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A New Intermittency in Coupled Dynamical Systems

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A coupled-chaos system consisting of two subsystems is numerically studied slightly below the transition point separating the uniform chaos from a non-uniform state. It is shown that the system exhibits two types of intermittency, depending on the statistical property of the uniform chaos. One is the tangential type and the other is a new one having a highly intermittent characteristics with abrupt insertion of temporally localized oscillations.

In previous papers[1)~3)] we have developed the stability theory of synchronized motion in coupled-oscillator system. In the present note we report some further results concerning the coupled-chaos system near the instability point of the uniform chaos.

We consider the two-dimensional map[2),*)]

$$x_{n+1}=f(x_n)+\xi\{f(y_n)-f(x_n)\}, \qquad (1a)$$

$$y_{n+1}=f(y_n)+\xi\{f(x_n)-f(y_n)\} \qquad (1b)$$

with $\xi\equiv(1-e^{-\alpha})/2$, where α is a certain positive constant. x_n and y_n are the state variables in the first and the second subsystems at the discrete time n, respectively. If we put $\alpha=0$, x_n and y_n are independent. So α is the quantity measuring the coupling strength. Hereafter we assume that the uncoupled system $x_{n+1}=f(x_n)$ is chaotic and has the Lyapunov exponent $\lambda(>0)$.

The system (1) has a uniform solution $x_n=y_n\equiv x_n^0$, which obeys $x_{n+1}^0=f(x_n^0)$ and is called Ψ_{unif}. The difference $|x_n-y_n|$ evolves in time under two conflicting effects, the trajectory instability and the coupling effect. The former specified by the positive λ accelerates the difference and the latter shrinks the difference at the rate of α. Therefore the quantity $\lambda'\equiv\lambda-\alpha$ is relevant to the stability of Ψ_{unif} against infinitesimal perturbations.[1)~3)] If $\lambda'<0$, the coupling effect overcomes the trajectory instability and Ψ_{unif} is stable. For $\lambda'>0$, Ψ_{unif} loses stability and the system changes into a non-uniform state. Thus as α is decreased from a value large enough, we meet the transition[1),2)] at

$$\alpha=\lambda\equiv\alpha_c. \qquad (2)$$

How can be the non-uniform state statistically characterized? In order to study this problem we tried to numerically solve (1) for α slightly below α_c.

By making use of the logistic parabolla $f(x)=ax(1-x)$ for two values of a, $a=4$, ($\lambda=\ln 2=0.693147\cdots$) and 3.8, ($\lambda=0.4323\cdots$), numerical calculations were carried out in two cases, case A ($a=4$ and $\alpha=0.69$), and case B ($a=3.8$ and

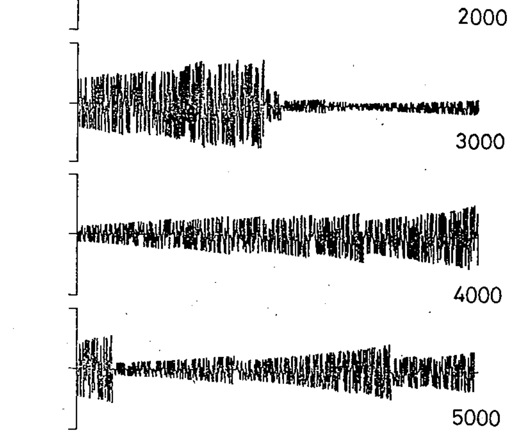

Fig. 1. Temporal evolution of $v_n(=(x_n-y_n)/2)$ for case A ($a=4$ and $\alpha=0.69$). The initial condition was chosen as $x_0=1/\sqrt{2}$ and $y_0=x_0+10^{-4}$. After initial 7×10^5 steps, 5000 steps are drawn.

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*) Recently a similar system was studied by Pikovsky.[4)]

$\alpha=0.43$). $\Psi_{\rm unif}$ is therefore unstable for both cases because $\lambda'>0$. In real calculations we solved

$$\bar{x}_{n+1}=a\bar{x}_n(1-\bar{x}_n)-ae^{2z_n}, \qquad (3a)$$

$$z_{n+1}=z_n+\ln|a(1-2\bar{x}_n)|-\alpha, \qquad (3b)$$

instead of (1), where $\bar{x}_n\equiv(x_n+y_n)/2$ and $z_n\equiv\ln(|x_n-y_n|/2)$. All calculations except on power spectra were carried out in the quadruple precision. Power spectra were calculated in the double precision.

Case A: The temporal evolution of $v_n\equiv(x_n-y_n)/2$ after a sufficiently large number of intial steps is drawn in Fig. 1. One easily observes that the amplitude $r_n=|v_n|$ has two typical behaviors: (i) the exponentially growing region approximately having the form

$$r_n\simeq e^{\beta n}r_0, \quad (\beta>0) \qquad (4)$$

with a constant β and (ii) the sudden diminishment of r_n. When $r_n\simeq 0$, (1) can be linearized in r_n as $r_{n+1}=|f'(\bar{x}_n)|e^{-\alpha}r_n$, $(f'(x)\equiv df(x)/dx)$. Solving this yields $r_n\simeq e^{\lambda' n}r_0$ if we put $\prod_{j=0}^{n-1}|f'(\bar{x}_j)|\propto e^{\lambda n}$. This suggests that $\beta=\alpha_c-\alpha=\lambda'$ holds. In fact the value $\lambda'=\ln 2-0.69=0.003\cdots$ coincides with the growth rate β estimated from Fig. 1. As r_n grows, the non-linearity with respect to r_n comes to be crucial and makes r_n suddenly decrease. Such behavior is similar to a sudden reversal known in the tangent bifurcation.[5]

We have calculated the probability density $P_L(r)$ for r_n(Fig. 2), where $P_L(r)$ is normalized in the region $0\leqq r\leqq L$ for a given L. To get $P_L(r)$ numerically, the r-space $(0\leqq r\leqq L)$ was divided into 200 subspaces, and the probability density was calculated by counting the number of phase points being in each space. For a sufficiently small L, $P_L(r)$ seems to obey the power law

$$P_L(r)\simeq cr^{-1}(r/L)^{\eta}\propto r^{-1+\eta}, \qquad (5)$$

where $\eta\simeq 0.8\sim 0.9\simeq c$.

Since, except the time regions corresponding to sudden diminishment of r_n, r_n can be well described by a single time scale $1/\beta(=1/\lambda')$, we can estimate the double time correlation function $C_n\equiv\langle r_n r_0\rangle-\langle r_0\rangle^2$ as $C_n\simeq C_0 e^{-\lambda' n}$.[6] This leads to the power spectrum

$$S(\omega)\simeq C_0\frac{\sinh(\lambda')}{\cosh(\lambda')-\cos(\omega)}. \qquad (6)$$

As is shown in Fig. 3, the theoretical result (6) is

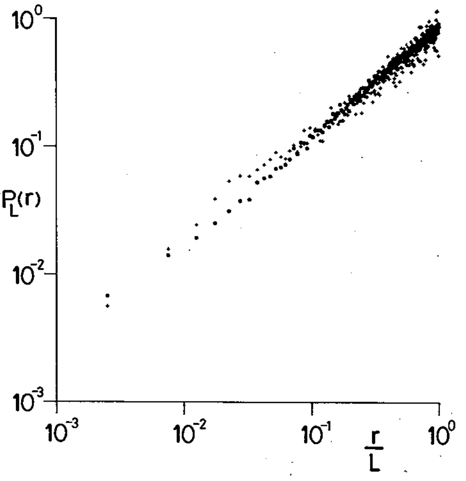

Fig. 2. The steady probability density $P_L(r)$ for $r_n(=|v_n|)$ for case A. It was calculated with 2×10^6 step data after intial 2×10^5 steps. The intial condition is the same as in Fig. 1. $P_L(r)$ is normalized in the region $0\leqq r\leqq L$, where $L=200\times 10^{-7}(\cdot)$ and $200\times 10^{-8}(+)$. By assuming (5), the least mean square method gives $\eta\simeq 0.86$ and $c\simeq 0.85$ for $L=2\times 10^{-5}$, and $\eta\simeq 0.81$ and $c\simeq 0.80$ for $L=2\times 10^{-6}$.

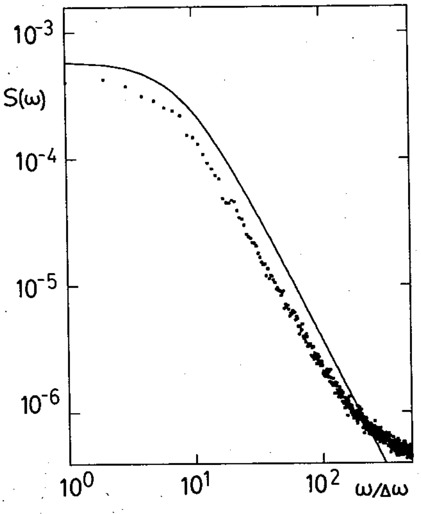

Fig. 3. The comparison of the theoretical power spectrum (6) (solid line) with the numerical results $(\cdot)$ (case A). In (6), C_0 has been used as the value numerically obtained $(\simeq 9.03\times 10^{-7})$. The power spectrum was calculated by averaging 200 power spectra each of which was obtained by using FFT for the sampling length 2^{14} which is much larger than the characteristic time $1/\lambda'\simeq 320$, $(\Delta\omega=2\pi/2^{14})$.

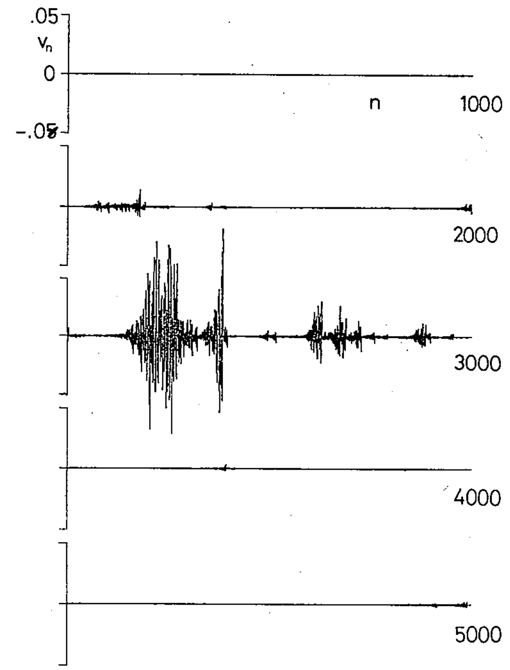

Fig. 4. Temporal evolution of v_n for case B ($a=3.8$ and $\alpha=0.43$). After initial 7×10^5 steps, 5000 steps are drawn. The initial condition is the same as in Fig. 1. v_n exhibits a highly intermittent behavior.

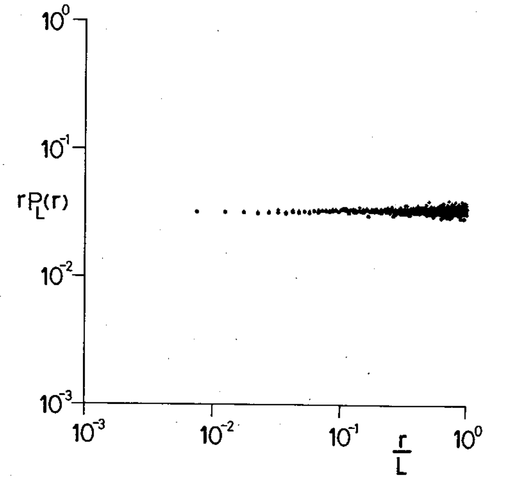

Fig. 5. The probability density $P_L(r)$ for r_n in case B. $P_L(r)$ was calculated with 2×10^6 step data after initial 2×10^5 steps. The initial condition is the same as in Fig. 1. $P_L(r)$ is normalized in the region $0\leq r\leq L$, where $L=200\times10^{-7}(\cdot)$ and $200\times10^{-8}(+)$. It seems that $P_L(r)$ obeys an inverse power law except an extremely small r.

in fairly good agreement with the numerical data.

Case B: The temporal evolution of v_n after a sufficiently large number of initial steps is plotted in Fig. 4. v_n exhibits a highly intermittent characteristics and there abruptly insert temporally localized oscillations. We call them *bursts*. One typical burst consists of two processes, the growing and the decaying of r_n. The present type of intermittency is extremely different from that in case A, but is similar to the characteristics observed by the high-pass filtered signal of velocity fluctuations in a turbulent flow.[7)]

The origin of such highly intermittent behavior may be interpreted as follows. When r_n is sufficiently small, ond has $r_n=e^{\zeta_n}r_0$, where $\zeta_n=\sum_{j=0}^{n-1}\{\ln|f'(\bar{x}_j)|-\alpha\}$. In contrast to case A where ζ_n is approximately written as $\zeta_n=\lambda' n+o(n)$ for n of order $1/\lambda'$, irrespective of the initial condition, ζ_n in the present case strongly fluctuates and cannot be written as $\simeq\lambda' n+o(n)$.[6)] If ζ_n starts to be positive, r_n exponentially grows and results in the ignition of a burst. In the other situation ($\zeta_n<0$) r_n shrinks, leading to the extinction of the burst.

The probability density $P_L(r)$ numerically obtained is shown in Fig 5. One observes that it has the inverse power law behavior

$$P_L(r)\propto r^{-1+\eta},\ (\eta\simeq0) \tag{7}$$

for $r/L>k$ $(=1/200)$. It is hard to detect the deviation of the exponent from unity. One easily sees that $P_L(r)$ has a remarkable deviation from the power law (7) for an extremely small $r(r/L<k)$. This deviation may be understood as follows. Since (7) is not integrable at $r=0$, for any tiny subspace unit l_m $(=L/N$, N being the number of subspaces) the probability density integrated from 0 to l_m diverges. In real calculations, however, it remains finite, but is outstandingly large.

As was discussed above, the temporal evolution of r_n cannot be described by a single time scale $1/\lambda'$ because ζ_n strongly fluctuates in the course of time. In Fig. 6, (6) is compared with the numerical result to recognize the inapplicability of the single time approximation for the double time correlation function. It seems that the numerical power spectrum is rather close to a ω^{-1} form.

We have found two different intermittency characteristics slightly below the transition point separating the uniform chaos from the nonuniform one. It seems that in case A the tempo-

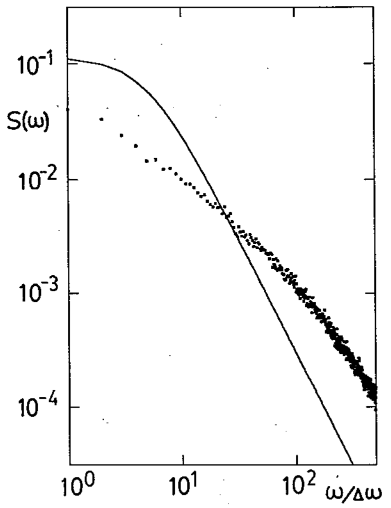

Fig. 6. The power spectrum for case B. The power spectrum was calculated by averaging 200 power spectra each of which was obtained by using FFT for the sampling length 2^{14}, ($\Delta\omega=2\pi/2^{14}$). One easily sees that the single time approximation (6) (solid line) is far from the numerical data, where C_0 has been used the numerical result ($\simeq 9.82\times10^{-5}$). The numerical power spectrum seems to be rather close to a ω^{-1} form.

ral evolution of r_n can be well described by the single time $1/\lambda'$, while in case B there appear many time scales. The difference of the intermittency characteristics stems from the difference of the statistical property of the uniform chaos $x_{n+1}^0=ax_n^0(1-x_n^0)$, depending on the value of a.[6] Namely in contrast to the fact that the trajectory instability of $x_{n+1}^0=4x_n^0(1-x_n^0)$ can be described by only one exponent $\lambda(=\ln 2)$, we need an infinite number of exponents to characterize the trajectory instability of $x_{n+1}^0=3.8x_n^0(1-x_n^0)$. The appearance of such an infinite number of exponents is due to the non-uniformity of the local divergence rate $\ln|f'(x_n^0)|$.[6] Furthermore as was already shown, such difference is also seen in the asymptotic forms of the probability density $P_L(r)$. Further analyses and more details will be given in the forthcoming paper.[6]

Finally we note that another type of intermittency has been studied by Kaneko[8] in a coupled logistic lattice with a similar coupling form as in Eq. (1) but for different parameter values.

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