A NEW INTEGRAL TRANSFORM WITH AN APPLICATION IN HEAT-TRANSFER PROBLEM

by

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In this paper, an new integral transform $J[\phi(\tau)] = 1/\mu \int_0^\infty \phi(\tau) \mathrm{e}^{-\mu\tau} \mathrm{d}\tau$ *is proposed for the first time. The integral transform is used to solve the differential equation arising in heat-transfer problem.*

Key words: *heat transfer, integral transform, differential equation, analytical solution*

Introduction

Integral transforms have played important roles in the practical problems involving computational heat and fluid [1]. In mathematics, any transform is expressed by [2]:

$$\Phi(\theta) = \int_{\theta_1}^{\theta_2} \phi(\tau)\Xi(\tau, \theta)\mathrm{d}\tau \tag{1}$$

where $\Xi(\tau, \theta)$ is the kernel function, the input of the transform (4) is a function $\Xi(\tau, \theta)$ and the output of the transform (4) is another function $\Phi(\theta)$.

The inverse transform associated inverse kernel, denoted as $\Xi^{-1}(\tau, \theta)$, is suggested as [2]:

$$\phi(\tau) = \int_{\tau_1}^{\tau_2} \Phi(\theta)\Xi^{-1}(\tau, \theta)\mathrm{d}\theta \tag{2}$$

In engineering practice, the Laplace transform was used to solve the heat transfer problems in [3, 4]. The Laplace-Carson transform, as a generalized Laplace transform, was considered to handle the heat-exchange problems [5]. We now recall the Laplace and Laplace-Carson transforms.

The Laplace transform of the function $\phi(\tau)$ is defined as [6]:

$$\Phi(s) = L[\phi(\tau)] = \int_0^\infty \phi(\tau)e^{-s\tau}\mathrm{d}\tau \tag{3}$$

provided the integral exists for some s, where L is the Laplace transform operator.

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The inverse operator of eq. (3) is written:

$$\phi(\tau) = L^{-1}[\Phi(s)] = \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} \phi(\tau)\mathrm{e}^{s\tau}\mathrm{d}s \tag{4}$$

where ω is a real-valued constant.

The Laplace-Carson transform of the function $\phi(\tau)$ is defined as [7]:

$$\Phi(p) = \Pi[\phi(\tau)] = p\int_0^\infty \phi(\tau)\mathrm{e}^{-p\tau}\mathrm{d}\tau \tag{5}$$

provided the integral exists for some p, where Π is the Laplace-Carson transform operator.

The inverse operator of eq. (5) is written:

$$\phi(\tau) = \Pi^{-1}[\Phi(p)] = \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} \frac{\phi(\tau)}{p}\mathrm{e}^{p\tau}\mathrm{d}p \tag{6}$$

where ω is a real-valued constant.

A new integral transform

By taking the kernel function $\Xi(\tau, \theta) = \mathrm{e}^{-s\tau}$ in eq. (3) into $\Xi(\tau, \theta) = \mathrm{e}^{-\mu\tau}/\mu$, a new integral transform of the function $\phi(\tau)$ is defined:

$$\Phi(\mu) = J[\phi(\tau)] = \frac{1}{\mu}\int_0^\infty \phi(\tau)\mathrm{e}^{-\mu\tau}\mathrm{d}\tau \tag{7}$$

provided the integral exists for some μ, where J is the new integral transform operator.

The inverse operator of eq. (7) is defined:

$$\phi(\tau) = J^{-1}[\Phi(\mu)] = \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} \phi(\tau)\mu\mathrm{e}^{\mu\tau}\mathrm{d}\mu \tag{8}$$

where ω is a real-valued constant.

The properties of the integral transform are:

(M1) Suppose that $\Phi_1(\mu)=J[\phi_1(\tau)]$ and $\Phi_2(\mu) = J[\phi_2(\tau)]$, then, we have:

$$J[a\phi_1(\tau) + b\phi_2(\tau)] = a\Phi_1(\mu) + b\Phi_2(\mu) \tag{9}$$

where a and b are two constants.

(M2) Suppose that $\Phi(\mu) = J[\phi(c\tau)]$, then, we have:

$$J[\phi(c\tau)] = \frac{1}{c}\Phi\left(\frac{\mu}{c}\right), \tag{10}$$

where c is a constant.

(M3) Suppose that $\Phi(\varpi) = J[\phi(\tau)]$ and the derivative of $\phi(\tau)$ is $\phi^{(1)}(\tau)$. Then, we have:

$$J[\phi^{(1)}(\tau)]=\mu\,\Phi(\mu)-\frac{\phi(0)}{\mu} \tag{11}$$

(M4) Suppose that $\Phi(\varpi)=J[\phi(\tau)]$ and let the integral of $\phi(\tau)$ is $\int_0^\tau \phi(\tau)\mathrm{d}\tau$. Then, we have:

$$J\left[\int_0^\tau \phi(\tau)\mathrm{d}\tau\right]=\frac{1}{\mu}\Phi(\mu) \tag{12}$$

Proof

(M1): By using the definition of the integral transform (7), we directly reduce to (M1).

(M2):

$$J[\phi(c\tau)]=\frac{1}{\mu}\int_0^\infty \phi(c\tau)\mathrm{e}^{-\mu\tau}\mathrm{d}\tau=\frac{1}{c\mu}\int_0^\infty \phi(c\tau)\mathrm{e}^{-\frac{\mu c\tau}{c}}\mathrm{d}(c\tau)=\frac{1}{c}\Phi\left(\frac{\mu}{c}\right)$$

(M3):

$$J[\phi^{(1)}(\tau)]=\frac{1}{\mu}\int_0^\infty \phi^{(1)}(\tau)\mathrm{e}^{-\mu\tau}\mathrm{d}\tau=\frac{1}{\mu}[\phi(\tau)\mathrm{e}^{-\mu\tau}]_0^\infty+\int_0^\infty \phi(\tau)\mathrm{e}^{-\mu\tau}\mathrm{d}\tau=\mu\,\Phi(\mu)-\frac{\phi(0)}{\mu}$$

(M4):

$$J\left[\int_0^\tau \phi(\tau)\mathrm{d}\tau\right]=\frac{1}{\mu}\left\{\left[\int_0^\tau \phi(\tau)\mathrm{d}\tau\right]\mathrm{e}^{-\mu\tau}\right\}_0^\infty+\frac{1}{\mu^2}\int_0^\infty \phi(\tau)\mathrm{e}^{-\mu\tau}\mathrm{d}\tau=\frac{1}{\mu}\Phi(\mu)$$

Thus, we finish the proof.

The integral transforms of the functions are given:

(V1) $$J[1]=\frac{1}{\mu^2}$$

(V2) $$J[\mathrm{e}^{c\tau}]=\frac{1}{\mu}\int_0^\infty \mathrm{e}^{c\tau}\mathrm{e}^{-\mu\tau}\mathrm{d}\tau=\frac{1}{\mu(\mu-c)}$$

(V3) $$J[\sin(c\tau)]=\frac{c}{\mu(\mu^2+c^2)}$$

(V4) $$J[\cos(c\tau)]=\frac{1}{\mu^2+c^2}$$

(V5) $$J[\tau]=\frac{1}{\mu}\int_0^\infty \tau\mathrm{e}^{-\mu\tau}\mathrm{d}\tau=\frac{1}{\mu^3}$$

Proof

$$J[1]=\frac{1}{\mu}\int_0^\infty \mathrm{e}^{-\mu\tau}\mathrm{d}\tau=\frac{1}{\mu^2} \tag{13}$$

$$J[e^{c\tau}]=\frac{1}{\mu}\int_0^\infty \mathrm{e}^{c\tau}\mathrm{e}^{-\mu\tau}\mathrm{d}\tau=\frac{1}{\mu(\mu-c)} \tag{14}$$

$$J[\sin(c\tau)]=\frac{1}{\mu}\int_0^\infty \sin(c\tau)\mathrm{e}^{-\mu\tau}\mathrm{d}\tau=\frac{1}{2i}\left[\frac{1}{\mu(\mu-ci)}-\frac{1}{\mu(\mu+ci)}\right]=\frac{c}{\mu(\mu^2+c^2)} \tag{15}$$

$$J[\cos(c\tau)]=\frac{1}{\mu}\int_0^\infty \cos(c\tau)\mathrm{e}^{-\mu\tau}\mathrm{d}\tau=\frac{1}{2}\left[\frac{1}{\mu(\mu-ci)}+\frac{1}{\mu(\mu+ci)}\right]=\frac{1}{\mu^2+c^2} \tag{16}$$

$$J[\tau]=\frac{1}{\mu}\int_0^\infty \tau\mathrm{e}^{-\mu\tau}\mathrm{d}\tau=\frac{1}{\mu^3} \tag{17}$$

Thus, the proof is finished.

Solving the heat-transfer problem

Let us consider the differential equation in heat-transfer problem [8, 9]:

$$-hM\Theta(x)=\rho Vc_p\Theta^{(1)}(x) \tag{18}$$

subject to the initial condition:

$$\Theta(0)=\beta \tag{19}$$

where $\Theta(x)$ is the temperature.

Taking the integral transform of eq. (18) gives:

$$-hM\Theta(\mu)=\rho Vc_p\left[\mu\,\Theta(\mu)-\frac{\beta}{\mu}\right] \tag{20}$$

From eq. (20), we have:

$$\Theta(\mu)=\frac{\beta}{\mu\left(\mu+\dfrac{hM}{\rho Vc_p}\right)} \tag{21}$$

Thus, by taking the inverse transform of eq. (21), the solution of eq. (13) takes the form:

$$\Theta(x)=\beta\mathrm{e}^{-\frac{hM}{\rho Vc_p}x} \tag{22}$$

which is in agreement with the result in [9].

Conclusion

A new integral transform based on the kernel function $\Xi(\tau, \theta) = e^{-\mu\tau}/\mu$ was presented for the first time. The analytical solution of the differential equation involving the heat-transfer was obtained. The technology is proposed, as a powerful approach, to solve the differential equations.

Nomenclature

c_p – specific heat of the material, [$Jkg^{-1}K^{-1}$]
h – convection heat transfer coefficient, [$Wm^{-2}K^{-1}$]
M – surface area of the body, [m^2]
V – volume, [m^3]
x – space co-ordinate, [m]

Greek symbols

$\Theta(x)$ – temperature, [Km^{-3}]
ρ – density, [kgm^{-3}]

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