xi^2+\eta^2)\}^2}, \tag{4·7}$$

$$q=\frac{-2\eta}{\sqrt{\xi^2+\eta^2}}\frac{\cosh[2\eta(x+4\xi t)+2\eta\varepsilon_+(x)+i\alpha]}{\cosh^2[2\eta(x+4\xi t)+2\eta\varepsilon_+(x)]-2\eta^2/(\xi^2+\eta^2)}$$

$$\times\exp[-2i(\xi x+2(\xi^2-\eta^2)t+\xi\varepsilon_+(x))]\cdot\exp(\mu_+{}^*-\mu_+), \tag{4·8}$$

where

$$\alpha=\tan^{-1}(\eta/\xi). \tag{4·9}$$

Here and in the following, the constant δ_0 is omitted for simplicity.

Let

$$u=x+4\xi t. \tag{4·10}$$

We can prove that ε_+ is a function of u. Hereafter we shall write $\varepsilon_+(u)$ instead of $\varepsilon_+(x)$ to emphasize the u-dependence of ε_+. There should be no confusion about this notation. Then from Eq. (2·22b) with Eq. (2·16a), we have

$$\frac{d}{du}(u+\varepsilon_+(u))=\sqrt{1+|q|^2}. \tag{4·11}$$

Substitution of Eq. (4·7) into Eq. (4·11) gives

$$\frac{d}{du}(u+\varepsilon_+(u))=\frac{\cosh^2[2\eta(u+\varepsilon_+(u))]}{\cosh^2[2\eta(u+\varepsilon_+(u))]-2\eta^2/(\xi^2+\eta^2)}, \tag{4·12}$$

where we have assumed that

$$\xi^2>\eta^2. \tag{4·13}$$

Integrating Eq. (4·12), we obtain

$$\varepsilon_+(u)=\frac{\eta}{\xi^2+\eta^2}\{\tanh[2\eta(u+\varepsilon_+(u))]-1\}. \tag{4·14}$$

This relation can be rewritten as

$$u=-\varepsilon_++\frac{1}{4\eta}\log\left[-\frac{2\eta}{\xi^2+\eta^2}\frac{1}{\varepsilon_+}-1\right]. \tag{4·15}$$

We can also prove that $(\mu_+^*-\mu_+)$ is a function of u. From Eq. (2·23b) with Eq. (2·16b), we have

$$\mu_+^*-\mu_+=\frac{1}{2}\int_x^\infty\left(\log\frac{q}{q^*}\right)_x\left(\frac{1}{\sqrt{1+|q|^2}}-1\right)dx, \tag{4·16}$$

Substitution of Eqs. (4·7) and (4·8) into Eq. (4·16) gives

$$\begin{aligned}\mu_+^*-\mu_+=&\frac{4i\xi\eta^2}{\xi^2+\eta^2}\int_x^\infty\frac{dx}{\cosh^2[2\eta(u+\varepsilon_+(u))]-\eta^2/(\xi^2+\eta^2)}\\&-\frac{2\eta^2}{\xi^2+\eta^2}\int_x^\infty\frac{(\mu_+^*-\mu_+)_x dx}{\cosh^2[2\eta(u+\varepsilon_+(u))]}.\end{aligned} \tag{4·17}$$

Solving Eq. (4·17), we obtain

$$\mu_+^*-\mu_+=\log\left\{\frac{\cosh[2\eta(u+\varepsilon_+(u))-i\alpha]}{\cosh[2\eta(u+\varepsilon_+(u))+i\alpha]}\right\}. \tag{4·18}$$

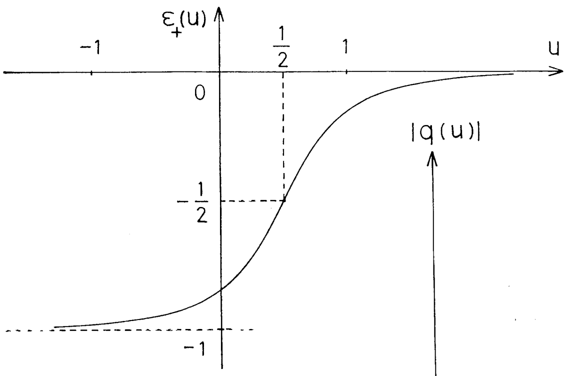

Fig. 1. The curve of $\varepsilon_+(u)$ for $\eta=1/2$ and $\xi/\eta=\sqrt{3}$.

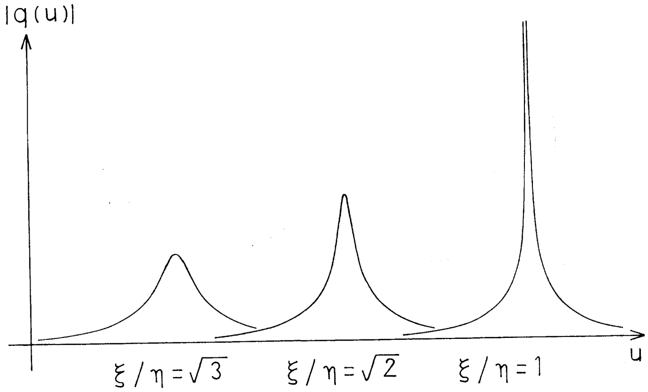

Fig. 2. The envelope of one-soliton solution for $\eta=1/2$ and $\xi/\eta=\sqrt{3}$, $\sqrt{2}$ and 1.

Substituting Eq. (4·18) into Eq. (4·8), we obtain an expression for one-soliton solution:

$$q(x,t)=\frac{-2\eta}{\sqrt{\xi^2+\eta^2}}\frac{\cosh[2\eta(x+4\xi t)+2\eta\varepsilon_+(x+4\xi t)-i\alpha]}{\cosh^2[2\eta(x+4\xi t)+2\eta\varepsilon_+(x+4\xi t)]-2\eta^2/(\xi^2+\eta^2)}$$
$$\times\exp[-2i\{\xi x+2(\xi^2-\eta^2)t+\xi\varepsilon_+(x+4\xi t)\}]. \qquad (4\cdot19)$$

Since $\varepsilon_+(u)$ is given by Eq. (4·15), we can evaluate $q(x,t)$ numerically. The function $\varepsilon_+(u)$ for $\eta=1/2$ and $\xi/\eta=\sqrt{3}$ and the one-soliton envelope $|q(x,t)|=|q(u)|$ for $\eta=1/2$ and various values of ξ/η are shown in Figs. 1 and 2, respectively. The peak of the soliton is obtained by the condition $d|q|^2/du=0$:

$$|q|_{\max}=\frac{2|\xi|\eta}{\xi^2-\eta^2}=\frac{2|\eta/\xi|}{1-(\eta/\xi)^2}. \qquad (4\cdot20)$$

From Eqs. (4·13) and (4·20), we find that the larger the peak amplitude of the soliton is, the smaller its speed 4ξ for constant η, as shown in Fig. 2.

Another expression for one-soliton solution is obtained as follows. Let

$$q(x,t)=|q|\exp(i\theta). \qquad (4\cdot21)$$

From Eqs. (4·11) and (4·12), we have

$$\frac{d}{du}\sqrt{1+|q|^2}$$
$$=-\frac{8\eta^3}{\xi^2+\eta^2}\sqrt{1+|q|^2}\frac{\sinh[2\eta(u+\varepsilon_+(u))]\cosh[2\eta(u+\varepsilon_+(u))]}{\{\cosh^2[2\eta(u+\varepsilon_+(u))]-2\eta^2/(\xi^2+\eta^2)\}^2}. \qquad (4\cdot22)$$

Since Eq. (4·12) yields

$$\cosh^2[2\eta(u+\varepsilon_+(u))]=\frac{2\eta^2}{\xi^2+\eta^2}\frac{\sqrt{1+|q|^2}}{\sqrt{1+|q|^2}-1}, \qquad (4\cdot 23)$$

we obtain

$$\frac{1}{2}\left(\frac{d}{du}\Phi\right)^2=-\omega\Phi^3(\Phi-1)^2(\Phi-\Phi_0), \qquad (4\cdot 24)$$

where

$$\Phi\equiv\sqrt{1+|q|^2}, \qquad (4\cdot 25a)$$

$$\omega=4(\xi^2-\eta^2), \qquad (4\cdot 25b)$$

$$\phi_0=(\xi^2+\eta^2)/(\xi^2-\eta^2). \qquad (4\cdot 25c)$$

The solitary wave solution exists under the condition (4·13), and then the integration of Eq. (4·24) gives

$$\log\left(\frac{\sqrt{\Phi_0-1}\sqrt{\Phi}-\sqrt{\Phi_0-\Phi}}{\sqrt{\Phi_0-1}\sqrt{\Phi}+\sqrt{\Phi_0-\Phi}}\right)+2\frac{\sqrt{\Phi_0-1}}{\Phi_0}\frac{(\Phi_0\Phi-\Phi^2)^{1/2}}{\Phi}$$
$$=\pm\sqrt{2\omega(\Phi_0-1)}\,(u-u_0). \qquad (4\cdot 26)$$

Here u_0 is an integration constant and $+(-)$ is for $y>y_0$ $(y<y_0)$. Equation (4·26) determines the shape of the envelope $|q|\equiv\sqrt{\Phi^2-1}$. Next we consider the phase part of $q(x,t)$. From Eq. (4·19) we find that

$$\theta(x,t)=-2\xi x-4(\xi^2-\eta^2)t+f(u), \qquad (4\cdot 27)$$

where

$$f(u)=-2\xi\varepsilon_+(u)-\arctan\left(\frac{\eta}{\xi}\tanh f\right). \qquad (4\cdot 28)$$

Differentiating Eq. (4·28) and using Eq. (4·23), we obtain

$$\frac{df}{du}=-4\xi\frac{\Phi^2}{1+\Phi}+2\xi\,. \qquad (4\cdot 29)$$

Therefore, the envelope and the phase of $q(x,t)$ can be determined by Eqs. (4·26) and (4·27) with Eq. (4·29), respectively.

§ 5. Discussion

In the present paper, using the inverse scattering method, we have solved a newly found integrable equation

$$iq_t+(q/\sqrt{1+|q|^2})_{xx}=0\,, \qquad (1\cdot 1)$$

under the boundary condition

$$q(x,t)\to 0 \quad \text{as} \quad |x|\to\infty . \tag{1·2}$$

In particular, we have examined one soliton-solution in detail. Here we point out two interesting limits of one-soliton solution.

First, when $\xi\gg\eta$, Eq. (4·19) or Eq. (4·26) reduces to

$$q(x,t)=\frac{2\eta}{\xi}\operatorname{sech}[2\eta(x+4\xi t)]\exp[-2i\xi x-4i(\xi^2-\eta^2)t], \tag{5·1}$$

which is in the same form as one-soliton solution of the nonlinear Schrödinger equation. The phase factor $\exp[-4i(-\eta^2)t]$ in Eq. (5·1) is left for comparison. From Eq. (5·1), we see that the soliton with small amplitude may be approximately described by the nonlinear Schrödinger equation. We show this fact directly from the original nonlinear evolution equation. Assuming that $|q|$ is small, we have

$$iq_t+\left[q\left(1-\frac{1}{2}|q|^2\right)\right]_{xx}=0 . \tag{5·2}$$

Suppose that the phase changes much faster than the envelope does. Then, for $q\sim\exp(-2i\xi x)$, we set

$$(|q|^2q)_{xx}\approx-4\xi^2|q|^2q . \tag{5·3}$$

Therefore, we obtain

$$iq_t+q_{xx}+2\xi^2|q|^2q=0 . \tag{5·4}$$

Equation (5·4) is the nonlinear Schrödinger equation whose one-soliton solution is given by Eq. (5·1).

Second, we consider the limit $|\xi|\to\eta+0$. We have shown that one-soliton solution has the properties:

$$q(x,t)\sim\exp(-i\omega t), \tag{5·5}$$

$$|q|_{\max}=8|\xi|\eta/\omega , \tag{5·6}$$

where

$$\omega=4(\xi^2-\eta^2). \tag{5·7}$$

We notice that as the frequency ω approaches zero the peak amplitude of the soliton increases infinitely, a bursting soliton. This situation reminds us of a kind of resonance phenomena. From this viewpoint, the application of Eq. (1·1) to the physical system, such as nonlinear optics, is under research.

References

1) M. Wadati, K. Konno and Y. H. Ichikawa, J. Phys. Soc. Japan (in press).
2) M. Wadati, K. Konno and Y. H. Ichikawa, J. Phys. Soc. Japan **46** (1979), 1965.
3) V. E. Zakharov and A. B. Shabat, Soviet Phys.-JETP **34** (1972), 62.
4) D. J. Kaup and A. C. Newell, J. Math. Phys. **19** (1978), 798.
5) C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, Phys. Rev. Letters **19** (1967), 1095.
6) M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, Stud. Appl. Math. **53** (1974), 249.