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A New Interior Point Method for Linear Complementarity Problem

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Abstract

For a given n-vector q and a real square matrix $M \in \mathrm{IR}^{n\times n}$, the linear complementarity problem, denoted $LCP(M,q)$, is that of finding nonnegative vector $z \in \mathrm{IR}^n$ such that $z^T(Mz+q)=0$ and $Mz+q \geqslant 0$. In this paper we suppose that the matrix M must be a symmetric and positive definite and the set

$$S = \{z \in \ \mathrm{IR}^n \ / \ z > 0 \text{ and } Mz+q > 0\};$$

named interior points set of the $LCP(M,q)$ must be nonempty.

The aim of this paper is to show that the $LCP(M,q)$ is completely equivalent to a convex quadratic programming problem ($CQPP$) under linear constraints. To solve the second problem, we propose an iterative method of interior points which converge in polynomial time to the exact solution; this convergence requires at most $o(n^{0,5}L)$ iterations, where n is the number of the variables and L is the length of a binary coding of the input; furthermore, the algorithm does not exceed $o(n^{3,5}L)$ arithmetic operations until its convergence and in the end, we close our paper with some numerical examples which illustrate our theoretical results.

Keywords: Linear Complementarity Problem, Convex Quadratic Programming with Equilibrium Constraints, Matrix symmetric and positive definite, Interior Point Algorithm

1. Introduction

We consider the linear complementarity problem $LCP(M, q)$ that is, given a real square matrix $M \in \mathrm{IR}^{n\times n}$ and q an element of IR^n, find vectors $z \in \mathrm{IR}^n$ such that

$$\begin{cases} < z, Mz + q >= 0 \\ Mz + q \geqslant 0 \\ z \geqslant 0 \end{cases}$$

This problem has important applications in game theory, operational research, and some other areas of engineering (see [3], [4], [5], [6], [7], [9], [10], [13], [14], [15], [16] and [17]). For solving this problem, many results exist, for instance Lemke[11] have developed an algorithm for solving a linear complementarity problem which is based on pivot steps. Mukherjee[12] gave an iterative method for finding a solution to a linear and quasi-linear complementarity problem. Kojima[8] and Achache[1] have showed that the linear complementarity problem is completely equivalent to solving quadratic convex problem (QCP); each of them gave a different iterative method to solve the second problem.

Our notation in this paper is the usual one. In particular, IR^n denotes the space of real $n-$dimensional vectors,

$\mathrm{IR}^n_+ := \{x \in \mathrm{IR}^n : x_i \geqslant 0, i = 1..n\}$ is the nonnegative orthant and its interior is $\mathrm{IR}^n_{++} := \{x \in \mathrm{IR}^n : x_i > 0, i = 1..n\}$.

With $x \in \mathrm{IR}^n$ we define $|x| = (|x_1|, .., |x_n|)^T \in \mathrm{IR}^n$.

We denote by I the identity matrix.

Let x, $y \in \mathrm{IR}^n$, $x^T y$ or $< x, y >$ is the inner product of the x and y; $||x||$ is the Euclidean norm.

For $x \in \mathrm{IR}^n$ and k a nonnegative integer, $x^{(k)}$ refers to the vector obtained after k iterations; for $1 \leqslant i \leqslant n$, x_i refers to the i^{th} element of x, and $x_i^{(k)}$ refers to the i^{th} element of the vector obtained after k iterations.

For $A \in \mathrm{IR}^{n\times n}$ and k a nonnegative integer, $A^{(k)}$ refers to the matrix obtained after k iterations; for p a nonnegative integer, $A^{p,(k)}$ refers to the matrix at puissance p obtained after k iterations and $A^{-p,(k)}$ denotes the inverse of the matrix $A^{p,(k)}$.

Let x, $y \in \mathrm{IR}^n$, the expression $x \leqslant y$ (respectively $x < y$) meaning that $x_i \leqslant y_i$ (respectively $x_i < y_i$) for each $i = 1..n$.

Given a vector x in IR^n, $X = diag(x)$ is the $n \times n$ diagonal matrix with $X_{ii} = x_i$ for all i and $X_{ij} = 0$ for all $i \neq j$.

The transpose of a vector (respectively matrix) is denoted by super script T, such as the transpose of the vector x (respectively the matrix A) is given by x^T (respectively A^T).

Remember that the spectrum $\sigma(A)$ of the matrix A is the set of its eigenvalues and its spectral radius ρ is given by $\rho(A) := \sup\{|\lambda|$ such that $\lambda \in \sigma(A)\}$.

We recall that a matrix M is called symmetric and positive definite matrix if and only if

$$x^t Mx > 0, \forall x \neq 0.$$

and a matrix M is called symmetric and positive matrix if and only if

$$x^t Mx \geqslant 0, \forall x.$$

The paper is organized as follows. In the next section, we show that solving linear complementarity problem associated with a matrix M and a vector q is completely equivalent to finding the minimum of a convex quadratic programming problem $(CQPP)$; for solving the second problem we propose to construct a sequence of vectors $\{z^{(k)}\}_{k=0,1,..}$ which converges to a vector z^* (the exact solution of linear complementarity problem LCP). In the third section, we show that the convergence of this method requires $o(\sqrt{n}L)$ number by iteration where L is the length of a binary coding of the input data of the problem and in the end, we close our paper with some numerical examples which illustrate our theoretical results.

2. Equivalent reformulation of the problem

In this section, we show that solving a linear complementarity problem (LCP) is equivalent to finding the minimum of a convex quadratic programming problem $(CQPP)$ under linear constraints.

Consider the linear complementarity problem as follows

Find $z \in \mathrm{IR}^n$ such that:

$$\begin{cases} < z, Mz + q >= 0 \\ Mz + q \geqslant 0 \\ z \geqslant 0 \end{cases} \tag{1}$$

where $M \in \mathrm{IR}^{n\times n}$ and $q \in \mathrm{IR}^n$ are given data.

and let's consider

$$S = \{z \in \mathrm{IR}^n \ / \ z > 0 \text{ and } Mz + q > 0\};$$

named interior points set of the $LCP(M, q)$.

Theorem 1 *: If M is symmetric positive matrix and the interior points set of the linear complementarity problem*

$$S := \{z \in \mathrm{IR}^n / z > 0 \ \ \textit{and } Mz + q > 0\}$$

is nonempty then the problem $LCP(M, q)$ has one and only one solution.

Proof. : For a proof of the above theorem we refer to [1]. ∎

Consider the minimization problem under linear constraints $(CQPP)$ as follows

$$\begin{cases} \min \ f(z) := < z, Mz + q > \\ \text{Subject to:} \\ \qquad\qquad\qquad Mz + q \geqslant 0 \\ \qquad\qquad\qquad\qquad z \geqslant 0 \end{cases} \tag{2}$$

We note that if z^* is a solution of the linear complementarity problem $LCP(M, q)$ then *zero* is the global minimum of the problem(2) (see [18]).

Inversely, if z^* is the minimum of the problem(2) then we have two situations:

- If $f(z^*) = 0$, then z^* is a solution of the $LCP(M, q)$.
- Otherwise (ie $f(z^*) > 0$), then the linear complementarity problem $LCP(M, q)$ admits no solution.

That is to say the problem(1) and the problem(2) are equivalent.

The Lagrange function associated with the problem(2) is given by

$$L(z, \lambda_1, \lambda_2) := f(z) - < \lambda_1, Mz + q > - < \lambda_2, z >$$

where λ_1 and λ_2 are the Lagrange multipliers associated with the problem(2).

The KKT conditions applied to the problem(2) imply that if z is a stationary point, then there exists $\lambda_1 \in IR^n$ and $\lambda_2 \in IR^n$ such that

$$\left\{\begin{array}{l} 2Mz + q - M\lambda_1 - \lambda_2 = 0 \\ \lambda_1^t(Mz+q) = 0 \\ \lambda_2^t z = 0 \\ Mz + q \geqslant 0 \\ z \geqslant 0 \\ \lambda_1 \geqslant 0 \\ \lambda_2 \geqslant 0. \end{array}\right. \tag{3}$$

Let's consider

$$T := \{(z, \lambda_1, \lambda_2) \ / \ 2Mz + q - M\lambda_1 - \lambda_2 = 0 \ ; \ z, Mz+q, \lambda_1, \lambda_2 \geqslant 0\}.$$

In this paper we propose to construct a sequence of vectors

$$\{(z^{(k)}, \lambda_1^{(k)}, \lambda_2^{(k)})\}_{k=0\ldots} \in T$$

such that

$$\lim_{k\to+\infty}(< \lambda_1^{(k)}, Mz^{(k)} + q > + < \lambda_2^{(k)}, z^{(k)} >) = 0.$$

To achieve this objective we consider the iterative method which starts from an arbitrary point $(z^{(0)}, \lambda_1^{(0)}, \lambda_2^{(0)}) \in T$ and generates successively points

$$z^{(k+1)} := z^{(k)} + d^{(k)}$$

where the vectors $d^{(k)}$ are the directions chosen to generate the vectors $z^{(k+1)}$, they are defined by the Newton directions associated with the following penalized problem (see [2])

$$\left\{\begin{array}{l} \min \ f_\mu(z) \\ \text{Subjects to:} \\ \qquad\qquad Mz + q > 0 \\ \qquad\qquad z > 0 \end{array}\right.$$

where

$$f_\mu(z) :=< z, Mz > + < q, z > -\mu\sum_{i=1}^{n}\log(z_i) - \mu\sum_{i=1}^{n}\log(w_i)$$

and

$$w = Mz + q.$$

The Newton direction at $z^{(k)}$ is the optimal solution of the following quadratic problem

$$\min \; g_{\mu,z^{(k)}}(d^{(k)}) \tag{4}$$

where

$$g_\mu(z^{(k)}) := \frac{1}{2} < d^{(k)}, \bigtriangledown^2 f_{\mu_k}(z^{(k)}) d^{(k)} > + < \bigtriangledown f_{\mu_k}(z^{(k)}), d^{(k)} > .$$

Now we show that

Theorem 2 *: The problem(4) admits one and only one solution; this solution is given by*

$$d^{(k)} = -H^{-1,(k)} \bigtriangledown f_{\mu_k}(z^{(k)})$$

where

$$\begin{cases} H^{(k)} := 2M + \mu_k(MW^{-2,(k)}M + Z^{-2,(k)}) \\ Z := diag(z) \\ W := diag(w) \\ w = Mz + q. \end{cases}$$

Proof. : The Hessian of the penalty function f_μ at $z^{(k)}$ noted by $H^{(k)}$ is given by

$$H^{(k)} := 2M + \mu_k(MW^{-2,(k)}M + Z^{-2,(k)}).$$

Since the matrix M is symmetric and positive definite, then the matrix $H^{(k)}$ is symmetric definite positive.

Therefore, the problem(4) admits one and only one solution.

This solution is given by

$$H^{(k)}d^{(k)} + \bigtriangledown f_{\mu_k}(z^{(k)}) = 0$$

this implies

$$d^{(k)} = -H^{-1,(k)} \bigtriangledown f_{\mu_k}(z^{(k)}).$$

■

After calculating the direction $d^{(k)}$, we can write

$$\begin{cases} z^{(k+1)} := z^{(k)} + d^{(k)} \\ \lambda_1^{(k+1)} := \mu_k W^{-1,(k)}(e - W^{-1,(k)} M d^{(k)}) \\ \lambda_2^{(k+1)} := \mu_k Z^{-1,(k)}(e - Z^{-1,(k)} d^{(k)}) \end{cases}$$

Now we give the following algorithm for solving our problem

Algorithm:

(Initialization):

$k = 0$

$\mu_0 > 0$: Parameter penalty.

$\epsilon > 0$: Tolerance wanted.

(Calculation of the vector $z^{(k+1)}$, $\lambda_1^{(k+1)}$, $\lambda_2^{(k+1)}$**)**

$$\begin{cases} d^{(k)} = -H^{-1,(k)} \bigtriangledown f_{\mu_k}(z^{(k)}) \\ z^{(k+1)} := z^{(k)} + d^{(k)} \\ w^{(k+1)} := M z^{(k+1)} + q \\ \lambda_1^{(k+1)} := \mu_k W^{-1,(k)}(e - W^{-1,(k)} M d^{(k)}) \\ \lambda_2^{(k+1)} := \mu_k Z^{-1,(k)}(e - Z^{-1,(k)} d^{(k)}) \\ \mu_{k+1} := 2\mu_k \frac{\delta^2 + \sqrt{n}}{\delta + 2\sqrt{n}} \end{cases}$$

(Stopping criterion):

if $< \lambda_1^{(k+1)}, w^{(k+1)} > + < \lambda_2^{(k+1)}, z^{(k+1)} > < \epsilon$ then STOP.

otherwise:

$k := k + 1$.

GO TO the previous step.

3. Convergence

In this section, in the one hand, we show that all points generated by this algorithm are in the set T and $(< \lambda_1^{(k)}, w^{(k)} > + < \lambda_2^{(k)}, z^{(k)} >)$ converges to zero when $k \to +\infty$; on the other hand, we prove that our algorithm has $o(\sqrt{n}L)$ iteration complexity; more precisely we will show that the algorithm does not exeed $O(n^{3.5}L)$ arithmetic operations until its convergence. To so do, we suppose that there exist $z^{(0)} > 0$ and

$0 < \delta < 1/2$ such that: $||Z^{-1,(0)}d^{(0)}|| \leqslant \delta$ and $||W^{-1,(0)}Md^{(0)}|| \leqslant \delta$ and we prove that if k is positive integer then we have

$$||Z^{-1,(k)}d^{(k)}|| \leqslant \delta$$

and

$$||W^{-1,(k)}Md^{(k)}|| \leqslant \delta$$

to achieve this goal, we will need the following four lemmas

Lemma 3 *: Let k be a positive integer, if*

$$||Z^{-1,(k)}d^{(k)}|| \leqslant \delta$$

then we have

$$||Z^{(k+1)}Z^{-1,(k)} - I|| \leqslant \delta.$$

where I is the identity matrix.

Proof. : For each $i = 1, .., n$ we have

$$\begin{aligned}
((Z^{(k+1)}Z^{-1,(k)} - I)e)_i &= \frac{z_i^{(k+1)}}{z_i^{(k)}} - 1 \\
&= \frac{d_i^{(k)}}{z_i^{(k)}} \\
&= (Z^{-1,(k)}d^{(k)})_i.
\end{aligned}$$

■

Lemma 4 *: Let k be a positive integer, if*

$$||W^{-1,(k)}Md^{(k)}|| \leqslant \delta$$

then we have

$$||W^{(k+1)}W^{-1,(k)} - I|| \leqslant \delta.$$

Proof. : For each $i = 1, .., n$ we have

$$((W^{(k+1)}W^{-1,(k)} - I)e)_i = \frac{w_i^{(k+1)}}{w_i^{(k)}} - 1 \quad (5)$$
$$= \frac{\sum_{j=1}^{n} m_{ij} d_i^{(k)}}{w_i^{(k)}}$$
$$= (W^{-1,(k)} M d^{(k)})_i.$$

■

Lemma 5 *: Let k be a positive integer, if*

$$||Z^{-1,(k)} d^{(k)}|| \leqslant \delta$$

then we have

$$||Z^{(k+1)} Z^{-2,(k)} d^{(k)} + \frac{\mu_{k+1}}{\mu_k} e - Z^{(k+1)} Z^{-1,(k)} e|| \leqslant \delta \frac{\delta^2 + \sqrt{n}}{\delta + 2\sqrt{n}}.$$

where

$$\mu_{k+1} := 2\mu_k \frac{\delta^2 + \sqrt{n}}{\delta + 2\sqrt{n}} \quad (6)$$

Proof. To show that we use the definition

$$\frac{\mu_{k+1}}{\mu_k} e := e - \frac{\delta - \delta^2}{\delta + 2\sqrt{n}} e; \quad ((1))$$

and the relation

$$Z^{(k+1)} = Z^{(k)} + D^{(k)}$$

where $Z^{(k)}$, $D^{(k)}$ are respectively the diagonal matrix of the vector $z^{(k)}$ and $d^{(k)}$.

By multiplying the last relation by the matrix $Z^{-1,(k)}$ we get

$$Z^{(k+1)} Z^{-1,(k)} = I + D^{(k)} Z^{-1,(k)}$$
$$= I + Z^{-1,(k)} D^{(k)}$$

this implies

$$(Z^{(k+1)} Z^{-1,(k)} - I)e = Z^{-1,(k)} D^{(k)} e \quad ((2))$$
$$= Z^{-1,(k)} d^{(k)}$$

if we note that

$$\Delta_1 := ||Z^{(k+1)}Z^{-2,(k)}d^{(k)} + \frac{\mu_{k+1}}{\mu_k}e - Z^{(k+1)}Z^{-1,(k)}e||$$

then we have

$$\begin{aligned}\Delta_1 &= ||(Z^{(k+1)}Z^{-1,(k)} - I)Z^{-1,(k)}d^{(k)} - \frac{\delta - 2\delta^2}{\delta + 2\sqrt{n}}e|| \\ &\leqslant ||Z^{(k+1)}Z^{-1,(k)} - I||.||Z^{-1,(k)}d^{(k)}|| + \sqrt{n}\frac{\delta - 2\delta^2}{\delta + 2\sqrt{n}} \\ &\leqslant \delta.\ \delta + \sqrt{n}\frac{\delta - 2\delta^2}{\delta + 2\sqrt{n}} \\ &= \delta\frac{\delta^2 + \sqrt{n}}{\delta + 2\sqrt{n}}.\end{aligned}$$

■

Lemma 6 *: Let k be a positive integer, if*

$$||W^{-1,(k)}Md^{(k)}|| \leqslant \delta$$

then we have

$$||W^{(k+1)}W^{-2,(k)}Md^{(k)} + \frac{\mu_{k+1}}{\mu_k}e - W^{(k+1)}W^{-1,(k)}e|| \leqslant \delta\frac{\delta^2 + \sqrt{n}}{\delta + 2\sqrt{n}}.$$

Proof. : In the one hand, if we use (5) we can write

$$(I - W^{(k+1)}W^{-1,(k)})e = -W^{-1,(k)}Md^{(k)}$$

and in the other hand, if we note that

$$\Delta_2 := ||W^{(k+1)}W^{-2,(k)}Md^{(k)} + \frac{\mu_{k+1}}{\mu_k}e - W^{(k+1)}W^{-1,(k)}e||$$

then we get

$$\begin{aligned}
\Delta_2 &= ||W^{(k+1)}W^{-2,(k)}Md^{(k)} + e - W^{(k+1)}W^{-1,(k)}e - \frac{\delta - 2\delta^2}{\delta + 2\sqrt{n}}e|| \\
&= ||W^{(k+1)}W^{-2,(k)}Md^{(k)} + (I - W^{(k+1)}W^{-1,(k)})e - \frac{\delta - 2\delta^2}{\delta + 2\sqrt{n}}e|| \\
&\leqslant ||W^{(k+1)}W^{-2,(k)}Md^{(k)} - W^{-1,(k)}Md^{(k)} - \frac{\delta - 2\delta^2}{\delta + 2\sqrt{n}}e|| \\
&\leqslant ||(W^{(k+1)}W^{-1,(k)} - I)W^{-1,(k)}Md^{(k)} - \frac{\delta - 2\delta^2}{\delta + 2\sqrt{n}}e|| \\
&\leqslant ||W^{(k+1)}W^{-1,(k)} - I||.||W^{-1,(k)}Md^{(k)}|| + \sqrt{n}\frac{\delta - 2\delta^2}{\delta + 2\sqrt{n}}e|| \\
&\leqslant \delta.\ \delta + \sqrt{n}\frac{\delta - 2\delta^2}{\delta + 2\sqrt{n}} \\
&= \delta\frac{\delta^2 + \sqrt{n}}{\delta + 2\sqrt{n}}.
\end{aligned}$$

■

Now we show that

Theorem 7 *: If we suppose that*

$$\begin{cases} ||Z^{-1,(k)}d^{(k)}|| \leqslant \delta \\ ||W^{-1,(k)}Md^{(k)}|| \leqslant \delta \end{cases}$$

then we have

$$\begin{cases} ||Z^{-1,(k+1)}d^{(k+1)}|| \leqslant \delta \\ ||W^{-1,(k+1)}Md^{(k+1)}|| \leqslant \delta \end{cases}$$

Proof. : Since

$$H^{(k)}d^{(k)} + \bigtriangledown f_{\mu_k}(z^{(k)}) = 0$$

we have

$$\begin{cases} H^{(k)}d^{(k)} + \bigtriangledown f_{\mu_k}(z^{(k)}) = 0 \\ H^{(k+1)}d^{(k+1)} + \bigtriangledown f_{\mu_{k+1}}(z^{(k+1)}) = 0 \end{cases}$$

By multiplying the two equations by $d^{(k+1)}$ we get

$$\begin{cases} < d^{(k+1)}, (2M + \mu_k(MW^{-2,(k)}M + Z^{-2,(k)}))d^{(k)} > + \\ \qquad\qquad < d^{(k+1)}, 2Mz^{(k)} + q - \mu_k(MW^{-1,(k)} + Z^{-1,(k)})e >= 0 \\ \\ < d^{(k+1)}, (2M + \mu_{k+1}(MW^{-2,(k+1)}M + Z^{-2,(k+1)}))d^{(k+1)} > + \\ \qquad\qquad < d^{(k+1)}, 2Mz^{(k+1)} + q - \mu_{k+1}(MW^{-1,(k+1)} + Z^{-1,(k+1)})e >= 0 \end{cases}$$

if we note

$$\Delta_3 := \mu_{k+1}(||W^{-1,(k+1)}Md^{(k+1)}||^2 + ||Z^{-1,(k+1)}d^{(k+1)}||^2)$$

then we have

$$\begin{aligned}
&\Delta_3 \leqslant \mu_{k+1} < d^{(k+1)}, (MW^{-1,(k+1)}+Z^{-1,(k+1)})e > - < d^{(k+1)}, 2Mz^{(k+1)}+q > \\
&= \mu_{k+1} < W^{-1,(k+1)}Md^{(k+1)}, e > +\mu_{k+1} < Z^{-1,(k+1)}d^{(k+1)}, e > \\
&\qquad - (\mu_k < d^{(k+1)}, (MW^{-1,(k)} + Z^{-1,(k)})e > \\
&\qquad - \mu_k < d^{(k+1)}, (MW^{-2,(k)}M + Z^{-2,(k)})d^{(k)} >) \\
&= \mu_k < d^{(k+1)}, \frac{\mu_{k+1}}{\mu_k}MW^{-1,(k+1)}e - MW^{-1,(k)}e + MW^{-2,(k)}Md^{(k)} > \\
&\qquad + \mu_k < d^{(k+1)}, \frac{\mu_{k+1}}{\mu_k}Z^{-1,(k+1)}e - Z^{-1,(k)}e + Z^{-2,(k)}d^{(k)} > \\
&= \mu_k < d^{(k+1)}, MW^{-1,(k+1)}(W^{(k+1)}W^{-2,(k)}Md^{(k)}+\frac{\mu_{k+1}}{\mu_k}e-W^{(k+1)}W^{-1,(k)}e > \\
&\qquad +\mu_k < d^{(k+1)}, Z^{-1,(k+1)}(Z^{(k+1)}Z^{-2,(k)}d^{(k)}+\frac{\mu_{k+1}}{\mu_k}e-Z^{(k+1)}Z^{-1,(k)}e > \\
&\leqslant \mu_k\delta\frac{\delta^2+\sqrt{n}}{\delta^2+2\sqrt{n}}(||W^{-1,(k+1)}Md^{(k+1)}||+||Z^{-1,(k+1)}d^{(k+1)}||)
\end{aligned}$$

so

$$\mu_{k+1}(||W^{-1,(k+1)}Md^{(k+1)}|| + ||Z^{-1,(k+1)}d^{(k+1)}||) \leqslant 2\mu_k\delta\frac{\delta^2+\sqrt{n}}{\delta^2+2\sqrt{n}}$$

finally

$$||W^{-1,(k+1)}Md^{(k+1)}|| + ||Z^{-1,(k+1)}d^{(k+1)}|| \leqslant \delta$$

hence the result. ■

Now we show that all points generated by the algorithm $(z^{(k)}, \lambda_1^{(k)}, \lambda_2^{(k)})$ remain in the set T.

For that, let's assume that:

$$(z^{(k)}, \lambda_1^{(k)}, \lambda_2^{(k)}) \in T$$

then we have

-
$$\begin{aligned} z^{(k+1)} &= z^{(k)} + d^{(k)} \\ &= Z^{(k)}(e + Z^{-1,(k)}d^{(k)}) \\ &> 0. \end{aligned}$$

-
$$\begin{aligned} w^{(k+1)} &= Mz^{(k+1)} + q \\ &= w^{(k)} + Md^{(k)} \\ &= W^{(k)}(e + W^{-1,(k)}Md^{(k)}) \\ &> 0. \end{aligned}$$

-
$$\begin{aligned} \lambda_1^{(k+1)} &:= \mu_k W^{-1,(k)}(e - W^{-1,(k)}Md^{(k)}) \\ &> 0. \end{aligned}$$

-
$$\begin{aligned} \lambda_2^{(k+1)} &:= \mu_k Z^{-1,(k)}(e - Z^{-1,(k)}d^{(k)}) \\ &> 0. \end{aligned}$$

-
$$\begin{aligned} \lambda_2^{(k+1)} &= \mu_k Z^{-1,(k)}(e - Z^{-1,(k)}d^{(k)}) \\ &= \mu_k Z^{-1,(k)}e - \mu_k Z^{-2,(k)}d^{(k)} \\ &= \mu_k Z^{-1,(k)}e - (H^{(k)} - 2M - \mu_k MW^{-2,(k)}M)d^{(k)} \\ &= \mu_k Z^{-1,(k)}e - H^{(k)}d^{(k)} + (2Mz^{(k+1)} + q) - (2Mz^{(k)} + q) \\ &\qquad + \mu_k MW^{-2,(k)}Md^{(k)} \\ &= \mu_k Z^{-1,(k)}e + (2Mz^{(k)} + q - \mu_k MW^{-1,(k)}e - \mu_k Z^{-1,(k)}e) \\ &\qquad + (2Mz^{(k+1)} + q) - (2Mz^{(k)} + q) + \mu_k MW^{-2,(k)}Md^{(k)} \\ &= -\mu_k MW^{-1,(k)}e + (2Mz^{(k+1)} + q) + \mu_k MW^{-2,(k)}Md^{(k)} \\ &= (2Mz^{(k+1)} + q) - \mu_k MW^{-1,(k)}(e - W^{-1,(k)}Md^{(k)}) \\ &= (2Mz^{(k+1)} + q) - M\lambda_1^{(k+1)}. \end{aligned}$$

Now we give the following proposition

Proposition 8 *: For any positive integer k, we have*

$$< w^{(k)}, \lambda_1^{(k)} > + < z^{(k)}, \lambda_2^{(k)} > \leqslant 2\mu_{k-1}(\delta + \sqrt{n})^2.$$

Proof. : From the above we can write

$$\begin{aligned}
&< w^{(k+1)}, \lambda_1^{(k+1)} > + < z^{(k+1)}, \lambda_2^{(k+1)} > \\
&\quad =< W^{(k)}(e + W^{-1,(k)}Md^{(k)}), \mu_k W^{-1,(k)}(e - W^{-1,(k)}Md^{(k)}) > + \\
&\qquad\quad < Z^{(k)}(e + Z^{-1,(k)}d^{(k)}), \mu_k Z^{-1,(k)}(e - Z^{-1,(k)}d^{(k)}) > \\
&\quad = \mu_k(< e + W^{-1,(k)}Md^{(k)}, e - W^{-1,(k)}Md^{(k)} > + \\
&\qquad\qquad < e + Z^{-1,(k)}d^{(k)}, e - Z^{-1,(k)}d^{(k)} >) \\
&\leqslant 2\mu_k(\delta+\sqrt{n})^2.
\end{aligned}$$

■

Hence the quantity $(< w^{(k)}, \lambda_1^{(k)} > + < z^{(k)}, \lambda_2^{(k)} >)$ converges to zero. Now we prove that our algorithm has $o(\sqrt{n}L)$ iteration complexity; more precisely we will show that the algorithm does not exceed $o(n^{3.5}L)$ arithmetic operations until its convergence, and in the end we will establish the total number of iterations performed by the algorithm so that its convergence is reached.

Proposition 9 *: The total number of iterations performed by the algorithm is less than or equal to*

$$k^* = \frac{\delta + 2\sqrt{n}}{\delta - \delta^2 + \sqrt{n}} \ln \frac{2(\delta + \sqrt{n})^2\mu_0}{\epsilon}$$

where ϵ denotes the tolerance of the problem and μ_0 is the parameter of the initial penalty.

Proof. : The algorithm ends when

$$2(\delta + \sqrt{n})^2\mu_k \leqslant \epsilon$$

thus, it is enough to show that k^* satisfies this inequality.

From the definition of k^* we can write

$$\begin{aligned}\ln(\epsilon) &= -\frac{\delta-\delta^2+\sqrt{n}}{\delta+2\sqrt{n}}k^* + \ln[2(\delta+\sqrt{n})^2\mu_0] \\ &\geqslant k^*\ln(\frac{\delta^2+\sqrt{n}}{\delta+2\sqrt{n}}) + \ln[2(\delta+\sqrt{n})^2\mu_0] \\ &= \ln(\frac{\delta^2+\sqrt{n}}{\delta+2\sqrt{n}})^{k^*} + \ln[2(\delta+\sqrt{n})^2\mu_0] \\ &= \ln[2(\delta+\sqrt{n})^2\mu_0(\frac{\delta^2+\sqrt{n}}{\delta+2\sqrt{n}})^{k^*}] \\ &= \ln[2(\delta+\sqrt{n})^2\mu_{k^*}]\end{aligned}$$

this implies

$$2(\delta+\sqrt{n})^2\mu_{k^*} \leqslant \epsilon$$

■

If we denote by L the length of a binary coding of the input data of the problem $LCP(M,q)$, then we have

Corollary 10 *: If $Log_2(\mu_0) = o(L)$ and $Log_2(\epsilon) = -o(L)$ then the algorithm stops in at most $o(\sqrt{n}L)$ iterations with $o(n^{3.5}L)$ arithmetic operations.*

Proof. : From this assumption we have

$$\begin{cases} Log_2(\mu_0) = o(L) \\ \text{and} \\ Log_2(\epsilon) = -o(L) \end{cases}$$

then

$$k^* = o(\sqrt{n}L)$$

But to solve a linear system we need $o(n^3)$ arithmetic operations, then the algorithm converges in at most $o(n^{3.5}L)$ arithmetic operations. ■

1. Numerical example

In this part, we consider the following example to test our method. Consider the following linear complementarity problem

Example 11 *Find vector z in IR^n satisfying $z^T(Mz+q)=0$, $Mz+q \geqslant 0$, $z \geqslant 0$,*

$$\textit{where } M = \begin{bmatrix} 100 & -2 & -3 & -4 \\ -2 & 50 & -6 & -7 \\ -3 & -6 & 100 & -11 \\ -4 & -7 & -11 & 200 \end{bmatrix} \textit{ and } q = \begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix}.$$

The exact solution is $z^* = (0, \frac{4}{93}, 0, \frac{2}{93})^T$.

The solution of this problem with six significant digits is presented in following Table.

Iteration	x_1	x_2	x_3	x_4
k=01	*1,217700*	*4,030283*	*10,748913*	*6,834140*
k=05	*1,094700*	*3,627983*	*9,659913*	*6,147865*
k=10	*0,971700*	*3,225683*	*8,570913*	*5,461590*
k=15	*0,848700*	*2,823383*	*7,481913*	*4,775315*
k=20	*0,725700*	*2,421083*	*6,392913*	*4,089040*
k=25	*0,602700*	*2,018783*	*5,303913*	*3,402765*
k=30	*0,479700*	*1,616483*	*4,214913*	*2,716490*
k=35	*0,356700*	*1,214183*	*3,125913*	*2,030215*
k=40	*0,233700*	*0,811883*	*2,036913*	*1,343940*
k=45	*0,110700*	*0,409583*	*0,947913*	*0,657665*
k=50	*0,073800*	*0,288893*	*0,621213*	*0,451783*
k=52	*0,036900*	*0,168203*	*0,294513*	*0,245900*
k=53	*0,012300*	*0,096750*	*0,012340*	*0,145670*
k=54	*0,000000*	*0,043010*	*0,000000*	*0,021505*

Conclusion:

In this paper we have showed that, on the one hand, linear complementarity problem can be written as a quadratic convex programming problem; on the other, we have built a method to solve it; the convergence of this method requires $o(\sqrt{n}L)$ number of iterations where L is the length of a binary coding of the input data of the problem $LCP(M,q)$.

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